ALGEBRAIC GEOMETRY AND THE GENERALISATION OF BEZOUT'S THEOREM

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Introduction

It is my aim in this thesis to provide an understanding of some of the concepts and ideas behind algebraic geometry.

I have discovered throughout the course of this year that algebraic geometry is an extremely rich and well developed field. I now fully understand my supervisor, Daniel Chan, when he tells me that it has been said that it is possible to give two hundred introductory courses on algebraic geometry with no overlapping material.

Rene Descartes, in the seventeenth century was the first to observe that the conic sections, and other more general curves on the plane, can be described as the set of solutions to a particular polynomial $f(x, y)$ in two variables. This is an algebraic description of a geometric object and I feel that this discovery by Descartes must rank among the most revolutionary ever made in mathematics.

Algebraic geometry is essentially the study of the solutions of systems of polynomials $f(x_1, x_2, \ldots, x_n)$ in n variables over some field.

I decided that the best way to get a feel for the subject was to look at a specific problem which uses results above the introductory level. This provides a motivation for studying some of the more abstract concepts, since it is easy to see the use of such concepts when dealing with a specific application.

Bezout's theorem provides a good motivation.

The theorem states that

> Any two distinct curves, f and g, on the projective plane, of degrees d and e respectively, will meet in exactly de points, counting multiplicities.

Bezout's theorem theorem is due to Isaac Newton, who proved a version of the theorem in the seventeenth century, and Etienne Bezout, who proved the result in his Phd thesis in 1779 in Paris. That is to say, the theorem is over two hundred years old.

Algebraic geometry provides us with the tools to state and prove Bezout's theorem using a modern language and approach. Moreover, this language allows us to prove the existence of a natural generalisation of the theorem to more complicated projective surfaces.

The generalisation of Bezout's theorem is part of the area of algebraic geometry known as intersection theory. Intersection theory is a very rich field in itself, as can be noted from browsing through William Fulton's two volume, *Intersection Theory*, (not to be confused with the introductory survey by Fulton mentioned in the bibliography).

I aim to do two things in this thesis. Firstly, to provide an explanation of the basic concepts and nature of algebraic geometry. Secondly, to look at some of the more advanced ideas, such as sheaf theory and divisors, and understand how they can be used to prove the existence of a remarkable generalisation to Bezout's theorem.

I feel that it would have been impossible to gain a fully rigourous understanding of all the intricacies involved in sheaf theory and cohomology in the time available. Instead, I felt it more instructive to try and provide an understanding of what sheaves are and how they can be used in algebraic geometry.

In chapters 1, 2 and 3 I explain the concepts of affine and projective varieties. Varieties are the one fundamental objects studied in algebraic geometry. Mr Rene Descartes himself was studying a variety when considering the solutions to a polynomial $f(x, y)$.

In chapters 4 and 5 I introduce divisors and explain how they are in some sense a generalisation of the notion of curves.

In chapter 6 I introduce sheaves and discuss some of their properties.

In chapter 7 the concepts from the previous chapters are used, along with some rules of cohomology, to show the existence of a generalisation to Bezout's theorem, which then drops out as a trivial corollary.

I assume the reader is familiar with the notions of a group, ring, field and module and some of their basic properties.

Unless indicated, k will denote an algebraically closed field.

Contents

Chapter 1

Hilbert's basis theorem

1.1 Noetherian rings

In this chapter we meet the concept of a Noetherian ring. Most of the rings of importance in algebraic geometry are Noetherian and it is necessary the basic properties of these rings. The characterisation of rings in this manner allows us to give a simple statement and proof of the useful Hilbert Basis Theorem. The theorem tells us that any ideal of the polynomial ring $k[x_1, x_2, \ldots x_n]$ where k is an algebraically closed field, is finitely generated.

Definition 1.1.1. A ring R satisfies the *ascending chain condition* or *a.c.c* if for ascending chain $\{J_1\,,\ J_2\,,\ \ldots\} \subset R$ satisfying

$$J_1 \subset J_2 \subset \ldots$$

of ideals in R, there exists an n such that $J_n = J_{n+1} = \ldots$ We say that the ascending chain of ideals *stabilises*.

Note that the definition above can be applied more generally to any partially ordered set. Saying that such a set satisfies the a.c.c is then just the same as saying that every totally ordered subset S has a maximal element, i.e an $m \in S$ such that $m < s$ does not hold for any $s \in S$.

Proposition-Definition 1.1.2. *Let R be a ring. The following are equivalent:*

1. *R satisfies the a.c.c on ideals.*
2. *Every non empty set S of ideals of R contains a maximal element.*
3. *Every ideal I of R is finitely generated, that is, there exist $r_1, r_2, \ldots, r_n \in R$ such that $I = (r_1, r_2, \ldots, r_n)$.*

A ring R satisfying the above conditions is called a Noetherian ring

Proof. $1 \to 2$. Suppose not, say S does not have a maximal element. Then given any $J_1 \in S$ there exists a $J_2 \in S$ satisfying $J_1 \subset J_2$. Continuing inductively we can form a non terminating ascending chain $J_1 \subset J_2 \subset \ldots$ contradicting 1.

$2 \to 3$. Let S be the set consisting of all finitely generated ideals J which are contained in I. Let J^* be the maximal element of S. Then J^* must equal I. For if $r \in J^* \backslash I$ then (J^*, r) contains both J^* and I and is finitely generated, contradicting the maximality of J^*.

$3 \to 1$. Given an ascending chain of ideals

$$J_1 \subset J_2 \subset \ldots$$

we let $J^* = \cup J_i$. By 3. there exists $r_1, r_2, \ldots, r_n \in R$ such that $J^* = (r_1, r_2, \ldots, r_n)$. From the definition of J^* and the fact that the J_i form an ascending chain, we must have that $J_k = (r_1, r_2, \ldots, r_n)$ for some k. This clearly implies that

$J_k = J_{k+1} = \ldots$ which completes the proof. □

Corollary 1.1.3. *If R is Noetherian and I is an ideal of R, then the quotient R/I is Noetherian.*

Proof. From results in ring theory we have the fact that ideals $\bar{J}$ of R/I are in one to one correspondence with ideals J of R containing I. So, given an ascending chain

$$\bar{J}_1 \subset \bar{J}_2 \subset ... \subset R/I$$

of ideals in R/I, we can form the corresponding chain,

$$J_1 \subset J_2 \subset ... \subset R$$

of ideals in R, which will be an ascending chain since inclusion is preserved by the correspondence between ideals of R/I and ideals of R containing I. The chain in R stabilises, giving a J_k such that $J_k = J_{k+1} = ...$. By the correspondence we then have that $\bar{J}_k = \bar{J}_{k+1} = ...$, which implies that R/I is Noetherian. □

Remark 1.1.4. We observe that if R is a principal ideal domain (every ideal is generated by a single element), then R is Noetherian. To see this let the ideals $J_i \mathit{subseteq} R$ form an ascending chain and let r_i be the generator of J_i. Now if $J_1 \subset J_2$ is strict, then $r_1 = a_1 r_2$ for some non-unit a_1. We repeat this argument to see that $r_2 = a_2 r_3$ and so on. If $b_1 b_2 .... b_k$ is the prime factorisation of r_1 into irreducible non-units b_i, then it is easy to see that any strictly ascending chain of ideals starting at J_1 has maximum length k. An example of such a chain is shown by letting $r_1 = b_1 r_2,\ r_2 = b_2 r_3,\ ....$ and we find $r_1 = (b_1 b_2 ... b_k) r_{k+1}$ which implies that r_{k+1} is a unit, thus $J_{k+1} = R$.
A field k is Noetherian since every element is a unit and hence 1 is in every ideal of k. That is to say that k has no proper ideals and thus no strictly ascending chains of ideals apart from $0 \subset k$. □

Whilst many rings, and especially those of interest to us, are Noetherian it is worth seeing some that are not, for example:

1. Let k be an algebraically closed field. Let $R = k[x_1, x_2, \ldots]$ be the polynomial ring with an infinite number of indeterminates. Then R is certainly not Noetherian since, for example, the ideal $(x_1, x_2 \ldots)$ is not finitely generated.
2. Let $R = k[x, y, x/y, x/y^2, x/y^3, \ldots]$. Then R is not Noetherian since the ascending chain

$$(x) \subset (x/y) \subset (x/y^2) \subset \ldots.$$

is non terminating.

1.2 Hilbert's basis theorem

The polynomial rings $R[x_1, x_2 \ldots]$ are of most importance in algebraic geometry. In most cases we let R be an algebraically closed field k. We would like to know whether R being Noetherian implies that $R[x]$ is Noetherian, for if so then we will know that ideals of $R[x_1, x_2 \ldots]$ are finitely generated which often simplifies computations considerably. Hilbert's basis theorem verifies this.

Theorem 1.2.1. *(Hilbert's Basis Theorem)*

If R is a Noetherian ring, then so is $R[x]$

Proof. See [**R1**] chapter II, Theorem 3.3. It is a very straightforward proof, once one knows how. □

The following simple corollary contains the most important use as far as we are concerned of Hilbert's Basis theorem. We will see that the polynomial rings with a finite number of variables and their quotient rings are Noetherian. Most importantly, any ideal is finitely generated.

Corollary 1.2.2.

1. *The ring $R = k[x_1, x_2, ... x_n]$ with k an algebraically closed field is Noetherian.*
2. *If I is an ideal of $R = k[x_1, x_2, ... x_n]$ then R/I is Noetherian.*

Proof.

1. An obvious appeal to induction and the fact, see 1.1.4, that k is Noetherian makes this clear.
2. Follows immediately from 1.1.3 and part 1.

□

Chapter 2

Affine Varieties and the Nullstellensatz

2.1 Affine Varieties

We define affine n-space to be the vector space over k denoted by $\mathbb{A}^n$ consisting of all elements of the form $(a_1, a_2, \ldots, a_n)$. The a_i are called the *coordinates* and we call the elements of $\mathbb{A}^n$ *points*. The operations are coordinate-wise addition and scalar multiplication.

Definition 2.1.1. Let S be a set of elements of $k[x_1, x_2, \ldots, x_n]$. S then determines a subset of $\mathbb{A}^n$ denoted $V(S)$ and defined by

$$V(S) = \{ p \in \mathbb{A}^n \mid f(p) = 0 \text{ for all } f \in S\}$$

We call any subset of A^n which is of the form $V(S)$ an *affine variety*. We drop the affine when there is no confusion and just call such a set a variety. If $f \in R$ we write $V(f)$ for the variety determined by f.

The next proposition shows that we can take S to be an ideal.

Proposition 2.1.2. *If S is a set of elements in $R = k[x_1, x_2, \ldots, x_n]$, then $V(S) = V(I)$ for some ideal I in R.*

Proof. Let J be the ideal generated by S. Then clearly $V(S) \subseteq V(J)$. Conversely if a point is a zero of all the polynomials in J then it is a zero of the generating set of J, that is, a zero of all the polynomials in S. So $V(S) \supseteq V(J)$. Completing the proof. $\square$

Remark 2.1.3. Since R is Noetherian we observe that any ideal J in R is finitely generated, say by $f_1, f_2, \dots, f_n$. Thus

$$V(J) = \{p \in A^n \mid f_i(p) = 0 \quad \text{for} i = 1, 2, \dots, n\}$$

That is, a variety is always specified by a finite number of polynomials $\square$

Some basic properties of varieties are given in the following proposition

Proposition 2.1.4. *Let J_1 and J_2 be ideals in $R = k[x_1, x_2, \dots, x_n]$, then*

1. *If $J_1 \subseteq J_2$ then $V(J_1) \supseteq V(J_2)$*
2. *$V(J_1 J_2) = V(J_1) \cup V(J_2)$*
3. *If $\{J_\lambda\}$ is any collection of ideals in R,*

   then $V(\sum_\lambda J_\lambda) = \bigcap_\lambda V(J_\lambda)$
4. *$V(0) = A^n$. $V(1) = \varnothing$.*

Proof. 1. Clear.

2. If $f \in J_1 J_2$ then $f = f_1 f_2$ with $f_1 \in J_1$ and $f_2 \in J_2$. So if $p \in V(J_1 J_2)$ but $p \notin V(J_1)$ then we must have $p \in V(J_2)$ since if not there exist $f_1 \in J_1$ and $f_2 \in J_2$ with $f(p) = f_1(p) f_2(p) \neq 0$, but $f \in J_1 J_2$ which contradicts $p \in V(J_1 J_2)$. Thus $V(J_1 J_2) \subseteq V(J_1) \cup V(J_2)$. The reverse equality is easy to see.

3. $V(\sum_\lambda J_\lambda) \subseteq \bigcap_\lambda V(J_\lambda)$ since for any $J \in \{J_\lambda\}$ we have that $J \subseteq \sum_\lambda J_\lambda$ and hence $V(\sum_\lambda J_\lambda) \subseteq V(J)$ from 1. Since this is the case for all $J \in \{J_\lambda\}$ we have that $V(\sum_\lambda J_\lambda) \subseteq \bigcap_\lambda V(J_\lambda)$. The reverse inequality is easy to see.
4. Every point is a zero of the polynomial 0 and no point is a zero of the polynomial 1.

□

Definition 2.1.5. We define the *Zariski topology* on $\mathbb{A}^n$ by taking the closed sets to be the affine varieties. The fact that this defines a topology follows from applying De-morgan's laws to see that both the whole set and the empty set are open, the union of any family of open sets is open and the intersection of any two open sets is open.

Remark 2.1.6. We will not be using this topology in a significant way in this thesis. The main point is to keep in mind what is meant by open affine set, that is, the complement of an affine variety. □

Example 2.1.7. Working in $\mathbb{A}^2$ we see that the zero set of any $f \in k[x_1, x_2]$ determines a variety which we write as $V(f)$. We call such varieties in $\mathbb{A}^2$ *curves*. □

2.2 The ideal associated with a subset of affine space

We can associate an ideal in $k[x_1, x_2, \ldots x_n]$ with a subset of $\mathbb{A}^n$

Definition 2.2.1. Let $R = k[x_1, x_2, \ldots x_n]$ and let X be a subset of $\mathbb{A}^n$. The *ideal of* X, $I(X)$ is defined by,

$$I(X) = \{ f \in R \mid f(p) = 0 \text{ for all } p \text{ in } X \}$$

That is, $I(X)$ is the set of all polynomials vanishing at all points of X.

Remark 2.2.2. We observe that $I(X)$ is an ideal of R. For if f and g vanish on X, then so does $f + g$ and if $h \in R$ then hf vanishes on X. □

Some simple properties are given in the following proposition.

Proposition 2.2.3. *Let $X_1 \subseteq X_2 \subseteq \mathbb{A}^n$ and let J_1 and J_2 be ideals in R. Then*

1. *If $X_1 \subseteq X_2$*

   then $I(X_1) \supseteq I(X_2)$.

2. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$
3. $J \subseteq I(V(J))$, $X \subseteq V(I(X))$
4. $V(J) = V(I(V(J)))$.

Proof. I omit the proofs of 1 - 3 as they are fairly obvious. To prove the 4 we have from 3 that $J \subseteq I(V(J))$. We apply V to get $V(J) \supseteq V(I(V(J)))$. However from 3 we also have $X \subseteq V(I(X))$. Letting $X = V(J)$ we have $V(J) \subseteq V(I(V(J)))$. which finishes the proof of 4. □

Corollary 2.2.4. *If X and Y are varieties, then $I(X) = I(Y)$ iff $X = Y$.*

Proof. Assume that $I(X) = I(Y)$, then $V(I(X)) = V(I(Y))$. Since X and Y are varieties (that is, of the form $V(J)$), we get from part 4 that $X = V(I(X)) = V(I(Y)) = Y$ completing the proof □

Remark 2.2.5. We will see shortly that the inclusions in 3 can be strict. This follows from a theorem of Hilbert known as the nullstellensatz. □

2.3 Irreducible Varieties

Definition 2.3.1. Let X be a variety. A subset $Y \subseteq X$ is called a *subvariety* of X if Y is also a variety. A subvariety is *proper* if the inclusion is strict.

Example 2.3.2. Let $X = A^2$. If $f \in k[x_1, x_2]$ is any polynomial, then $V(f) \subseteq X$ is a subvariety. □

Definition 2.3.3. A variety X is called *irreducible* if it can not be written as the union of two proper subvarieties.

Proposition 2.3.4. *Every variety can be written as a union of irreducible varieties.*

Proof. Let X be a variety. If X is irreducible then we are done. If not $X = Y_1 \cup Y_2$. Now consider Y_1 and Y_2. If they are both irreducible then we are done. If not then continue on. We claim that the process must stop after a finite number of steps. If not we will obtain an infinite descending chain $X \supset X_1 \supset X_2 \supset ...$

where the inclusions are strict. Applying I we get an infinite ascending chain $I(X) \subset I(X_1) \subset I(X_2) \subset \ldots$ of ideals in $R = k[x_1, x_2, \ldots, x_n]$ where by 2.2.4 the inclusions are again strict. This contradicts the fact that R is Noetherian which completes the proof. □

We now prove a correspondence between irreducible varieties and prime ideals.

Proposition 2.3.5. *Let $R = k[x_1, x_2, \ldots, x_n]$ and let $X \subseteq \mathbb{A}^n$ be a variety. Then $I(X)$ is a prime ideal iff X is irreducible.*

Proof. $\Rightarrow$. Suppose not and let $X = X_1 \cup X_2$ with $X_1, X_2 \neq X$. We have from 2.2.4 that $I(X_1), I(X_2) \neq I(X)$, so, choose $f \in I(X_1) - I(X)$ and $g \in I(X_2) - I(X)$. Then fg vanishes on all of $X = X_1 \cup X_2$, that is, $fg \in I(X)$, however by definition neither f nor g are in $I(X)$, contradicting 1.

$\Leftarrow$. Let $fg \in I(X)$, then $V(I(X)) \subseteq V(fg) = V(f) \cup V(g)$. Since X is a variety $V(I(X)) = X$ and so $X \subseteq V(f) \cup V(g)$ which means $X = (X \cap V(f)) \cup (X \cap V(g))$ is the union of two subvarieties (the intersection of two varieties is a variety). Since X is irreducible we have that $(X \cap V(f)) = X$ which implies $X \subseteq V(f)$ so that $I(V(f)) \subseteq I(X)$. Since $(f) \subseteq I(V(f))$ we have that $(f) \subseteq I(X)$ which means that $f \in I(X)$. Similarly for g which completes the proof. □

2.4 Hilbert's Nullstellensatz

Definition 2.4.1. Let $J \subseteq R$ be an ideal. We define the *radical* of J, denoted $\sqrt{J}$ by

$$\sqrt{J} = \{ f \in R \mid f^n \in I \quad \text{for some positive integer } n \}$$

An ideal J is *radical* if $\sqrt{J} = J$

I now state the Nullstellensatz which, in particular, provides a correspondence between radical ideals and varieties.

Theorem 2.4.2. *Let J be an ideal in $R = k[x_1, x_2, \dots, x_n]$. Then $I(V(J)) = \sqrt{J}$. In particular, if J is radical then $I(V(J)) = J$*

Proof. Most books on algebraic geometry or commutative algebra will contain a proof. See for example [**R3**] chapter 5. The proof is omitted due to the fact that a number of lemmas are needed and it may take up a little too much space. □

Corollary 2.4.3. *If J is an ideal of R, then $V(J) = \varnothing$ iff $1 \in J$*

Proof. If $V(J) = \varnothing$ then $R = I(\varnothing) = I(V(J)) = \sqrt{J}$. That is to say that $1^n = 1$ is in J. For the converse we note that no point is a zero of the constant polynomial 1. □

Corollary 2.4.4. *Every maximal ideal of R is of the form $(x_1 - a_1, \dots, x_n - a_n)$. For some $a_1, \dots, a_n \in \mathbb{A}^n$. Similarly, every ideal of this form is maximal.*

Proof. Let I be a proper ideal of R. Then from the above corollary $V(I)$ contains at least one point, say $(a_1, \dots, a_n)$. Now if M is the ideal $(x_1 - a_1, \dots, x_n - a_n)$ then M is maximal since x_i is $a_i \in k$ in the quotient R/M and so R/M is a field (in fact isomorphic to k). It is clear that $V(M)$ consists of the single point $(a_1, \dots, a_n)$ Thus $V(M) \subseteq V(I)$. We then get $I \subseteq I(V(I)) \subseteq I(V(M)) = \sqrt{M} = M$ where the last equality follows from M being maximal, hence prime, hence radical. It follows that every proper ideal is contained inside one of the same form as M which completes the proof. □

The material in the above two sections now gives us the following 1-1 correspondences induces by applying V and I to get from left to right or right to left respectively.

$$\{ \text{ radical ideals in R } \} \quad \longleftrightarrow \quad \{ \text{ varieties in } \mathbb{A}^n \}$$

$$\{ \text{ prime ideals in R } \} \quad \longleftrightarrow \quad \{ \text{ irreducible varieties in } \mathbb{A}^n \}$$

$$\{ \text{ maximal ideals in R } \} \quad \longleftrightarrow \quad \{ \text{ points of } \mathbb{A}^n \}$$

Remark 2.4.5.

Observe that there inclusion relations between the rows, That is, every maximal ideal is a prime ideal and every prime ideal is a radical ideal. Similarly, every point is an irreducible variety and every irreducible variety is a variety. □

2.5 The coordinate ring of a variety

Let $X = \mathbb{A}^n$ and $R = k[x_1, x_2, \dots, x_n]$.

Definition 2.5.1. If $Y \subseteq X$ is any subvariety then we define the *coordinate ring* $k[Y]$ of Y, by $k[Y] = R/I(Y)$ where $I(Y)$ is the ideal of polynomial functions vanishing on Y. Observe that $k[X] = R$ since only the zero polynomial vanishes on all of X.

Remark 2.5.2. If Y is an irreducible subvariety then $I(Y)$ is prime so $k[Y]$ is an integral domain. The coordinate ring of any variety is a finitely generated k-algebra. For example $k[Y]$ is generated by 1 and the residues (or images) of the x_i in the natural surjection $R :\longrightarrow R/I(Y)$. Also, by the Nullstellensatz $I(Y)$ is a radical ideal so $k[Y]$ has no nilpotent elements (ie elements f s.t. $f^n = 0$ for some n). Conversely, given any finitely generated k-algebra A with no nilpotents, we write A as a quotient $k[x_1, x_2, \dots, x_m]/J$ for some m and radical ideal J (this is the def'n of a finitely generated k-algebra with no nilpotents). Take $Y = V(J)$. Since J is radical $I(Y) = J$ from the Nullstellensatz thus A is the coordinate ring of some variety Y. $\square$

Definition 2.5.3. Let $Y \subseteq X$ be an irreducible variety. A *rational function* on Y is an element of the quotient field of the domain $k[Y]$. Denote the set of rational functions on Y by $k(Y)$. More precisely

$$k(Y) = \{ \frac{f}{g} \mid f, g \in k[Y] \}$$

t **Remark 2.5.4.** Note that there may be more than one way to write and element $h \in k(Y)$. For example, if $Y = V(x_1x_2 - x_3x_4) \subseteq \mathbb{A}^4$ then $\frac{x_1}{x_3} = \frac{x_4}{x_2}$ in $k(Y)$. $\square$

Definition 2.5.5. Let $Y \subseteq X$ be an irreducible variety. Let $p \in Y$. An element $h \in k(Y)$ is *regular* at p if it is possible to write h in the form $\frac{f}{g}$ (with $f, g \in k[Y]$) such that $g(p) \neq 0$. A rational function $h \in k(Y)$ is *regular on* Y if it is regular at all $p \in Y$.

We let $\mathcal{O}_{Y,p}$ denote the set of all rational functions regular at $p \in Y$ and we let $\mathcal{O}_Y$ denote the set of all rational functions regular on Y.

Remark 2.5.6. It is not hard to see that both $\mathcal{O}_{Y,p}$ and $\mathcal{O}_Y$ are rings under the usual addition and multiplication. $\square$

Proposition 2.5.7. $\mathcal{O}_Y \simeq k[Y]$

Proof. Suppose $f \in \mathcal{O}_Y$ is given. Define the ideal

$$J_f = \{ g \in k[Y] \mid gf \in k[Y] \}$$

J_f is the ideal of denominators of f. Since $f \in \mathcal{O}_Y$ we have that $V(J_f) = \varnothing$. From the Nullstellensatz we have that $1 \in J_f$. That is, $f \in k[X]$. □

Chapter 3

Projective Varieties

3.1 Projective space

In some respects affine geometry is unsatisfactory. In particular the fact that there exist parallel lines. They are the lines which do not intersect anywhere in the space. Projective geometry gets around this by adding "points at infinity" to affine space so that lines which might be considered parallel in the affine case then meet at infinity.

Definition 3.1.1. We define $\mathbb{P}^n$ projective n-space $\mathbb{P}^n$ in the following manner

$$\mathbb{P}^n = \{ x = (x_0, x_1, \dots, x_n) \in \mathbb{A}^{n+1} - 0 \}/\sim$$

where $\sim$ is the equivalence relation defined by $x \sim y$ if $x = \lambda y$ for some $\lambda \in k$. That is, x and y lie on the same line through the origin in $\mathbb{A}^{n+1}$. The elements $x \in \mathbb{P}^n$ are called *points* and the $(x_0, x_1, \dots, x_n)$ are the *homogeneous coordinates* of x. Let R be the polynomial ring $k[x_0, x_1, \dots, x_n]$. If we are interested in the zero set of a polynomial over projective space we need to deal with special polynomials f such that if $x \in \mathbb{P}^n$ then $f(x) = 0$ iff $f(\lambda x) = 0$ for all non zero $\lambda \in k$.

Definition 3.1.2. A *homogeneous polynomial* in n variables is a polynomial $f \in k[x_1, x_2, \ldots, x_n]$ such that there exists a $d \in \mathbb{N}$ such that $f(\lambda x) = \lambda^d f(x)$ for all $\lambda \in k$ and all $x \in \mathbb{A}^n$.

Remark 3.1.3. It can be shown that any homogeneous polynomial f in n variables is a linear combination of elements of the form $x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}$ where $r_1 + \ldots r_n = d$ in which case we call d the *degree* of f. □

Definition 3.1.4. A *line* in $\mathbb{P}^2$ is the set of solutions to a homogeneous polynomial of degree 1.

Proposition 3.1.5. *Any two distinct lines in $\mathbb{P}^2$ meet in exactly one point.*

Proof. Let x, y, z be the coordinates on $\mathbb{P}^2$ and let $a_1 x + a_2 y + a_3 z = 0$ and $b_1 x + b_2 y + b_3 z = 0$ define two distinct lines. To find their intersection we row reduce the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

assuming w.l.o.g. that $(a_1, a_2) \neq (b_1, b_2)$ this reduces to

$$\begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \end{pmatrix} \qquad \text{for some fixed } \alpha,\ \beta \in k .$$

Thus the two lines meet at $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix} \qquad \lambda \in k$$

This is a line through the origin in $\mathbb{A}^3$ and thus a single point in $\mathbb{P}^2$. □

3.2 Projective varieties

A *homogeneous ideal* in $R = k[x_0, x_1, \dots, x_n]$ is an ideal generated by a collection of homogeneous elements of R. It can be shown that the sum, product, intersection and radical of a homogeneous ideal are homogenous ideals.

We now define projective varieties.

Definition 3.2.1. A subset $X \subseteq \mathbb{P}^n$ is a *projective variety* if there exists a homogenous ideal $J \subseteq R$ such that $X = V(J)$

Remark 3.2.2. We observe that projective varieties are well defined. For if f is a generator of the homogeneous ideal J then f is homogeneous of some degree d and so the fact that $f(\lambda x) = \lambda^d f(x)$ for all $\lambda \in k$ tells us that $f(x) = 0$ iff $f(\lambda x) = 0$ for non zero $\lambda \in k$. That is, f is zero on x implies that f is zero on all elements equivalent to $x \in X \subseteq \mathbb{P}^n$. □

Similarly we can define the ideal of a subset $X \subseteq \mathbb{P}^n$.

Definition 3.2.3. Given a subset $X \subseteq \mathbb{P}^n$ we define the homogeneous ideal $I(X)$ of functions vanishing on X by

$$I(X) = \{ f \in R \mid f \text{ is homogeneous and } f(p) = 0 \text{ for all } p \in X \}$$

Remark 3.2.4. Many of the properties of affine varieties and ideals carry over to the projective case with almost no change in the proofs. I state some properties below and omit the proofs. □

Proposition 3.2.5. *Let $X_1, X_2 \subseteq \mathbb{P}^n$ and let J_1 and J_2 be homogeneous ideals in $R = k[x_0, x_1, \ldots, x_n]$. Then*

1. $V(J_1 J_2) = V(J_1) \cup V(J_2)$
2. *If $\{J_\lambda\}$ is any collection of homogeneous ideals, then $V(\sum_\lambda J_\lambda) = \bigcap_\lambda V(J_\lambda)$*
3. $V(0) = A^n$. $V(1) = \varnothing$.
4. *If $J_1 \subseteq J_2$, then $V(J_1) \supseteq V(J_2)$*
5. *If $X_1 \subseteq X_2$, then $I(X_1) \supseteq I(X_2)$.*
6. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$

Definition 3.2.6. As in the affine case we define the *Zariski* topology on $X \subseteq \mathbb{P}^n$ by taking the open sets to be the complements of varieties.

The following proposition gives a very useful correspondence between certain subsets of $\mathbb{P}^n$ and $\mathbb{A}^n$.

Proposition 3.2.7. *$\mathbb{P}^n$ is the union of $n+1$ affine sets U_i, the points of each in one to one correspondence with the points of $\mathbb{A}^n$.*

Proof. Let $U_i = \{ x \in \mathbb{P}^n \mid x_i \neq 0 \}$. It is clear that $\mathbb{P}^n$ is the union of the U_i $i = 0, 1, \ldots, n$.

Let $\varphi_0 : U_0 \longrightarrow \mathbb{A}^n$ be the map sending $(x_0, x_1, \ldots, x_n)$ to $(x_1/x_0, \ldots, x_n/x_0)$ The map is well defined since $x_i/x_0 = (\lambda x_i)/(\lambda x_0)$ for all non zero $\lambda \in k$. φ_0 establishes a one to one correspondence with $\mathbb{A}^n$ since we can assume that $x_0 = 1$ and then vary the x_i as we wish, obtaining all points of $\mathbb{A}^n$. The map easily generalises to give φ_i and the correspondence for each of the U_i. □

Remark 3.2.8. Observe that each U_i is an open set with respect to the Zariski topology, U_i being the complement of the variety $V(x_i)$. We call the union $\bigcup U_i$ an *affine open cover* of $\mathbb{P}^n$. If $Y \subseteq \mathbb{P}^n$ is a subvariety then we obtain an affine open cover $\bigcup Y_i$ of Y by letting $Y_i = Y \cap U_i$. □

3.3 Rational functions

Let $X = \mathbb{P}^n$ and let $R = k[x_0, x_1, \ldots, x_n]$.

We observe that a non constant homogeneous polynomial of in R does not determine a function on X since it will not be constant on equivalent elements in X (recall $f(\lambda x) = \lambda^d f(x)$ for some $d \in \mathbb{N}$ for all $\lambda \in k$). However, the quotient of two homogeneous polynomials of the same degree will define a function on X. To see this we observe that if f and g are homogeneous of the same degree d, then $f(\lambda x)/g(\lambda x) = \lambda^d f(x)/\lambda^d g(x) = f(x)/g(x)$ for all non zero $\lambda \in k$.

Definition 3.3.1. Let $Y \subseteq X$ be an irreducible variety with homogeneous ideal $I(Y)$. A *rational function* on Y is the quotient f/g of two homogeneous polynomials $f, g \in R$ both of the same degree d and reduced modulo $I(Y)$ with the requirement that $g \notin I(Y)$. We denote the set of all rational functions by $k(Y)$.

The notions of regularity at a point $p \in Y$ and regularity on all of Y carry over directly from the affine case.

If $Y \subseteq X$ is a variety with affine open cover $\bigcup Y_i$ we will often be interested in the coordinate rings $k[Y_i]$.

Definition 3.3.2. We define $k[X_i] = k[x_0/x_i\,,\, x_1/x_i\,,\, \ldots\,,\, x_n/x_i]$ as an affine coordinate ring. For $Y \subset X$ with ideal $I(Y)$ we define $k[Y_i] = k[X_i]/I(Y_i)$ where $I(Y_i)$ is the image of the ideal $I(Y)$ under the mapping $x_j \longmapsto x_j/x_i$ for $j = 1, 2, \ldots, n$. That is $f(x_0\,,\, x_1\,,\, \ldots\,,\, x_n) \longmapsto f(x_0/x_i\,,\, x_1/x_i\,,\, \ldots\,,\, x_n/x_i)$.

Remark 3.3.3. Observe that since Y_i is affine $k[Y_i]$ is the ring of all rational functions regular on Y_i. See chapter 2.5.

It is also worth noting, or taking as a definition that $k[X_i \cap X_j] = k[X_i][x_i/x_j]$ and similarly $k[X_i \cap X_j \cap X_k] = k[X_i \cap X_j][x_i/x_k]$ and so on. □

3.4 Dimension

We include here some facts about dimension

Definition 3.4.1. Let X be a variety. The *dimension* of X as a variety is the smallest non negative integer n such that for any ascending chain

$$X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_p$$

of subvarieties, the length p of the chain is $\leqslant n$. We let $dim(X)$ denote the dimension of X.

Varieties of dimension 1 are called *curves*.

Varieties of dimension 2 are called *surfaces*.

Fact 3.4.2.

1. $dim(\mathbb{A}^n) = dim(\mathbb{P}^n) = n$.
2. An affine variety $X \subseteq \mathbb{A}^n$ has dimension $n-1$ iff $X = V(f)$ for some irreducible nonconstant polynomial f.

3. An projective variety $X \subseteq \mathbb{P}^n$ has dimension $n-1$ iff $X = V(f)$ for some irreducible nonconstant homogeneous polynomial f.

□

See [**H**] Chapter I.2 or [**S**] Chapter I.6 for more details.

Remark 3.4.3. Observe that $\mathbb{P}^2$ is a surface according to the above definition of dimension. Similarly a variety $X = V(f) \subseteq \mathbb{P}^3$ where f is an irreducible homogeneous polynomial, is a surface. □

Chapter 4

Discrete Valuation Rings

4.1 Local rings

Definition 4.1.1. A *unit* in a ring R is an element $u \in R$ such that there exists a $v \in R$ such that $uv = 1$.

A *local ring* is a ring in which the set of non-units forms an ideal.

The following proposition gives an equivalent definition of a local ring.

Proposition 4.1.2. *A ring R is local if and only if it contains a unique maximal ideal.*

Proof. If R is local then the set of non units is an ideal. This ideal is clearly maximal since any non unit generates R as an ideal. Furthermore it contains all proper ideals of R , hence is the unique maximal ideal of R.

Conversely, let M be the unique maximal ideal of R and assume there exists a non-unit $a \in R \setminus M$. Then the ideal (M, a) is strictly larger than M but is not equal to all of R since it contains no units, contradicting the maximality of M. So M is the set of all non-units of R. □

Remark 4.1.3. Let X be an affine variety and let $p \in X$. The ring $\mathcal{O}_{X,p}$ of rational functions regular at P is easily seen to be a local ring. Since if f is a non-unit in $\mathcal{O}_{X,p}$ then there exist homogeneous polynomials $f,\ g$ of the same degree such that

$$f \;=\; \frac{g}{h} \qquad \text{with} \quad g(p) = 0\,,\ f(p) \neq 0$$

It follows that the set of non-units of $\mathcal{O}_{X,p}$ forms an ideal. □

We can go a little further in describing the structure of $\mathcal{O}_{X,p}$.

Proposition 4.1.4. *Let X be an irreducible affine variety and let $p \in X$. Then $\mathcal{O}_{X,p}$ is a local, Noetherian domain.*

Proof. That $\mathcal{O}_{X,p}$ is local has been shown. To see that $\mathcal{O}_{X,p}$ is a domain we consider the product of two elements f and g in $\mathcal{O}_{X,p}$. If $fg = 0$ in $\mathcal{O}_{X,p}$ then the product of their numerators is zero. The numerators are elements of $k[X]$ and so the fact that $k[X]$ is a domain tells us that $\mathcal{O}_{X,p}$ is also a domain. Thus it remains to show that any ideal of $\mathcal{O}_{X,p}$ is finitely generated. Let $I \subseteq \mathcal{O}_{X,p}$ be an ideal. We know that $k[X]$ is Noetherian. Suppose the generators of $k[X]$ are $f_1\,, f_2\,, \ldots f_n$. We claim that I is generated by at most n elements. This follows from the fact that for all $h \in \mathcal{O}_{X,p}$ there exists a $g \in k[X]$, $g(p) \neq 0$ and such that $gh \in k[X]$ (such a g exists from the definition of a rational function). We write gf as a linear combination of the generators f_i. That is,

$$gf \;=\; a_1 f_1 \;+\; \ldots \;+\; a_n f_n \qquad \text{with } a_i \in k[X]$$

We then divide by g (which is valid since $g(p) \neq 0$) to get

$$f \;=\; (\frac{a_1}{g}) f_1 \;+\; \ldots \;+\; (\frac{a_n}{g}) f_n \qquad \text{with } \frac{a_1}{g} \in \mathcal{O}_{X,p}$$

which, since $f \in I$ was arbitrary, implies that we can generate I with less than or equal to n elements, where n is the number of generators of $k[X]$. □

4.2 Discrete valuation rings

Proposition 4.2.1. *If R is a domain which is not a field, then the following conditions are equivalent:*

1. *R is Noetherian and local, and the maximal ideal M is principal.*
2. *There exists an irreducible element t in R such that if r is any non zero element in R, then $r = ut^n$ for some non negative integer n.*

Proof. $1 \Rightarrow 2$. Let t be a generator for the maximal ideal M and let r be a non zero element in R. If r is a unit there is nothing to prove. So assume r is a non unit. Then $r \in M$ since R is local. So $r = a_1 t$ for some non zero $a_1 \in R$. If a_1 is a unit we are done. If not a_1 *in* M and so $a_1 = a_2 t$ for some non zero $a_2 \in R$. This process must eventually stop and we will end up with a_n being a unit for some n. If not then we can continue inductively and form the ascending chain of ideals

$$(a) \subset (a_1) \subset (a_2) \dots$$

where $a_i = a_{i+1} t$. Since R is Noetherian the chain stabilises, that is there exists a k such that $(a_k) = (a_{k+1}) = \dots$ If $(a_k) = (a_{k+1})$ then $ba_k = a_{k+1}$ for some b in R. Thus $a_k = a_{k+1} t$ implies that $bt = 1$ contradicting the fact that t being the generator of the maximal ideal, is a non unit. Hence our assumption is wrong and we will eventually find an n such that a_n is a unit. Thus $a = a_1 t = a_2 t^2 = \dots = a_n t^n = ut^n$ where u is a unit.

$2 \Rightarrow 1$. If $r \in R$ is a non unit then $r = ut^n$ for some positive integer n and unit u. Thus the set of all non units in R is the principal ideal (t). To see that

R is Noetherian we observe that if $I \subseteq R$ is an ideal, then $I = (t^n)$ where $n = min\{ a \mid t^a \in I \}$. Thus I is finitely generated, in fact principal. □

Definition 4.2.2. A ring R satisfying either of the two conditions in the proposition above is called a *discrete valuation ring*. The irreducible element t generating the maximal ideal is called a *uniformising parameter*. It is clear that any other uniformising parameter has the form ut with u a unit in R, this follows from the property that the maximal ideal contains all the non units.

4.3 General valuations

Let R be a discrete valuation ring (and thus a domain) with uniformising parameter t.

Let $Q(R)$ be the quotient field or R, that is

$$Q(R) = \{ a/b \mid a\,,\, b \in R\setminus 0 \} \cup \{0\}$$

We define a map

$$\mathcal{V} : R\setminus 0 \longrightarrow \mathbb{Z}$$

$$a = ut^n \longmapsto n$$

We extend this map to $Q(R)$ by defining

$$\mathcal{V}(a/b) = \mathcal{V}(a) - \mathcal{V}(b) \qquad \text{for non zero } a/b \in Q(R)$$

$$\text{and} \qquad \mathcal{V}(0) = \infty$$

We observe that for $a\,,\, b \in R$ we have

$$\mathcal{V}(ab) = \mathcal{V}(a) + \mathcal{V}(b) \qquad \text{and} \qquad \mathcal{V}(a + b) \geqslant min\,\{\mathcal{V}(a)\,,\, \mathcal{V}(b)\}$$

It then follows that these two properties hold for $a, b \in Q(R)$.

A map of this form is called a *valuation*.

Example 4.3.1. Let $X = \mathbb{A}^1$ and let $p \in X$. The ring $\mathcal{O}_{X,p}$ is the ring of rational functions regular at p. Since we are working over an algebraically closed field, any element $h \in \mathcal{O}_{X,p}$ can be written as

$$h = (x - p)^n (f/g) \qquad \text{where } f(p) \neq 0 \; g(p) \neq 0$$

We observe that (f/g) is a unit in $\mathcal{O}_{X,p}$ and it then follows that $\mathcal{O}_{X,p}$ is a discrete valuation ring with uniformising parameter $(x - p)$.

Observing that the quotient field of $\mathcal{O}_{X,p}$ is $k(X)$, we obtain a valuation $\mathcal{V}$ on $k(X)$ where $\mathcal{V}(f)$ is the order of the zero (if positive) or pole (if negative) of f at p (as we are familiar with from complex analysis). $\square$

Chapter 5

Divisors

5.1 Definition

We observe that a polynomial function of one complex variable defined on all of $\mathbb{C}$ is determined up to multiplication by a constant by its set of zeros and their multiplicities.

For example, if $f \in \mathbb{C}[z]$ then, by the fundamental theorem of algebra we can write

$$f = \prod (z - \alpha_i)^{n_i} \qquad \alpha_i \in \mathbb{C} \quad n_i \in \mathbb{Z}^+$$

From which we note that the zeros α_i together with their multiplicities n_i allow us to construct the function to within multiplication by a constant. Similarly the quotient h of two complex polynomials f , g will be determined by its set of zeros and poles and their respective multiplicities, where pole takes its usual meaning from complex analysis.

We would like to extend this concept to functions defined on an arbitrary variety X. That is, we would like to associate a set of subvarieties to a polynomial function such that those subvarieties represent the zeros and poles of the function.

Fact 5.1.1. Let X be a variety. The set of points at which a polynomial in $k[X]$ vanishes forms a subvariety of codimension 1. □

See [**S**] Chapter I.6.2.

Example 5.1.2. We observe immediately that a polynomial in one variable (that is, a polynomial on the affine line) will vanish at a finite set of points. The fact tells us that a polynomial in two variables (say, on the affine plane) will vanish on a finite collection of lines. □

We are led to associate with a function on a variety X a finite collection of irreducible subvarieties C_i of codimension 1 with preassigned integral multiplicities a_i

Definition 5.1.3. Let X be a variety. A collection of irreducible codimension 1 subvarieties $C_i \subseteq X$ with preassigned integral multiplicities a_i is called a *divisor*.

A divisor D is usually written in the form

$$D = a_1 C_1 + \ \ldots \ + a_k C_k$$

If all the $a_i \geq 0$ then we say that D is an *effective divisor* or equivalently we write $D \geqslant 0$

An irreducible codimension 1 subvariety C is called a *prime divisor*.

We define addition of divisors in the natural way.

$$\text{if } D = \sum a_i C_i \text{ and } E = \sum b_i C_i \text{ are divisors}$$

then we define the divisor

$$D + E := \sum (a_i + b_i) C_i$$

where we observe that if D and E are divisors then $D = E$ is also a divisor (finite sum + finite sum = finite sum).

We denote the set of all divisors on a variety X by $Div(X)$ and we observe that with addition as defined above, $Div(X)$ is the free abelian group generated by the irreducible subvarieties of codimension 1.

5.2 Smooth varieties

Definition 5.2.1. Let $X \subseteq \mathbb{P}^n$ be a variety. We say that X is a *smooth* variety if the following three conditions hold

1. For an irreducible subvariety $Y \subseteq X$ the ring $\mathcal{O}_Y$ of rational functions regular on Y is a discrete valuation ring.
2. For a point $p \in X$ the local ring at p, $\mathcal{O}_{X,p}$, is a unique factorisation domain.
3. Curves on X are specified by local equations, that is, if C is a curve on X, then for any point $p \in X$ there exists a neighbourhood U (with respect to the Zariski topology) and a polynomial f such that on U, $C = V(f)$.

See [**S**] Chapter II.3

Remark 5.2.2. There exist other definitions of smooth and the definition given above is really a consequence of the usual definitions. The three properties above however, are the properties of smoothness that are most important when associating a divisor to a rational function. □

Example 5.2.3. The varieties (surfaces) $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $V(f) \subset \mathbb{P}^3$ where f is a homogeneous cubic are all smooth. □

5.3 The divisor of a rational function

Let $X \subseteq \mathbb{P}^n$ be a smooth variety and let $k(X)^*$ denote $k(X) \setminus \{0\}$.

Let C be an irreducible codimension 1 subvariety. From the smoothness condition on X we have that $\mathcal{O}_C$ is a discrete valuation ring. That is, there exists a surjective map

$$\mathcal{V}_C : k(X)^* \longrightarrow \mathbb{Z} \qquad \text{satisfying}$$

$$\mathcal{V}_C(ab) = \mathcal{V}_C(a) + \mathcal{V}_C(b) , \qquad \mathcal{V}_C(a+b) \geqslant min\{\mathcal{V}_C(a) + \mathcal{V}_C(b)\}$$

$$\text{and} \qquad \mathcal{O}_C = \{ h \in k(X)^* \mid \mathcal{V}_C(h) \geqslant 0 \} \bigcup \{0\}$$

Let $h \in k(X)^*$.

If $\mathcal{V}_C(h) > 0$ then we say that h has a zero of order $\mathcal{V}_C(h)$ along C.

If $\mathcal{V}_C(h) < 0$ then we say that h has a pole of order $\mathcal{V}_C(h)$ along C.

If $\mathcal{V}_C(h) = 0$ then h and h^{-1} are regular along C.

Proposition 5.3.1. *Let $h \in k(X)$. Then there are only a finite number of irreducible codimension 1 subvarieties C such that $\mathcal{V}_C(h) \neq 0$.*

Proof. First consider the case when X is affine. If C is not a component of $V(h)$, the zero set of f, then we have $\mathcal{V}_C(h) = 0$ since the local equation for C, π say, does not divide f, hence $f \notin (\pi^l)$ for any $l \in \mathbb{Z}^+$. Since the zero set of f has only finitely many irreducible components (f is a polynomial of finite degree) , we see that there exists only a finite number of C such that $\mathcal{V}_C(h) \neq 0$. When X is projective, we look at a finite affine open cover $\bigcup U^i$ of X. Every C intersects at least one of the U^i. We then repeat a similar argument to that above using local equations and the fact there are only finitely many U^i to show that there are only finitely many C such that $\mathcal{V}_C(h) \neq 0$. □

Definition 5.3.2. For $h \in k(X)^*$ we define the divisor

$$\sum \mathcal{V}_{C_i}(h) C_i$$

where the sum runs over the finite number of prime divisors C_i which satisfy $\mathcal{V}_C(h) \neq 0$. The above sum is called the *divisor of the function* h and is denoted D_h.

Remark 5.3.3. Let $D_h = \sum a_i C_i$. If we let

$$D_h^0 = \sum_{a_i > 0} a_i C_i \qquad \text{and } D_h^\infty = \sum_{a_i < 0} a_i C_i$$

The we can write D_h as the difference of effective divisors, that is , $D_h = D_h^0 - D_h^\infty$. We call D_h^0 the *divisor of zeros* of h and D_h^∞ the *divisor of poles* of h. (Note that most authors use (h) where we have used D_h, however (h) seems to be a little too similar to the notation used to denote a principal ideal generated by the element h). □

5.4 Principal divisors and the Picard group

We come now to the important notion of a principal divisor.

Definition 5.4.1. Let $X\,\mathbb{P}^n$ be a variety. A *principal divisor* is a divisor of the form D_h for $h \in k(X)^*$.

Remark 5.4.2. We observe that if $h_1, h_2 \in k(X)^*$ then $h_1h_2 \in k(X)^*$ and $h_1h_2^{-1} \in k(X)^*$ since $k(X)^*$ is a field. It follows from the properties of valuations and the definition of D_h that if D_{h_1} and D_{h_2} are principal divisors then $D_{h_1} + D_{h_2} = D_{h_1h_2}$ and $D_{h_1} - D_{h_2} = D_{h_1h_2^{-}1}$. Thus, if D and D' are principal divisors then so are $D + D'$ and $D - D'$. Observe also that 0 is a principal divisor, 0 being the divisor of any constant in $k(X)^*$. It follows that the set of principal divisors form a subgroup of $Div(X)$. □

Definition 5.4.3. For a variety $X \subseteq \mathbb{P}^n$ we define the *Picard group* on X, denoted $Pic(X)$ by

$$Pic(X) := Div(X)/\sim$$

where $\sim$ is the equivalence relation

$$D \sim D' \qquad \text{if } D - D' \text{ is a principal divisor}$$

If $D \sim D'$ we say that D is *linearly equivalent* to D'.

Remark 5.4.4. We observe that since the set of principal divisors is a subgroup the relation $\sim$ is easily seen to be an equivalence relation. $Pic(X)$ is just the quotient of $Div(X)$ by the subgroup of principal divisors however it is more customary to talk of the notion of linear equivalence and most authors define $Pic(X)$ as we have done above. $\square$

5.5 Divisors on the projective plane

Let $X = \mathbb{P}^2$. Subvarieties of codimension 1 on X are curves and X has the nice property that all curves are specified by a single homogeneous polynomial.

Irreducible curves, that is prime divisors, are defined by a single irreducible homogeneous polynomial.

Given a homogeneous polynomial $f \in k[x_0\,, x_1\,, x_2]$ we factor f as

$$f = \prod_{i=1}^{k} f_i^{a_i}$$

where the f_i are irreducible and the a_i positive integers. The f_i correspond to prime divisors C_i.

If the degree of f is d then we observe that $f/x_0^d \in k(X)$. Assuming f is non zero it is clear that the divisor associated to f/x_0^d is

$$D_{f/x_0^d} = \sum_{i=1}^{k} a_i C_i - dH \qquad \text{where } H = V(x_0)$$

This follows from observing that the uniformising parameter in the discrete valuation ring $\mathcal{O}_{C_i}$ can be taken to be f_i/g where $deg(g) = deg(f)$ and $f \neq g$. It is then clear that $f = (f_i/g)^{a_i} u$ where u is a unit in $\mathcal{O}_{C_i}$. It follows that $\mathcal{V}_{C_i}(f) = a_i$.

We observe that

$$\sum_{i=1}^{k} a_i C_i \sim dH \qquad \text{where } H \text{ is the prime divisor corresponding to } x_0$$

The above construction shows us that any effective divisor is linearly equivalent to some non negative integer multiple of H. For if

$$D = \sum_{i=1}^{r} b_i C_i \qquad \text{is an effective divisor}$$

$$\text{and we let } \; g = \prod_{i=1}^{r} f_i^{b_i}$$

then it follows from a procedure similar to that above that $D \sim eH$ where $e = deg(g)$ and we let the f_i be the irreducible homogeneous polynomial corresponding to the C_i.

It then follows that more generally any divisor is linearly equivalent to an integer multiple of H. For if D is any divisor, then we write $D = D' - D''$ where D and D' are effective. Let $D' \sim dH$ and $D'' \sim eH$, that is $D' - dH$ and $D'' - eH$ are principal. Hence

$$D - (d-e)H = (D' - dH) - (D'' - eH) \qquad \text{is also principal}$$

and so $D \sim (d-e)H$.

Remark 5.5.1. In the above procedure it is not hard to see that we may have replaced x_0 with any homogeneous degree 1 polynomial, that is replaced H with any line. It follows that any divisor on $\mathbb{P}^2$ is equivalent to an integer multiple of a line. □

Chapter 6

Sheaves

6.1 The Definition of a sheaf

Definition 6.1.1. Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups consists of the data

1. An abelian group $\mathcal{F}(U)$ associated with each open set $U \subseteq X$.
2. For every inclusion $V \subseteq U$ of open sets, a group homomorphism

   $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$,

   subject to the conditions
3. $\mathcal{F}(\varnothing) = 0$.
4. ρ_{UU} is the identity map $\mathcal{F}(U) \longrightarrow \mathcal{F}(U)$.
5. if $W \subseteq V \subseteq U$ are open sets, then $\rho_{UW} = \rho_{UV} \circ \rho_{VW}$.

We refer to an element $s \in \mathcal{F}(U)$ as a *section* of $\mathcal{F}$ over U. We will also often write $s|_V$ for $\rho_{UV}(s)$ if $s \in \mathcal{F}(U)$. We call the maps ρ_{UV} the *restrictions* of s.

Definition 6.1.2. A presheaf $\mathcal{F}$ on a topological space X is a *sheaf* if it satisfies the following additional conditions,

1. Let U be an open set with open cover $\{V_i\}$. If there exists an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = 0$ for all i then $s = 0$.

2. If U is an open set and $\{V_i\}$ an open covering of U and we have elements $s_i \in \mathcal{F}(V_i)$ for each i such that $s_i|_{V_i \cap V_j} = s_j|_{V_j \cap V_i}$ for each i, j, then there exists an element an element $s \in \mathcal{F}(U)$ such that the s_i come from restricting s.

Example 6.1.3. Let X be the real line and for open $U \subseteq X$ define F by letting $\mathcal{F}(U)$ be the set of continuous real valued functions defined on U.This set is an abelian group under addition of functions. For open $V \subseteq U$ and $s \in \mathcal{F}(U)$ define $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ to be the map which restricts the domain of s to V. Then $\mathcal{F}$ is easily seen to be a presheaf. In fact $\mathcal{F}$ defines a sheaf since if $\{V_i\}$ is an open cover of U and $s \in \mathcal{F}(U)$ is zero on each of the V_i then s is zero on U. To see that sheaf condition 2 is satisfied we note that since continuity of a real valued function s at a point $p \in X$ depends only upon the values of s in small neighbourhoods of p, an element of $\mathcal{F}(U)$ is fully determined by its restrictions to the $\mathcal{F}(V_i)$ for each i. That is, if we have s_i satisfying 6.1.2 condition 2 then the s_i determine a unique element of $\mathcal{F}(U)$ whose restrictions are the s_i. □

Definition 6.1.4. If $\mathcal{F}$ is a sheaf on X and $p \in X$ then we define the *stalk* of $\mathcal{F}$ at p to be the direct limit $\mathcal{F}_p$ of $F(U)$ where U runs over all the opens containing p. This is the set of all sections $s \in \mathcal{F}(U)$ over all opens containing p, modulo the equivalence relation $s \sim s'$ if they agree on some smaller open neighbourhood containing p. In the case of the structure sheaf $\mathcal{O}_X$ described above, the stalk at $p \in X$ is the local ring $\mathcal{O}_{X,p}$ of functions regular at p.

6.2 Sheaves of modules

In the discussion above, we required in the definition that a sheaf $\mathcal{F}$ on X consisted of assigning an abelian group $\mathcal{F}(U)$ to each open $U \subseteq X$ and that the restriction maps be abelian group homomorphisms. In fact we can define sheaves of rings in the same way.

A *sheaf of rings* on a topological space X is a sheaf $\mathcal{F}$ such that for each open $U \subseteq X$ the set $\mathcal{F}(U)$ is a ring and the restriction maps are ring homomorphisms.

Example 6.2.1. Let X be a variety over the field k. For open $U \subseteq X$ we define a sheaf of rings $\mathcal{O}_X$ by letting $\mathcal{O}_X(U)$ be the set of regular functions from U to k. This set is a ring under addition and multiplication of functions. If $V \subseteq U$ then a regular function on U clearly restricts to a regular function on V. To see that $\mathcal{O}_X$ defines a sheaf we observe that, as for continuity of real valued functions, regularity of a function at a point p is determined by the behaviour of the function on small open neighbourhoods of p. So if $\{V_i\}$ is an open cover of U, a regular function on U is determined by its values on each of the V_i. That is, given $s_i \in \mathcal{O}_X(V_i)$ for each i with equal restrictions $s_i|_{V_i \cap V_j} = s_j|_{V_j \cap V_i}$ for each i, j, the s_i determine a unique regular function s on U with $s|_{V_i} = s_i$ for each i. We call $\mathcal{O}_X$ the *structure sheaf* on X □

A similar concept is that of a sheaf of $\mathcal{O}_X$-modules.

Definition 6.2.2. Let X be a topological space. Let $\mathcal{O}_X$ be the structure sheaf on X and $\mathcal{F}$ be a sheaf of abelian groups on X. We say that $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules if each of the sets $\mathcal{F}(U)$ for open U has the structure of an $\mathcal{O}_X(U)$-module and the restriction maps are $\mathcal{O}_X(U)$-module homomorphisms. That is, for $a \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$ we have $(a.s)|_V = a|_V.s|_V$ where $a|_V$ is the restriction of a specified by the structure sheaf.

The sheaves which will be of most importance to us will be sheaves of $\mathcal{O}_X$-modules.

Before discussing morphisms and subsheaves we need to introduce the idea of the sheaf associated with a given $k[X]$-module M, where X is an affine variety. We omit the construction of $\tilde{M}$ and instead treat the following proposition as the definition of the sheaf associated to a $k[X]$-module M. The interested reader is invited to consult [**H**] chapter II, 5 or [**K**] chapter 5.2 for further details.

Definition 6.2.3. Let R be a domain and $P \subseteq R$ a prime ideal. If we define $U = R \setminus P$ we observe that U is a multiplicative set, that is, $1 \in U$ and $ab \in U$ for all $a, b \in U$. We define the *localisation* of R at P denoted R_P by

$$R_P = \{ a/b \mid a \in R, \ b \in U \}/ \sim$$

where $a/b \sim c/d$ if there exists a $u \in U$ such that $u(ad - bc) = 0$

We define addition and multiplication in the natural way induced from the multiplication and addition in R (as we do with $\mathbf{Q}$). The reader may check that $\sim$ is an equivalence relation.

If $f \in R$ we define the localisation at f denoted R_f in exactly the same way, with U replaced by $\{1, f, f^2, \ .... \}$.

If M is an R-module then we can define an R_P-module M_P by

$$M_P = \{ a/b \mid a \in M, \ b \in U \}/ \sim$$

with $\sim$ and U as above. We define scalar (module) multiplication by elements in R_P in the natural way.

See [**R3**] Chapter 6, or any book on commutative algebra for further details.

Proposition-Definition 6.2.4. *Let X be an affine variety, M a $k[X]$-module. Consider a base $\{U_f = V(f)^c \mid f \in k[X]\}$ for the Zariski topology on X. There exists a unique sheaf $\tilde{M}$ with the following properties:*

1. *$\tilde{M}(U_f) := M_f$, the localisation of M at f.*
2. *In particular $\tilde{M}(X) = M$*
3. *For each $p \in X$ we define the stalk of $\tilde{M}$ at p, $\tilde{M}_p := M_{\mathfrak{m}_p}$, the localisation of M at the ideal of all functions vanishing at p.*
4. *(Definition), $\tilde{M}$ is the sheaf associated with M.*

Definition 6.2.5. If U is an open subset of a topological space X and $\mathcal{F}$ is a sheaf on X, we define a sheaf $\mathcal{F}|_U$ on U by setting $\mathcal{F}|_U(W) = \mathcal{F}(W)$ for all open $W \subseteq U$ and we give $\mathcal{F}|_U$ the same restriction maps as $\mathcal{F}$.

Definition 6.2.6. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasi-coherent* if X can be covered by affine open subsets U_i such that for each i there is a $k[U_i]$-module M_i with $\mathcal{F}|_{U_i} \simeq \tilde{M}_i$. Furthermore, $\mathcal{F}$ is said to be *coherent* if each $\tilde{M}_i$ is finitely generated. I will write (quasi-)coherent $\mathcal{O}_X$-module to mean (quasi-)coherent sheaf of $\mathcal{O}_X$-modules from now on.

Much of the data of the sheaf is hidden in the $\tilde{M}_i$ and it is perhaps more instructive to use the following equivalent definition of quasi-coherent.

Definition 6.2.7. Let X be a variety with affine open cover $\bigcup X_i$. A *quasi-coherent* sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is the data,

1. A $k[X_i]$-module $M_i := \mathcal{F}(X_i)$ for each i.

2. Isomorphisms of $k[X_i \cap X_j]$-modules $\alpha_{ij} : M_i|_{X_i \cap X_j} \longrightarrow M_j|_{X_i \cap X_j}$

   these are called the *patching* conditions.

3. The *compatibility* conditions, $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$

Example 6.2.8. Recall the structure sheaf $\mathcal{O}_X$ from 6.2.1. We now define the structure sheaf as a quasi-coherent sheaf of $\mathcal{O}_X$-modules, or $\mathcal{O}_X$-module for short. Let X be a variety with affine open cover $\bigcup X_i$. Let $M_i = k[X_i]$ be the set of functions regular on X_i. Now, $M_i|_{X_i \cap X_j} = k[X_i \cap X_j] = M_j|_{X_i \cap X_j}$. To complete the definition of $\mathcal{O}_X$ we need to specify $k[X_i \cap X_j]$-module isomorphisms $\alpha_{ij} : k[X_i \cap X_j] \longrightarrow k[X_i \cap X_j]$ in order to satisfy the patching conditions. Obviously there are are great number of possibilities since for any $a \in k[X_i \cap X_j]$ the map $1 \mapsto a$ is an isomorphism of $k[X_i \cap X_j]$ with $k[X_i \cap X_j]$. The structure sheaf is defined by letting the α_{ij} be the isomorphisms $1 \mapsto 1$ for all i, j. □

6.3 Morphisms and subsheaves

Definition 6.3.1. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules on variety X and let $\bigcup X_i$ be an affine open cover of X. Let $\mathcal{F}(X_i) = M_i$ and $\mathcal{G}(X_i) = N_i$ be the $k[X_i]$-modules associated with $\mathcal{F}$ and $\mathcal{G}$. Let α_{ij} and β_{ij} denote the patching maps on $\mathcal{F}$ and $\mathcal{G}$ respectively.

A *morphism* of quasi-coherent $\mathcal{O}_X$-modules on X, $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of $k[X_i]$-module homomorphisms $\varphi_i : M_i \longrightarrow N_i$ defined for all i such that the maps agree on all overlaps.

A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}'$ is a *subsheaf* of $\mathcal{F}$ if for each i the $k[X_i]$-modules $\mathcal{F}'(X_i) = M'_i$ are submodules of the M_i $(= \mathcal{F}(X_i))$ and the patching maps for $\mathcal{F}'$

are obtained by restricting the domains of the patching maps for $\mathcal{F}$, that is, they are essentially the same. We write $\mathcal{F}' \subseteq \mathcal{F}$

Given a morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ we define the *kernel* and *image* of φ denoted $ker(\varphi)$ and $im(\varphi)$ respectively, by defining

1. $ker(\varphi)(X_i) = ker(\varphi_i)$ and giving $ker(\varphi)$ the patching maps induced from $\mathcal{F}$.
2. $im(\varphi)(X_i) = im(\varphi_i)$ and giving $im(\varphi)$ the patching maps induced from $\mathcal{G}$.

It is then clear from the definition that $ker(\varphi)$ and $im(\varphi)$ are subsheaves of $\mathcal{F}$ and $\mathcal{G}$ respectively.

We recall the definition of an exact sequence of R-modules.

Definition 6.3.2. Let R be a ring and let the M_i be R-modules. A sequence of R-module homomorphisms

$$... \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow ...$$

is said to be *exact* at M_i if $ker(f_i) = im(f_{i-1})$. The sequence is exact if it is exact at each M_i.

An exact sequence of quasi-coherent $\mathcal{O}_X$-modules is defined in the same way with the M_i being replaced by sheaves and the R-module homomorphisms being replaced by morphisms of sheaves.

6.4 Tensor products

I sketch the definition of a tensor of two modules.

Definition 6.4.1. Let R be a commutative ring and let M and N be R-modules ($rm = mr \quad r \in R, m \in M$). The tensor product space of M and N over R, written $M \otimes_R N$ is the R-module consisting of the set of elements $m \otimes n$ with $m \in M$ and $n \in N$ and the relations

$$r(m \otimes n) = (rm \otimes n) = (m \otimes rn) \quad for\, r \in R$$

$$((m + m') \otimes n) = (m \otimes n) + (m' \otimes n)$$

$$(m \otimes (n + n')) = (m \otimes n) + (m \otimes n')$$

The tensor product space $M \otimes_R N$ is thus an R-module. Note that the first relation says that we allow elements of R to commute through the $\otimes$ if the tensor product is taken over R. We often just write $M \otimes N$ knowing that the product is over$R$.

Remark 6.4.2. The tensor product has the following useful properties,

1. It respects localisation, that is $(M \otimes N)_f = (M_f \otimes N_f)$
2. It is exact (meaning it respects exact sequences), that is, if

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is exact, then tensoring with N gives the right exact sequence

$$(M_1 \otimes N) \longrightarrow (M_2 \otimes N) \longrightarrow (M_3 \otimes N) \longrightarrow 0$$

□

We now define the tensor product of two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$.

Definition 6.4.3. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules over variety X. Recall that this means on an affine open cover $X = \bigcup X_i$ we have $k[X_i]$-modules M_i and N_i together with patching conditions. Let α_{ij} and β_{ij} denote the patching maps for $\mathcal{F}$ and $\mathcal{G}$ respectively. We define the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the quasi-coherent $\mathcal{O}_X$-module on X with the $k[X_i]$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}(X_i)$ given by $(M_i \otimes_{k[X_i]} N_i)$ for each i. We note that $(M_i \otimes_{k[X_i]} N_i)|_{X_i \cap X_j} = (M_i|_{X_i \cap X_j} \otimes_{k[X_i \cap X_j]} N_i|_{X_i \cap X_j})$ as $k[X_i \cap X_j]$-modules. We must define patching maps γ_{ij} giving isomorphisms

$$\gamma_{ij} : (M_i|_{X_i \cap X_j} \otimes N_i|_{X_i \cap X_j}) \longrightarrow (M_j|_{X_i \cap X_j} \otimes N_j|_{X_i \cap X_j})$$

We define the γ_{ij} in the natural way by letting $\gamma_{ij} = \alpha_{ij} \otimes \beta_{ij}$, where $\alpha_{ij} \otimes \beta_{ij}$ is defined by

$$\alpha_{ij} \otimes \beta_{ij} : (M_i|_{X_i \cap X_j} \otimes N_i|_{X_i \cap X_j}) \longrightarrow (M_j|_{X_i \cap X_j} \otimes N_j|_{X_i \cap X_j})$$

$$a \otimes b \longmapsto \alpha_{ij}(a) \otimes \beta_{ij}(b)$$

Since the α_{ij} and β_{ij} are the required isomorphisms on $\mathcal{F}$ and $\mathcal{G}$ respectively (that is, the patching maps) satisfying the compatibility condition, it is easy to see that the $\gamma_{ij} = \alpha_{ij} \otimes \beta_{ij}$, are patching maps on $\mathcal{F} \otimes \mathcal{G}$ satisfying compatibility. The surjectivity of γ_{ij} is clear. For injectivity we observe that if $\alpha_{ij}(a) \otimes \beta_{ij}(b) = 0$ then $\alpha_{ij}(a) = 0$ or $\beta_{ij}(b) = 0$. This implies that one of $a = 0$ or $b = 0$ and hence that $a \otimes b = 0$ which implies injectivity of γ_{ij}.

The patching maps γ_{ij}, together with the $k[X_i]$-modules $M_i \otimes_{k[X_i]} N_i$ define the quasi-coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Definition 6.4.4. The direct sum of two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, over a variety X, is defined in a similar manner. If $X = \bigcup X_i$ is an affine open cover and $\mathcal{F}(X_i) = M_i$ and $\mathcal{G}(X_i) = N_i$ then we define $\mathcal{F} \oplus \mathcal{G}$ so that $(\mathcal{F} \oplus \mathcal{G})(X_i) = (M_i \oplus N_i)$. With the same notation as above we define the patching maps γ_{ij} for $\mathcal{F} \oplus \mathcal{G}$ by

$$\gamma_{ij} : (M_i|_{X_i \cap X_j} \oplus N_i|_{X_i \cap X_j}) \longrightarrow (M_j|_{X_i \cap X_j} \oplus N_j|_{X_i \cap X_j})$$

$$(a\,,\, b) \;\mapsto\; (\alpha_{ij}(a)\,,\, \beta_{ij}(b))$$

It is then easy to see that the γ_{ij} are patching maps satisfying the compatibility conditions which, along with the $(M_i \oplus N_i)$ define the sheaf $\mathcal{F} \oplus \mathcal{G}$.

6.5 Some important sheaves

In this section we define the quasi-coherent $\mathcal{O}_X$-modules which will be of most use to us in this thesis. All of them are very closely related to the structure sheaf and as such are far less complicated than the definition allows.

Let X be a variety with affine open cover $\bigcup X_i$.

6.5.1 The structure sheaf

Recall the structure sheaf $\mathcal{O}_X$ which is defined by setting $M_i = \mathcal{O}_X(X_i) = k[X_i]$ the set of functions regular on X_i and defining patching maps α_{ij} giving $k[X_i \cap X_j]$-module isomorphisms of $M_i|_{X_i \cap X_j} = k[X_i \cap X_j]$ with $M_j|_{X_i \cap X_j} = k[X_i \cap X_j]$ defined by $1 \mapsto 1$.

If $Y \subseteq X$ is a subvariety with affine open cover $\bigcup Y_i$ then the structure sheaf $\mathcal{O}_Y$ considered as a sheaf on Y is defined in a similar. On Y_i we set $\mathcal{O}_Y(Y_i) = k[Y_i]$

and give $\mathcal{O}_Y$ the patching maps $1 \mapsto 1$ as was done for $\mathcal{O}_X$. We can consider $\mathcal{O}_Y$ as a sheaf on X by defining $\mathcal{O}_Y(X_i) = \mathcal{O}_Y(X_i \cap Y) = k[X_i \cap Y]$. We note that $k[X_i \cap Y] = k[X_i]/I_i$ where $I_i = I(X_i \cap Y)$ is the ideal of functions vanishing on $X_i \cap Y$.

6.5.2 The sheaf associated with a divisor

Let D be an effective divisor (recall this is a union of subvarieties of codimension 1) on an X. Assume that on X_i D is specified by a local equation f_i (which must be regular since D is effective), that is to say that $D \cap X_i$ is given by the set of solutions to $f_i = 0$ that is, $V(f_i)$. We define the sheaf $\mathcal{O}_X(-D)$ as follows, Let $M_i = \mathcal{O}_X(-D)(X_i) = f_i\, k[X_i]$. This is the set of regular functions on X_i which are zero on $D \cap X_i$. We define $\mathcal{O}_X(D)$ similarly by letting $\mathcal{O}_X(D)(X_i) = f_i^{-1}k[X_i]$ which is the set of rational functions regular on X_i except for poles at worst $D \cap X_i$. Now we must define the patching maps. For $\mathcal{O}_X(-D)$ we have $M_i = f_i\, k[X_i]$ and so $M_i|_{X_i \cap X_j} = f_i\, k[X_i \cap X_j]$ as a $k[X_i \cap X_j]$-module. Similarly for $M_j|_{X_i \cap X_j}$. Define $k[X_i \cap X_j]$-module homomorphisms

$$\alpha_{ij} : f_i\, k[X_i \cap X_j] \longrightarrow f_j\, k[X_i \cap X_j]$$

$$f_i \longmapsto f_i$$

We see that α_{ij} is defined as multiplication by 1 and is thus an injection. To see the surjectivity of α_{ij} we observe that since $X_i\, cap\, X_j$ is contained in both X_i and X_j we have that f_i and f_j are local equations for D on $X_i \cap X_j$. This means that (f_i/f_j) and $(f_i/f_j)^{-1}$ are both regular on $X_i \cap X_j$ that is, members of $k[X_i \cap X_j]$. In particular $f_j = f_i(f_i/f_j)^{-1} \in f_i k[X_i \cap X_j]$ and so $f_j \mapsto f_j$ under α_{ij} which shows surjectivity.

The maps α_{ij} satisfy the compatibility condition $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ since they are just multiplication by 1 in all cases. The patching maps α_{ij} together with the M_i define the quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X(-D)$.

To define $\mathcal{O}_X(D)$ when D is not an effective divisor, that is, $D = D' - D''$ with D' and D'' both effective, we let f_i' and f_i'' be the local equations for D' and D'' respectively on $D \cap X_i$. We then define $\mathcal{O}_X(-D)$ exactly as we did above but replacing f_i with f_i'/f_i''.

Proposition 6.5.1. *Let X be a variety with affine open cover $\bigcup X_i$ and let D and D' be divisors on X given by the local equations $f_i$and f_i' respectively on X_i. Then,*

1. *$\mathcal{O}_X(-D - D') \simeq \mathcal{O}_X(-D) \otimes \mathcal{O}_X(-D')$*
2. *$\mathcal{O}_X(-D) \simeq \mathcal{O}_X(-D')$ iff $D - D'$ is a principal divisor*

Proof. 1. From the definition of the tensor product and of the sheaf $\mathcal{O}_X(-D)$ we have that $(\mathcal{O}_X(-D) \otimes \mathcal{O}_X(-D'))(X_i) = (f_i\, k[X_i] \otimes f_i'\, k[X_i])$ which is generated by $(f_i \otimes f_i')$. Similarly $\mathcal{O}_X(-D - D')(X_i) = f_i f_i'\, k[X_i]$ is generated by $f_i f_i'$. Define $k[X_i]$-module homomorpisms

$$\varphi_i : (f_i\, k[X_i] \otimes f_i'\, k[X_i]) \longrightarrow f_i f_i'\, k[X_i]$$

$$a \otimes b \longmapsto ab$$

These homomorphisms are isomorphisms. Surjectivity is clear since $f_i \otimes f_i' \mapsto f_i f_j'$. To see injectivity we note that $(f_i a \otimes f_i' b) = (f_i ab \otimes f_i')$ (since we are tensoring over $k[X_i]$) hence $f_i f_i' ab = 0 \Leftrightarrow ab = 0 \Leftrightarrow (f_i ab \otimes f_i') = 0 \Leftrightarrow (f_i a \otimes f_i' b) = 0$. The maps φ_i are the same on each i and hence agree on the overlaps that is, they patch together. The collection of the φ_i thus give an isomorphism of sheaves

$\mathcal{O}_X(-D) \otimes \mathcal{O}_X(-D') \simeq \mathcal{O}_X(-D-D')$.

2. If $D - D'$ is a principal divisor then f/f' is a degree zero homogeneous polynomial and $D - D'$ is defined on all of X by f/f', that is $f/f' = f_i/f'_i$ for all i. We define isomorphisms $\varphi_i : f_i\, k[X_i] \longrightarrow f'_i\, k[X_i]$ by letting φ_i be multiplication by f_i/f'_i. These maps give an isomorphism of sheaves since the maps agree on overlaps since multiplying by f_i/f'_i is the same as multiplying by f_j/f'_j due to the fact that $D-D'$ is principal. Thus $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$. For the converse the reader is referred to [**H**] Chapter II.6 Prop. 6.13.

□

Remark 6.5.2. The above proposition shows that the Picard group of divisors modulo linear equivalence is naturally isomorphic to the group of sheaves associated with divisors where the group operation is the tensor product and the identity is $\mathcal{O}_X$, where we observe that if D is a principal divisor then $\mathcal{O}_X(D) \simeq \mathcal{O}_X$. This follows from noting that D will have the same local equation f on all X_i and hence the $k[X_i]$-modules M_i are generated by f for each i. Sending f to 1 gives an isomorphism $\mathcal{O}_X(D) \simeq \mathcal{O}_X$. □

Chapter 7

Intersection numbers on surfaces

7.1 Euler characteristic

The Euler characteristic is an integer associated to a sheaf on X which we will use in defining an intersection number.

We follow [**R2**] and introduce the Euler characteristic along with some rules of what is called coherent cohomology. Due to the nature of this thesis it was impractical to try to understand and prove the results of coherent cohomology. The important concepts are introduced as standard rules or axioms.

1.

For any variety X and any $\mathcal{O}_X$-module (recall this means sheaf of $\mathcal{O}_X$-modules) $\mathcal{F}$ on X there exist k-vector spaces $H^i(X, \mathcal{F})$ for $i = 1, 2, \dots$. A morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ gives rise to linear maps $H^i(X, \varphi) : H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{G})$.

2.

If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence of quasi-coherent $\mathcal{O}_X$-modules, then there exists a natural map $d_i : H^i(X, \mathcal{F}'') \longrightarrow H^{i+1}(X, \mathcal{F}')$.

3.

$H^0(X, \mathcal{F}) = \mathcal{F}(X)$ the space of sections of $\mathcal{F}$ over X.

4.

The maps specified in 1. and 2. are such that if $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence of quasi-coherent $\mathcal{O}_X$-modules then 1. gives a short exact sequence $0 \longrightarrow H^i(X, \mathcal{F}') \longrightarrow H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F}'') \longrightarrow 0$.

Similarly, 2. together with 1. gives a long exact sequence

$$... \longrightarrow H^i(X, \mathcal{F}') \longrightarrow H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F}'') \longrightarrow$$

$$\longrightarrow H^{i+1}(X, \mathcal{F}') \longrightarrow H^{i+1}(X, \mathcal{F}) \longrightarrow ...$$

5.

If $\mathcal{F}$ is coherent and X variety, then $H^i(X\,\mathcal{F})$ is finite dimensional over k for any i.

6.

$H^i(X, \mathcal{F}) = 0$ for all $i > \dim(X)$. In particular, if X is a finite set of points, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

7. The Euler characteristic.

Let X be a projective variety. We define the *Euler characteristic* $\chi(\mathcal{F})$ of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i dim_k H^i(X\,\mathcal{F})$$

Points 1. to 5. above were included mainly to make it easy to see how the Euler characteristic satisfies the following nice property:

If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is an exact sequence of quasi-coherent $\mathcal{O}_X$-modules then

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$$

This follows immediately from the long exact sequence (of k-vector spaces) in 4. Since given a long exact sequence of vector spaces

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow V_4 \longrightarrow V_5 \longrightarrow ...$$

we have that

$im(d_1) = ker(d_2)$

$im(d_2) = ker(d_3)$

$im(d_3) = ker(d_4)$

$im(d_4) = ker(d_5)$

etc..

where d_i is the map from V_i to V_{i+1}. Adding these equations alternately gives us the equation

$$(im(d_1)) - (ker(d_2)+im(d_2)) + (ker(d_3)+im(d_3)) - (ker(d_4)+im(d_4)) + ... = 0$$

which by the rank nullity theorem for vector spaces and the injectivity of d_1 (since the sequence is exact) is the same as

$$dim(V_1) - dim(V_2) + dim(V_3) - dim(V_4) + ... = 0$$

Applying this result to the long exact sequence in 4 gives

$$\sum_{i=0}^{\infty}(-1)^i dim_k H^i(X\,\mathcal{F}) = \sum_{i=0}^{\infty}(-1)^i dim_k H^i(X\,\mathcal{F}') + \sum_{i=0}^{\infty}(-1)^i dim_k H^i(X\,\mathcal{F}'')$$

Which is the same as $\chi(\mathcal{F}') = \chi(\mathcal{F}) + \chi(\mathcal{F}'')$ as required.

8. Riemann-Roch Theorem on curves.

Let C be a curve and D a divisor on C. That is, D is a finite sum of points p_i of C with integral multiplicities a_i. We write $D = \sum a_i p_i$ and $deg(D)|_C = \sum a_i$. The Riemann-Roch theorem states that

$$\chi(\mathcal{O}_C(D)) - \chi(\mathcal{O}_C) = deg(D)|_C$$

7.2 The multiplicity of a point

Let X be a smooth projective surface.

Definition 7.2.1. Let $p \in X$. Let C and D be two divisors on X with no common component and let f and g be the local equations for C and D respectively at p. We define the *intersection multiplicity* $(C.D)_p$ of C and D at p to be the dimension over k of the vector space $\mathcal{O}_{X,p}/(f,g)$, where $\mathcal{O}_{X,p}$ is the local ring at p.

Remark 7.2.2. Observe that $\mathcal{O}_{X,p}/(f,g) = 1$ if and only if f and g generate the unique maximal ideal of $\mathcal{O}_{X,p}$ (so that $\mathcal{O}_{X,p}/(f,g) \simeq k$). This corresponds to our intuitive notion of curves meeting transversally. For example if $X = \mathbb{P}^2$ with coordinates x, y, z and $p = (0,0,1)$ then $\mathcal{O}_{X,p} = k[x/z\, ,\, y/z] \simeq k[x\, ,\, y]$. Let $f = y$ and $g = y - x^n$. We then see that $dim_k(k[x\, ,\, y]/(f,g)) = dim_k(1\, ,\, x\, ,\, x^2\, ,\, ...\, x^{n-1}) = n$, agreeing in some sense with our notion of the multiplicity at the point $(0,0)$ of the intersection of the curves $y = 0$ and $y = x^n$ on the affine plane. In particular the multiplicity is 1 if and only if $n = 1$. □

Fact 7.2.3. If C and D are divisors on a nonsingular projective surface X then $C \cap D$ consists of a finite set of points. □

Corollary 7.2.4. *If $\mathcal{O}_{C \cap D}$ is the structure sheaf on $C \cap D$ then*

$$\chi(\mathcal{O}_{C \cap D}) = dim_k \, H^0(X \, , \, \mathcal{O}_{C \cap D})$$

Proof. . This is a direct consequence of the definition of the Euler characteristic and rule 6 above, since if $C \cap D$ is a finite set of points then its dimension is zero as a variety. □

Definition 7.2.5. Let X be a projective surface with affine open cover $\bigcup X_i$ and let C and D be divisors on X. We define the sheaf $\mathcal{O}_{C \cap D}$ by

$$\mathcal{O}_{C \cap D}(X_i) = \bigoplus_{p \in (C \cap D) \cap X_i} \mathcal{O}_{X,p}/(f, g)$$

where f and g are the local equations for C and D respectively at each p. The patching maps are defined to be the natural $1 \mapsto 1$ maps. That is $(1, 0, \, ... \, , 0) \mapsto (1, 0, \, ... \, , 0)$, $(0, 1, 0, \, ... \, , 0) \mapsto (0, 1, 0 \, ... \, , 0)$ and so on.

Remark 7.2.6. We observe that $H^0(X \, , \, \mathcal{O}_{C \cap D}) = \mathcal{O}_{C \cap D}(X) = \bigoplus_{p \in (C \cap D)} \mathcal{O}_{X,p}/(f, g)$. This follows from the definition of $\mathcal{O}_{C \cap D}$ and from rule 3 above. □

Proposition 7.2.7. *If C and D are divisors on a nonsingular projective surface X then*

$$dim_k\, H^0(X\,,\,\mathcal{O}_{C\,\cap\,D}) \;=\; \sum_{p\,\in\,C\,\cap\,D} dim_k\,(\mathcal{O}_{X,p}/(f,g))$$

Proof. We observe that $H^0(X\,,\,\mathcal{O}_{C\,\cap\,D}) \;=\; \bigoplus_{p\in C\cap D} \mathcal{O}_{X,p}/(f,g)$. The result follows immediately as we are calculating the dimension (as a k-vector space) of a direct sum of k-vector spaces. □

7.3 The intersection product

Given two divisors C and D on X, our main aim is to define an intersection product $(C.D)$ which is symmetric, bilinear and well defined on $Pic(X) \times Pic(X)$. That is, the number depends only upon the equivalence class of C and D. This is the most important property of the intersection product, for $Pic(X)$ is in most cases finitely generated whereas $Div(X)$ has an infinite number of generators. The calculation of the intersection product on the finite number of generators of $Pic(X)$ will allow us to calculate the intersection product on all of $Div(X)$ using linear equivalence.

Lemma 7.3.1. *The sequences*

1. $$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$
2. $$0 \longrightarrow \mathcal{O}_X(-C-D) \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_C(-D) \longrightarrow 0$$
3. $$0 \longrightarrow \mathcal{O}_D(-C) \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{C\,\cap\,D} \longrightarrow 0$$

are exact.

Proof. .

Let $\bigcup X_i$ be an affine open cover of X and let C and D be specified by the local equations f_i and g_i respectively on X_i. Since C and D

1. We need only to check that the sequence

$$0 \longrightarrow f_iR_i \longrightarrow R_i \longrightarrow R_i/f_iR_i \longrightarrow 0 \text{ is exact}$$

where $R_i = k[X_i]$ and that the patching conditions hold. This follows immediately, since f_iR_i injects into R_i and the kernel of the natural surjection from R_i into R_i/f_iR_i is of course f_iR_i. To see that the patching conditions hold we observe that the maps are all natural and so the same on each i.

2. we check that the sequence

$$0 \longrightarrow f_ig_iR_i \longrightarrow g_iR_i \longrightarrow (g_iR_i + f_iR_i)/f_iR_i \longrightarrow 0 \text{ is exact}$$

and that the patching conditions hold. This follows immediately from an argument similar to the above.

3. See [**H**] chapter V, lemma 1.3 for a verification. □

We are now ready to prove the existence of an intersection product on surfaces.

Theorem 7.3.2. *Let X be a smooth projective surface. Given two divisors C and D on X, with no common component, there exists an integer $(C.D)$ defined by,*

$$(C.D) = \chi(\mathcal{O}_X(-C - D)) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X)$$

satisfying

1. *$(C.D)$ is symmetric and bilinear, that is $(C.D) = (D.C)$ and if $C = C_1 + C_2$ then $(C.D) = (C_1.D) + (C_2.D)$*
2. *If C is linearly equivalent to C' then $(C.D) = (C'.D)$*
3. *$(C.D) = \sum_{p \in C \cap D} (C.D)_p$ where*

$$(C.D)_p = dim_k \, (\mathcal{O}_{X,p}/(f, g))$$

with f and g being the local equations for C and D respectively at p. In particular, if C and D meet transversally at all points in their intersection then $(C.D) = \#(C \cap D$ the number of points in their intersection.

Proof.

We prove 3 first.

We have the exact sequences

(a) $\quad 0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$

(b) $\quad 0 \longrightarrow \mathcal{O}_X(-C - D) \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_C(-D) \longrightarrow 0$

(c) $\quad 0 \longrightarrow \mathcal{O}_D(-C) \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{C \cap D} \longrightarrow 0$

Now we use the properties of the Euler characteristic (see rule 7) to get, from the above exact sequences, the relations

(1a) $$\chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X) + \chi(\mathcal{O}_C) = 0$$

(2a) $$\chi(\mathcal{O}_X(-D)) - \chi(\mathcal{O}_X) + \chi(\mathcal{O}_D) = 0$$

(1b) $$\chi(\mathcal{O}_X(-C-D)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_C(-D)) = 0$$

(2b) $$\chi(\mathcal{O}_X(-C-D)) - \chi(\mathcal{O}_X(-C)) + \chi(\mathcal{O}_D(-C)) = 0$$

(1c) $$\chi(\mathcal{O}_D(-C)) - \chi(\mathcal{O}_D) + \chi(\mathcal{O}_{C\cap D}) = 0$$

(2c) $$\chi(\mathcal{O}_C(-D)) - \chi(\mathcal{O}_C) + \chi(\mathcal{O}_{C\cap D}) = 0$$

Where we obtain the (2) from the (1) by letting $C = D$ in the (1) which the reader can see is perfectly valid from the construction of the exact sequences. That is to say, the sequences are still exact with C replaced by D.

Now we take

$(1b) + (2b) - (1a) - (2a) - (1c) - (2c)$

and divide by two to obtain

$\chi(\mathcal{O}_X(-C-D)) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{C\cap D})$

We now recall that from 7.2.4 and 7.2.7 we have that

$$\chi(\mathcal{O}_{C\cap D}) = dim_k\, H^0(X\,,\,\mathcal{O}_{C\cap D}) = \sum_{p\in C\cap D} dim_k\, \mathcal{O}_{X,p}/(f,g)$$

which proves 3.

To prove 2 we note that if C is linearly equivalent to C' then $-C$ is linearly equivalent to $-C'$ and so $\mathcal{O}_X(-C) \simeq \mathcal{O}_X(-C')$. Likewise $-C-D$ is linearly equivalent to $-C'-D$ and so from the definition it is easily seen that $(C.D) = (C'.D)$ since the Euler characteristic is constant on isomorphic sheaves.

Now to prove bilinearity we use the relationshps

(1a) $\quad \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X) + \chi(\mathcal{O}_C) = 0$

(1b) $\quad \chi(\mathcal{O}_X(-C-D)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_C(-D)) = 0$

obtained above.

(1b) - (1a) gives $(C.D) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_C(-D))$

thus, from the Riemann-Roch theorem on curves (rule 8) we obtain

$(C.D) = -deg(-D)|_C = deg(D)|_C$

Since $deg(D+D')|_C = deg(D)|_C + deg(D')|_C$ from the definition of degree, we see that $(C.D)$ is linear in D. The definition of $(C.D)$ immediately shows that $(C.D) = (D.C)$ (that is, (.) is symmetric) and hence we have that (.) is bilinear.

Completing the proof of the theorem. □

Corollary 7.3.3. *Bezout's theorem.*

Let $X = \mathbb{P}^2$. Let C and D be two curves on X with no common component defined by homogeneous polynomials f and g of degrees d and e respectively. Then C and D intersect in exactly de points, counting multiplicities.

Proof. Any curve of degree d on $\mathbb{P}^2$ is linearly equivalent to d multiples of a line. Let G and H be distinct lines. Then $C \sim dG$ and $D \sim eH$. We have

$(C.D) = (dG.eH)$ (using linear equivalence).

$= de(G.H)$ (using bilinearity).

$= de$ (using the fact that any two distinct lines on $\mathbb{P}^2$ meet exactly once).

$= \sum_{p \in C \cap D} (C.D)_p$ (using property 3 of the intersection number).

Which is Bezout's theorem. □

Remark 7.3.4. If $X = \mathbb{P}^1 \times \mathbb{P}^1$ then it is possible to show that $Pic(X) = \mathbb{Z}E \times \mathbb{Z}H$ with intersection products $(E.E) = 0$, $(E.H) = 1$, $(H.H) = 0$. Observe that this allows us to calculate the intersection product for any two curves on X. We first find their images in $Pic(X)$ and then use the bilinearity of the intersection product and the above three products to obtain the desired number.

The point is that we have reduced the calculation of an infinite number of intersection products (since there are an infinite number of distinct curves) to the calculation of in this case, just three.

As another example, if $X \subseteq \mathbb{P}^3$ is defined by a homogeneous cubic polynomial f, that is $X = V(f)$, then it is possible to show that

$$Pic(X) = \mathbb{Z}H \times \mathbb{Z}E_1 \times \mathbb{Z}E_2 \times \ ... \ \times \mathbb{Z}E_6$$

The intersection products of the H, E_i are found to be $(H.H) = 1$, $(H.E_i) = 0$ $(E_i.E_i) = -1$ and $(E_i.E_j) = 0$ for $i \neq j$. □

See [**H**] Chapter V, proposition 4.8.

As an interesting corollary to Bezout's theorem we have the following.

Theorem 7.3.5. *(Pascal's hexagon)*

Let A, B, C, A', B', C', be six points on an irreducible projective conic (a curve of degree 2 on $\mathbb{P}^2$). Let P be the intersection of AB' and $A'B$, let Q be the intersection of AC' and $A'C$ and let R be the intersection of BC' and $B'C$. Then P, Q and R are collinear.

Proof. Let D be the conic. Define lines $L_1 = AB'$, $L_2 = BC'$, $L_3 = A'C$ and let $L = L_1 + L_2 + L_3$ as a divisor. Similarly define $M_1 = A'B$, $M_2 = B'C$, $M_3 = AC'$ and let $M = M_1 + M_2 + M_3$. Now if f_i is the degree 1 homogeneous polynomial defining L_i then L is defined by the cubic $f = f_1 f_2 f_3$, that is $L = V(f)$. Similarly $M = V(g)$ for some cubic g.

Pick a point r which is on the conic D but not on L or M. Choose $\lambda \in k$ such that $f(r) + \lambda g(r) = 0$. We let N be the divisor defined by the cubic $f + \lambda g$. By the definition of N we have that all points in $L \cap M$ are on N. In particular the six points A, B, C, A', B' and C' which also lie on the conic D. Since r is on N and D we have that $\#(D \cap N) \geqslant 6 + 1 = 7$. However, since D is a conic an N a cubic we have that $\#(D \cap N)$ is at most $2.3 = 6$ by Bezout's theorem, a contradiction. Thus D and N must have a common component, which must be D since D is irreducible. Thus $N = D + K$ where K must be a line since N is a cubic and D a conic. Now, the three points P, Q and R lie on $L \cap M$ and hence must also lie on N. Since P, Q and R do not lie on D they must lie on the line K, completing the proof. □

Remark 7.3.6. If we let $k = \mathbb{C}$ and arrange the points such that none of them lie on the line at infinity and all have real coordinates then we obtain a theorem in real affine geometry. □

Pascals hexagon.

Picture taken from [**H**] page 407.

7.4 Remarks

The importance of the intersection product is, firstly, that it is well defined on $Pic(X) \times Pic(X)$, that is, the product depends only on the linear equivalence classes of divisors.

Secondly, it can be shown that the product is a numerical invariant of a surface from the point of view of classification of surfaces.

We have not studied surfaces in any detail in this thesis as the detailed study of surfaces is an entire field in itself and requires a large amount of background material. The book [**B**], by Beauville is a good introduction to this area.

Another interesting application of the intersection product is a consideration of those surfaces which have divisors C with the property that $(C.C) = -1$. For example, six of the seven generators of the Picard group of the cubic surface have this property. A result is that if a divisor C on a surface X satisfies $(C.C) = -1$ then we know that X is isomorphic to some other surface 'blown up' at a point p. For details of the blow up construction see for example [**B**] Chapter II.1. It is a fact that the cubic surface is isomorphic to the projective plane blown up at six different points, see [**H**] Chapter V.4.

We can by no means comment on all of the applications, or generalisations (yes, that's right, generalisations) of the intersection product to other areas of algebraic geometry, as most of these areas are rather unfamiliar territory at this stage.

What is worth noting, however, is that with the concept of divisors, the intersection product is a very natural generalisation of Bezout's theorem. The fact that it is possible at all to generalise in this manner illustrates the power of some of the tools of algebraic geometry. Of course it has taken a great number of people a lot of hard work to develop these concepts and tools and bring us to where we are today.

It is to them we owe the most thanks.

Antony Orton, UNSW, Sydney, November 2003.

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