Bicategories and Higher Categories

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Abstract

Category theory allows a precise formulation of relationships between structures that emerge from seemingly different contexts across mathematics. One would like to study not just the properties of one particular group, for example, but of the category of groups and group homomorphisms between them as a whole. Category theorists would then like to do the same with their own subject, and study not just some particular category but rather the whole picture of categories, functors, and natural transformations. This is a preliminary example of a 2-category.

Acknowledgements

This thesis would not have been possible without my wonderful support network. First and foremost my supervisor, A/Prof. Daniel Chan. He helped direct my attention and clarify my understanding on countless occasions throughout the year. I have learned so much from him over the past year and he has greatly helped me mature as a mathematician. His passion for the subject and incredible understanding made the experience instructive and enjoyable. Thanks also go to my partner, Anusha Nidigallu, who assisted me with typesetting most of the diagrams and in proof reading. To John Burke, Branko Nikolic, Alex Campbell, Daniel Lin, Ross Street, Stephen Lack, Richard Garner and Dominic Verity, whose helpful discussions and direction towards useful sources greatly assisted me over the course of the year. To Ed McDonald for helping me with formatting in LaTeX. Finally, to my family, who made sure I had a supportive environment in which to work and who helped keep me motivated when times were tough.

Plagiarism Declaration

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, nor material which to a substantial extent has been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.

Contents

Introduction

In a category, the most meaningful notion of 'sameness' is not strict equality of objects but rather an invertible morphism, or isomorphism, between them. Any properties of morphisms however, such as associativity and unit laws for composition, need to hold strictly. This is because there is no notion of a 'higher morphism', invertible or not, up to which such properties could hold. One of the aims of higher category theory is to formalise the notion of 'higher morphisms' so that structural conditions on k-morphisms in an n-category for $k < n$ only hold up to an invertible $(k+1)$-morphism.

In this thesis we wish to guide a reader familiar with categories to a rigorous understanding of this notion in the case of $n = 2$. We present a summary from first principles of the basic theory of weak 2-categories, or *bicategories* as they are commonly called, and the various notions of structure preserving maps that naturally arise between them. Detailed proofs involve many large diagrams which are difficult to find elsewhere in the literature. It turns out, however, that bicategories are in a sense equivalent to strict 2-categories in which structural properties hold 'on the nose', or up to an identity 2-cell. This greatly simplifies the computations involved in the study of bicategories. We will be following a similar line of proof to this result as in [2], [18] and [23], filling in details to do with the compositional structure of the structure preserving maps which they omit.

We then look at of some geometric and algebraic models for strict higher categorical structures, and show how bicategorical phenomena motivate the study of weak 3-categories, or tricategories, in a similar way to how categorical phenomena motivate the study of bicategories. The structure of tricategories is much richer than that of bicategories however, and indeed $n = 2$ is the highest 'dimension' in which every n-category is equivalent to a strict n-category.

Of particular interest for developing intuition will be special cases of bicategories which are degenerate in a similar sense to the following cases of ordinary categories.

1. A category with only identity morphisms is called *discrete*, and if it is small it may be identified with its set of objects, or equivalently its set of morphisms.
2. A category in which every morphism is invertible is called a *groupoid*.
3. A (locally small) category with only one object is just a monoid $(M, *)$: composition of morphisms corresponds to the monoid operation $*$, while the associativity and identity axioms coincide.
4. If the previous two both hold, then the morphisms are just a group under composition.
5. A category is called *skeletal* if it has no non-identity isomorphisms. With the axiom of choice, one can show that every small category is equivalent to its skeleton.
6. A small category whose hom-sets are singletons is just a preordered set. $(P, \preccurlyeq)$. The objects correspond to the elements in the preordered set, and a morphism exists in each hom-set if and only if $x \preccurlyeq y$. Composition of morphisms corresponds to transitivity, for which associativity holds vacuously, while the existence of identities for each object corresponds to reflexivity of the order.
7. If 5 and 6 both hold then what we get is just a partial order: antisymmetry is precisely the condition that no two distinct objects are isomorphic.
8. If 3 and 6 both hold then what we get is just an equivalence relation: symmetry is precisely the condition that every morphism is invertible.
9. If 3, 5 and 6 hold simultaneously, then 1 will also hold: every equivalence class collapses to a point.

Note the use of the word *just* above. In this paper, when we say "A is just a B" for some mathematical structures A and B, we will mean that both A and B uniquely determine one another. For example, we would say that a group is just a monoid in which every element is invertible, or a partial order is just a preorder that is antisymmetric. Indeed, more precisely, there will be an equivalence of categories between the category of As and the category of Bs in this case.

One concern other mathematicians may have when first learning category theory is that its foundations must necessarily be broader than standard ZFC set theory so as to meaningfully talk about **Cat**, the category of categories, or functor categories $[C, D]$, without encountering Russell's Paradox like issues. In particular, the theory of *grothendieck universes* is used. We will ignore such issues as is common in the theory, and use the words

- *small* to mean that the class of objects in question is an ordinary set of ZFC.
- *locally small* to mean that the hom-classes in question are ordinary sets of ZFC.

Note however that one can define **CAT**, the category of categories of size at most κ for any inaccessible cardinal κ, and its size will conveniently be greater than κ so as not to cause paradoxes. Readers who have further interest in foundational issues should consult [8].

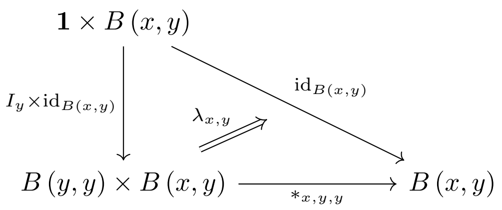
1 Bicategories

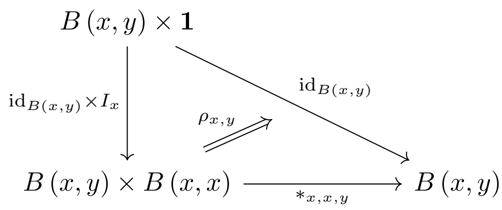
1.1 The Definition

Definition 1 (Bicategory)**.** A bicategory B consists of the following data subject to the following axioms:

DATA

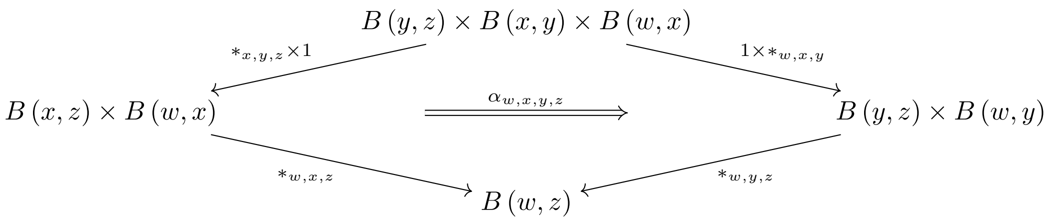
- A class of 0-*cells*, denoted B_0
- For every pair of 0-cells $x, y \in B_0$, a category $B(x, y)$ called the *hom-category* from x to y whose
  - objects $f \in B(x, y)$ are called 1-*cells* from x to y, written $f : x \to y$.
  - morphisms $\phi \in \mathrm{Hom}_{B(x,y)}(f, g)$ are called 2-*cells*, written $\phi : f \Rightarrow g$.
  - composition $\circ : \mathrm{Hom}_{B(x,y)}(g, h) \times \mathrm{Hom}_{B(x,y)}(f, g) \to \mathrm{Hom}_{B(x,y)}(f, h)$ is called *vertical composition.*
- For every $x \in B_0$ a functor $I_x : \mathbf{1} \to B(x, x)$ called the *identity* of x. Without ambiguity, we will identify this functor with the unique 1-cell in $B(x, x)$ to which it sends the unique object of $\mathbf{1}$.
- For every $x, y, z \in B_0$ a bifunctor $*_{x,y,z} : B(y, z) \times B(x, y) \to B(x, z)$ called *horizontal composition* whose action on 1-cells is called 1-*cell composition.*
- For every $x, y \in B_0$, two natural isomorphisms respectively called the *left unitor* and *right unitor*





whose components for 1-cells $f \in B(x,y)$ are given by $(\lambda_{x,y})_f := \lambda_f : f * I_y \Rightarrow f$ and $(\rho_{x,y})_f := \rho_f : I_x * f \Rightarrow f$.

- For every $w, x, y, z \in B_0$, a natural isomorphism called the *associator* given by

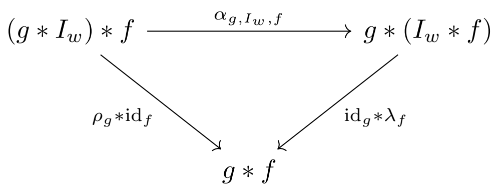


whose components for $f \in B(y,z), g \in B(x,y), h \in B(w,x)$ are given by $(\alpha_{w,x,y,z})_{h,g,f} := \alpha_{h,g,f} : (h * g) * f \to h * (g * f)$.

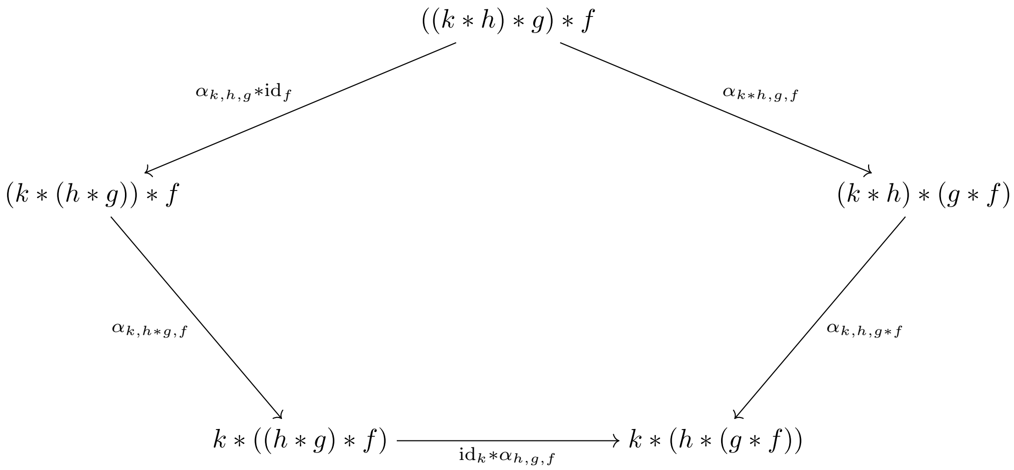
AXIOMS

The following diagrams commute for all composable quadruples of 1-cells $v \xrightarrow{f} w \xrightarrow{g} x \xrightarrow{h} y \xrightarrow{k} z$

- (Triangle identity)



- (Pentagon identity)



The classes of natural isomorphisms α, λ, and ρ are collectively called the *coherence constraints*. If they are all identity natural isomorphisms, then the bicategory is called *strict*, in which case we will refer to it as a 2-category. If B_0 is a set then B is called *small*, while if all hom-categories $B(x,y)$ are small then B is called *locally small*.

Notation 2. *From here on, we will omit subscripts on 2-cells that correspond to identities of 1-cells, as in the remark below.*

Remark 3. The associativity and unit laws in a bicategory need not hold exactly as in a category, but only up to isomorphism. That is, the following diagrams must commute for all 2-cells $\varphi : f \Rightarrow f', \phi : g \Rightarrow g', \psi : h \Rightarrow h'$:

- The naturality square for the associator α

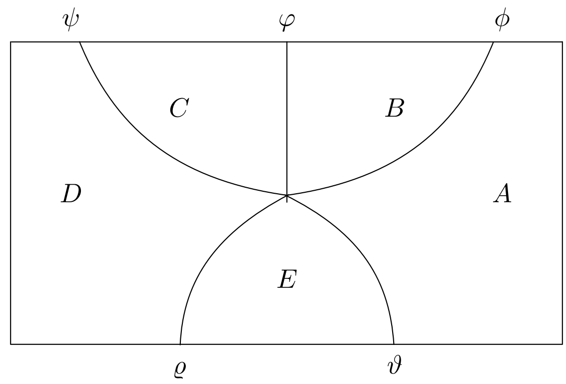
$$\begin{array}{ccc}
(h * g) * f & \xrightarrow{\alpha_{h,g,f}} & h * (g * f) \\
\Big\downarrow{\scriptstyle (\psi * \phi) * \varphi} & & \Big\downarrow{\scriptstyle \psi * (\phi * \varphi)} \\
(h' * g') * f' & \xrightarrow[\alpha_{h',g',f'}]{} & h' * (g' * f')
\end{array}$$

- The naturality squares for the unitors λ and ρ

$$\begin{array}{ccccc}
I_y * h & \xrightarrow{\lambda_h} & h & \xrightarrow{\rho_h} & h * I_x \\
\Big\downarrow{\scriptstyle \mathrm{id} * \psi} & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \psi * \mathrm{id}} \\
I_y * h' & \xrightarrow[\lambda_{h'}]{} & h' & \xrightarrow[\rho_{h'}]{} & h' * I_x
\end{array}$$

The fact that α, λ, and ρ have are natural isomorphisms and hence have invertible components means that any pentagon or triangle formed by reversing the directions of any of the arrows in the axioms above still commutes. The pentagon axiom is the 'associahedron' K_4, whose vertices are each possible parenthesised word on four letters such that the letters appear in the same order, and edges are an application of associativity. Its commutativity asserts that there is no ambiguity between the two possible ways to transform an expression involving four horizontally composed 1-cells bracketed to the left to one bracketed to the right, while the triangle axiom asserts that there is no ambiguity between the two ways to collapse the unit if it appears in the middle of an expression involving horizontally composed 1-cells. These axioms are often called the *coherence conditions* for bicategories, and they hold vacuously if the bicategory is strict.

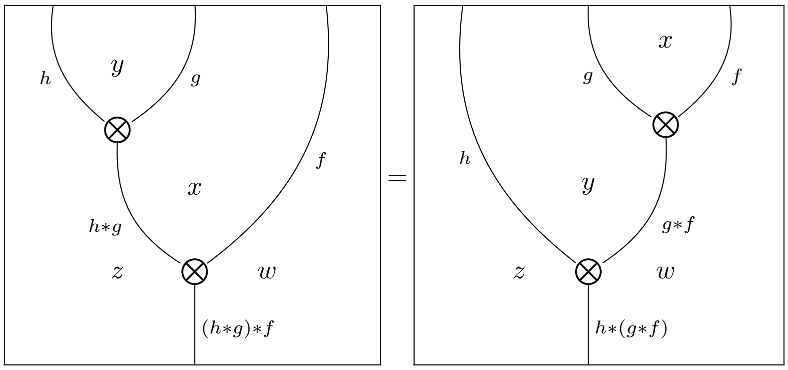
Remark 4. Often it may be useful to take the Poincare dual of the underlying graph of any diagram in a bicategory to see a *string diagram*. For example, letting $A = ((k * h) * g) * f$, $B = (k * (h * g)) * f$, $C = k * ((h * g) * f)$, $D = k * (h * (g * f))$, $E = (k * h) * (g * f)$, $\phi = \alpha_{k,h,g} * \mathrm{id}_f$, $\varphi = \alpha_{k,h*g,f}$, $\psi = \mathrm{id}_k * \alpha_{h,g,f}$, $\vartheta = \alpha_{k*h,g,f}$, $\varrho = \alpha_{k,h,g*f}$, we have the associativity axiom represented by the following string diagram in which by convention we read from right to left.



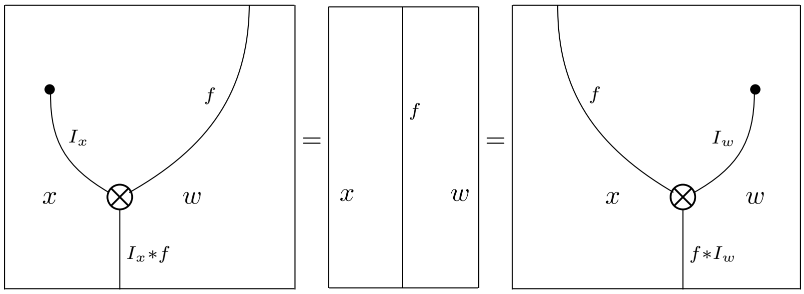
in which the regions correspond to the objects, the strings correspond to the morphisms, and the vertex corresponds to the commutativity of the pentagon diagram.

Remark 5. A similar construction can be used to produce *process network diagrams*, in which 0-cells are represented by regions of the plane, 1-cells are represented as strings and horizontal composition is represented by a 'black box' process $\bigotimes$ which transforms one string into another. The structure induced by the associator and unitors can be expressed by the following equivalence relations on process networks:

- Associativity coherence:

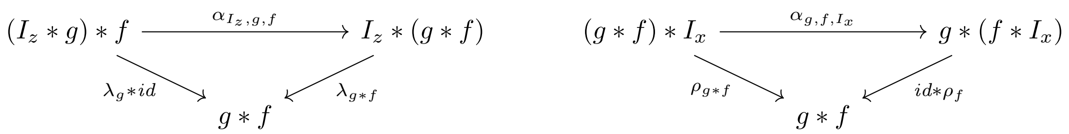


- Unit coherence:

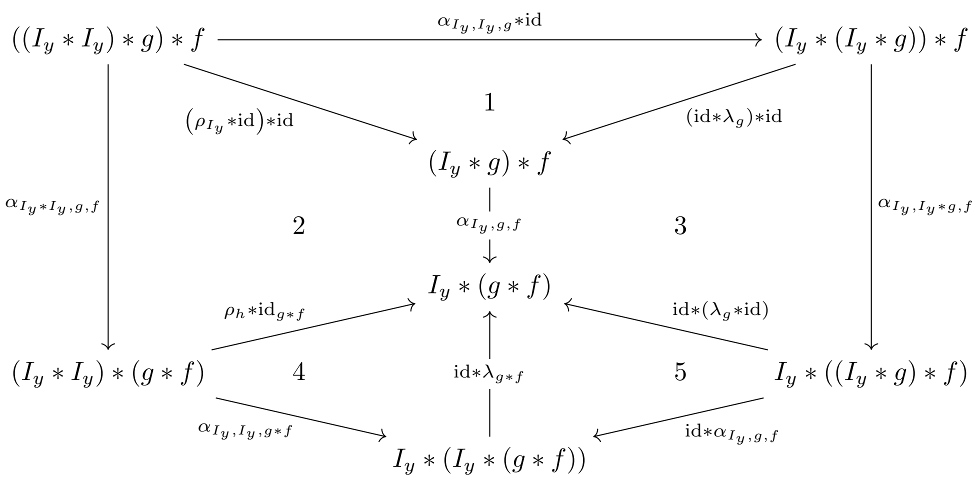


These diagrammatic tools are not just visual aids for intuition. A rigorous treatment was provided in the papers [11] and [12], where it was essentially shown that continuously 'pulling' strings from one embedding in $\mathbb{R}^2$ to another is possible if and only if the property described by the first diagram implies the property described by the second diagram. In the interests of being self contained, we will not use these constructions in proofs, but we nevertheless encourage the reader to experiment with string diagrams themselves. For example, proof of the proposition below is a lot less mysterious in light of of process network diagrams.

Proposition 6. *The following diagrams commute for all* $x \xrightarrow{f} y \xrightarrow{g} z$



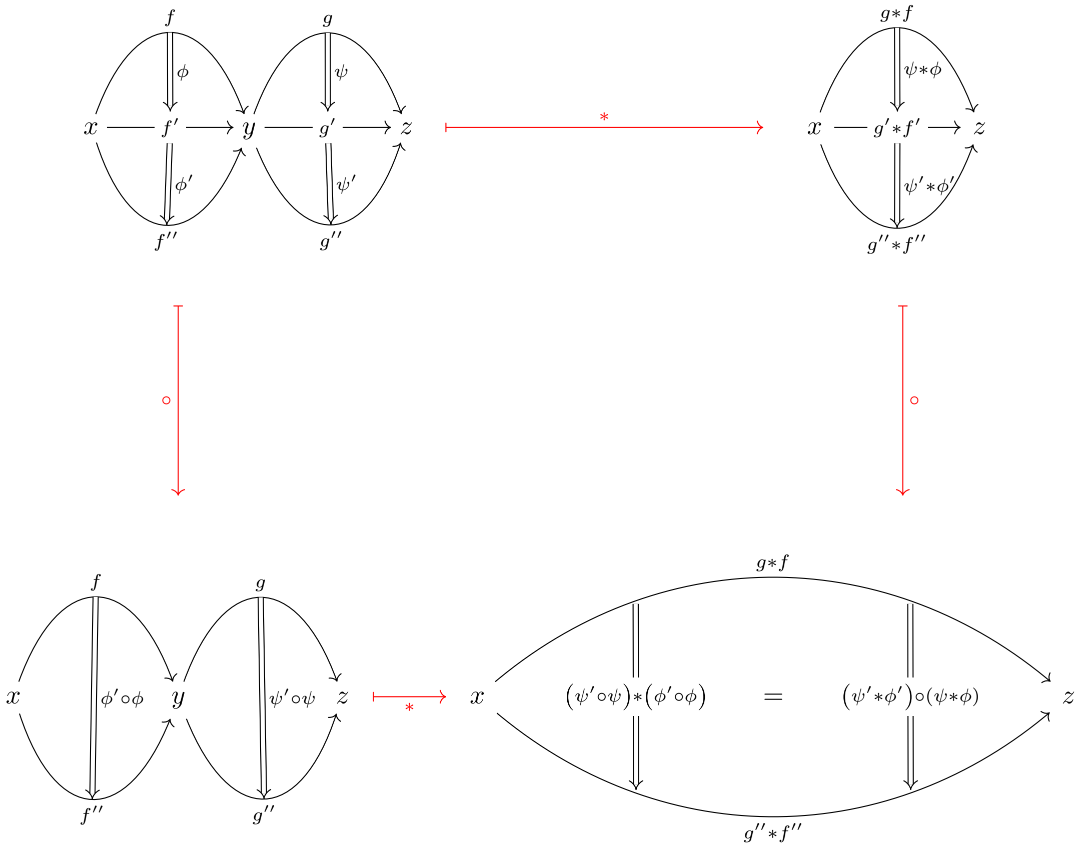
Proof. We prove the first diagram as the proof for the second is similar. Consider the following diagram:



We observe that the boundary commutes by the pentagon axiom. Region 4 is the triangle axiom while region 1 is the triangle axiom pre-composed with f. Regions 2 and 3 commute by naturality of α. Hence 5 commutes, by the diagram pasting lemma. The required diagram is the image of 5 under λ. □

In fact, we will show in chapter three that *all* diagrams formed entirely of coherences and identity 2-cells commute.

Remark 7. We elaborate on what the notion of functoriality means for horizontal composition. Let x, y, z be 0-cells in a bicategory, with $f, f', f'' : x \to y$, $g, g', g'' : y \to z$ 1-cells, and $\phi : f \Rightarrow f'$, $\phi' : f' \Rightarrow f''$, $\psi : g \Rightarrow g'$, $\psi' : g' \Rightarrow g''$ 2-cells as pictured on the top left. Then functoriality of horizontal composition is precisely the identity $(\psi' \circ \psi) * (\phi' \circ \phi) = (\psi' * \phi') \circ (\psi * \phi)$. That is, the two parallel 2-cells as pictured on the bottom right are equal.



This relationship between the different notions of composition in a higher categorical structure turns out to be very useful in describing some of their degenerate structures.

1.2 Degenerate Cases

Example 8. Note that just as any set can be viewed as a discrete category, a category is just a 2-category with no non-identity 2-cells. Hence a set is just a 2-category with no non identity 1 or 2 cells. There is also a bicategory with only one cell for all $n \in \{0, 1, 2\}$, which we will denote $\underline{\mathbf{1}}$ and whose unique n-cell will be denoted n.

Example 9. **Cat** is a strict 2-category whose 0-cells are categories, hom-categories **Cat** (C, D) are given by functor categories $[C, D]$, and horizontal composition is given by composition of functors and horizontal composition of natural transformations. Then each of the examples discussed in the introduction are the 0-cells of certain 2-categories. For example, **Grpd** is the full sub-2-category of **Cat** whose 0-cells are groupoids and hence, **Grp** may be considered as a 2-category, namely the full sub-2-category of **Grpd** whose 0-cells are one object groupoids.

Example 10. A *monoidal category* is just a bicategory with only one 0-cell. The underlying category is the unique hom-category, the unit of the monoidal category is the identity of the 0-cell, and the horizontal composition in the bicategory is just the monoidal product $\otimes$. So monoidal categories are to bicategories as monoids are to categories.

Monoidal categories can most simply be seen as categories whose objects have a monoid structure in which associativity and unit laws need to hold only up to isomorphism. Often monoidal categories also come equipped with a notion of commutativity known as a *braiding*. We briefly recall the definitions.

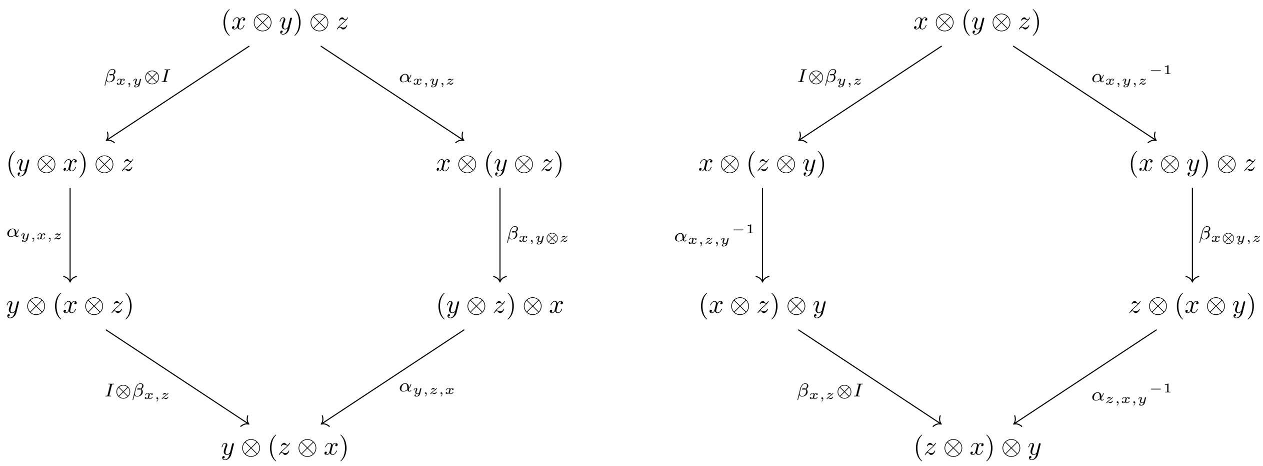
Definition 11 (Braided Monoidal Category)**.** A *braided monoidal category* $(M, \otimes, I, \lambda, \rho, \alpha, \beta)$ is given by the following data subject to the following axioms.

DATA

- A monoidal category $(M, \otimes, I, \lambda, \rho, \alpha)$
- $\beta : (-,-) \Rightarrow (-,-)$, a natural isomorphism called a *braiding*, with components $\beta_{x,y} : x \otimes y \to y \otimes x$ at $x, y \in M$.

AXIOMS

The following hexagons commute for all $x, y, z \in M$

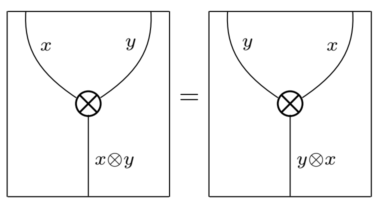


A braided monoidal category is called *symmetric* if $\beta_{x,y} \circ \beta_{y,x} = \mathrm{id}_{x \otimes y}$ for all $x, y \in M$.

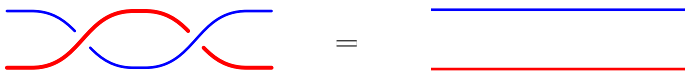
Remark 12. The naturality square for the braiding is given by the following diagram:

$$\begin{array}{ccc} x \otimes y & \xrightarrow{\beta_{x,y}} & y \otimes x \\ {\scriptstyle f \otimes g}\downarrow & & \downarrow{\scriptstyle g \otimes f} \\ x' \otimes y' & \xrightarrow[\beta_{x',y'}]{} & y' \otimes x' \end{array}$$

The coherence for braiding is represented by the equivalence on process network diagrams:

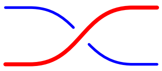


Remark 13. Braidings are so named because they model strings braiding over one another. The assertion that the braiding is symmetric is described by the equivalence relation:

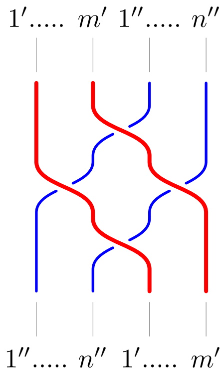


Example 14. Recall the *braid group* $\mathfrak{B}_n = \left\langle \sigma_1, ..., \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_k\sigma_l = \sigma_l\sigma_k \text{ if } |k-l| \geq 2 \\ \sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1} \text{ if } 1 \leq k \leq n-2 \end{array} \right\rangle$

A *braid* is an element of $\mathfrak{B}_n$, and may be visualised as a way of crossing a set of n strings over one another. Each σ_k can be visualised as crossing string k over string $k+1$ like so:



The category of braids **Braid** has natural numbers as objects, and non-empty hom-sets $\mathrm{Hom}_{\mathbf{Braid}}(n, m) = \mathfrak{B}_n$ if and only if $m = n$. It has a monoidal product which acts via $m \otimes n = m + n$ on objects, and via the group homomorphism $\mathfrak{B}_m \times \mathfrak{B}_n \to \mathfrak{B}_{m+n}$ on morphisms. It has 0 as its unit, and identity associator and unitors. The action of the braiding $\beta_{m,m}$ on morphisms is shown in the braid diagram below, where we have suppressed braids for $1 < k < m$ $1' < k' < n'$.



This category is the motivating example of a braided, but not symmetric, monoidal category.

Definition 15 (Binary Interchange Law)**.** Let X be a set and let $\oplus, \otimes : X \times X \to X$ be binary operations on X. Then $\oplus$ and $\otimes$ are said to satisfy the *binary interchange law* if $(w \oplus x) \otimes (y \oplus z) = (w \otimes y) \oplus (x \otimes z)$ for all $w, x, y, z \in X$.

Lemma 16 (Eckmann-Hilton Argument)**.** *Recall that* $(X, \otimes, 1)$ *is called a unital if* 1 *is a two-sided identity for* $\otimes$*, with* $x \otimes 1 = x = 1 \otimes x$ *for all* $x \in X$*. Let* X *be a set and let* $\oplus, \otimes : X \times X \to X$ *be binary operations such that* $(X, \oplus, 0)$ *and* $(X, \otimes, 1)$ *form unitals for some* $0, 1 \in X$*.*

If $\oplus$ *and* $\otimes$ *satisfy the binary interchange law, then* $\oplus = \otimes$ *and both operations are commutative and associative.*

Proof. We first show that $0 = 1$.

$0 = 0 \oplus 0 = (1 \otimes 0) \oplus (0 \otimes 1) = (1 \oplus 0) \otimes (0 \oplus 1) = 1 \otimes 1 = 1$

Then, for $x, y \in X$, we have that

$x \oplus y = (1 \otimes x) \oplus (y \otimes 1) = (1 \oplus y) \otimes (x \oplus 1) = y \otimes x$
$y \otimes x = (y \oplus 1) \otimes (1 \oplus x) = (y \otimes 1) \oplus (1 \otimes x) = y \oplus x$

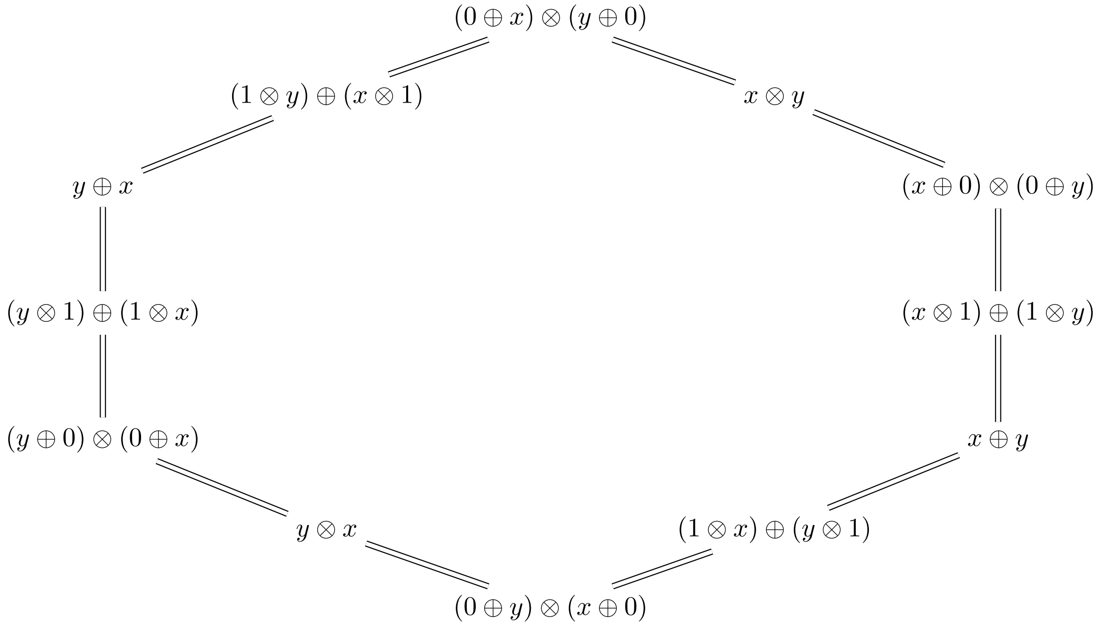
So we have that $\otimes = \oplus$, and the operations are commutative. Finally, for associativity, take

$x, y, z \in X$ and observe that

$$(x \otimes y) \otimes z = (x \otimes y) \otimes (1 \otimes z) = (x \otimes 1) \otimes (y \otimes z) = x \otimes (y \otimes z)$$

This completes the proof. $\square$

Remark 17. If we allow ourselves to assume associativity and that $0 = 1$, then the rest of the proof is clarified by the following clock diagram, due to Eugenia Cheng, in which each edge is an equality.



Indeed, we see that each 'hour' on the clock represents an application of one of the hypotheses.

Remark 18. Note that if X is a set endowed with n binary operations $\otimes_1 ... \otimes_n : X \times X \to X$ such that

- Each of the binary operations has a two sided identity
- For all $1 \leq i < i + 1 \leq n$, both $\otimes_i$ and $\otimes_{i+1}$ obey the binary interchange law.

Then by induction on n, the Eckmann-Hilton Argument shows that all the operations are commutative, associative, and equal to one another.

Corollary 19. *Bicategories with only one* 0*-cell and only one* 1*-cell are just commutative monoids with a distinguished invertible element which will be the identity if and only if B is strict.*

Proof. Let B be such a bicategory and call its 0 cell 0 and 1 cell 1. Th unique hom-category in B forms a monoidal category with only one object. Its morphisms will just be monoids under vertical composition. Further, note that the identity of the unique 1-cell is a two-sided identity for both vertical and horizontal composition. Hence applying the Eckmann-Hilton Argument above, we see that they coincide and form a commutative binary operation.

Since α, λ and ρ are natural isomorphisms, their components for must be invertible. The pentagon axiom then says that $(\alpha_1)^3 = (\alpha_1)^2$, and so $\alpha_1 = \mathrm{id}_1$. Then, the triangle axiom says that $\rho_1 = \lambda_1$. This unique unitor isomorphism is the required distinguished invertible element. As $\alpha_1 = \mathrm{id}_1$, the unitor isomorphism will be the identity if and only if B is strict. $\square$

1.3 2-diagrams

Diagrams in bicategories and hence proofs constructed from them can become quite large very quickly. We therefore summarise the theory of 2-diagrams developed in [20], which readers who are interested in further details should consult.

Notation 20. *We denote by* (V, E, F) *the triple of vertices, edges, and faces of a planar directed graph* G*. In a planar directed graph, a path* $p = (v_1, ..., v_n)$ *along directed edges* $(v_i, v_{i+1}) \in E$ *will be written* $p : v_1 \to v_n$*. It will by convention be assumed that paths have no repeated vertices. If* $q : v_n \to v_m$ *is also a path then we denote by* qp *the path* $(v_1, ..., v_n, ..., v_m)$ *formed by traversing* p *and then continuing along* q*. We denote by* $\bar{p}$ *the path* $(v_n, ..., v_1)$ *formed by reversing the direction of the edges. Lastly, we denote by* $ext(G)$ *the unique exterior face of* G*.*

Definition 21 (Directed planar graph with source and sink)**.** A *directed planar graph with source and sink* is a directed planar graph $G = (V, E, F)$ with two distinguished vertices s and t in the exterior face of G such that for every vertex $v \in V$, there is a path from s to v and a path from v to t.

Definition 22 (Pasting scheme)**.** A *pasting scheme* is a directed planar graph $G = (V, E, F)$ with source and sink such that every face $f \in F$ has two distinct vertices $s(f)$ and $t(f)$ and paths $p, q : s(f) \to t(f)$ such that the boundary of f is $\bar{q}p$. In this situation we write f as the pair (p, q).

Definition 23 (Labelling of a pasting scheme)**.** Given a pasting scheme as above, a *labelling* for that pasting scheme in a 2-category B is given by the following data subject to the following axioms.

DATA

- A function $g_0 : V \to B_0$
- A function $g_1 : E \to B_1$. For a path $(v_1, ..., v_n)$ we write $g_1(p)$ for the horizontal composition $g_1(v_n, v_{n-1}) * ... * g_1(v_2, v_1)$. This is unambiguous as B is strict.
- A function $g_2 : F \to B_2$

AXIOMS

- For every $e = (v, w) \in E$, we have that $g_1(e) : g_0(v) \to g_0(w)$.
- For every $f = (p, q) \in F \backslash \{\text{ext}(G)\}$, we have that $g_2(f) : g_1(p) \Rightarrow g_1(q)$.

These definitions essentially amount to saying that the faces of a pasting scheme have as their boundary two parallel paths, and that the faces are 'directed' from one of these paths to the other. Furthermore, the labelling of a pasting scheme in a 2-category is a way of assigning to each vertex, edge, or face in the pasting scheme, a 0, 1 or 2-cell in B respectively, such that the assignment respects the sources and targets in the pasting scheme by sending them respectively to domains and codomains of cells in B. We are now ready to state the 2-categorical pasting lemma.

Theorem 24 (Pasting lemma for 2-categories)**.** *Every labelling in a* 2*-category* B *of a pasting scheme has a unique composite.*

Note that the pasting lemma for 2-diagrams relies on strictness of the underlying bicategory. In fact, by the strictness and coherence theorems proven in chapter three, there is also a pasting lemma for general bicategories. This allows one to deduce the commutativity of a 2-diagram in bicategory by gluing together commuting 2-diagrams on their common 2-cells, or 'faces'. As with string diagrams, 2-diagrams will feature in this thesis as visual aids for intuition rather than as key ingredients in our proofs. The utility of 2-diagrams is similar to that of using 1-diagrams rather than just listing out equalities of composed morphisms, which can in some sense be thought of as a '0-diagram'. The 2-cells of a 2-diagram can be visualised as 'directed faces' of a two dimensional surface, and the 1-cells can be interpreted as their 'boundary edges'. See the discussion on directed n-graphs in chapter four for more details.

Just as commuting diagrams in categories can be thought of as collections of equal morphisms between some source and target, commuting 2-diagrams can be thought of as equal 2-cells going between 1-cells. A 2-diagram is said to

- 1-commute if its underlying 1-diagram commutes.
- 2-commute if all paths along 2-cells with the same source and target 1-cells are equal.

1.4 Basic Notions in a Bicategory

We now outline how some familiar constructions from ordinary categories generalise to bicategories.

Given a category, we can reverse its morphisms to form its opposite category. There are three distinct analogous constructions for bicategories.

Definition 25 (Duals of a bicategory)**.** Given a bicategory B, we also have the bicategories

- B^{op} the 1-cell dual in which 1-cells and constraints are reversed. Formally, $B^{op}(x, y) \cong B(y, x)$ is an isomorphism of categories and $\lambda^{op} = \rho$, $\rho^{op} = \lambda$, ${\alpha^{op}}_{f,g,h} = {\alpha_{h,g,f}}^{-1}$. Hence in particular, $f : x \to y$ is a 1-cell in B if and only if $f : y \to x$ is a 1-cell in B^{op}
- B^{co} the 2-cell dual in which 2-cells are reversed. Formally, $B^{co}(x, y) \cong (B(x, y))^{op}$ is an isomorphism of categories. Hence, $\phi : f \Rightarrow g$ is a 2-cell in B if and only if $\phi : g \Rightarrow f$ is a 2-cell in B^{co}
- B^{co-op} the bidual in which both 1-cells and 2-cells are reversed as above. Formally, $B^{co-op} := (B^{op})^{co}$.

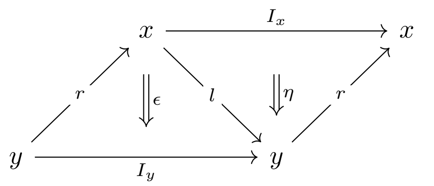
In a category, there is only really one way in which a morphism can be invertible. For functors, there is some flexibility in the notion of invertibility- isomorphisms, equivalences and adjunctions. In bicategories, there is only one way in which a 2-cell can be invertible as there are no '3-cells' up to which the property of invertibility could be asked to hold. However, for 1-cells, we do have some flexibility in this notion.

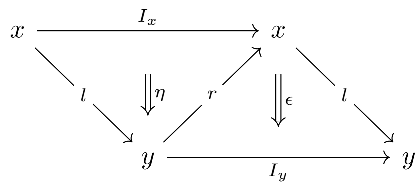
Definition 26 (Adjunction)**.** An *adjunction* (l, r, η, ϵ) between $x, y \in B_0$ is given by the following data subject to the following axiom.

DATA

- A pair of 1-cells $l \in B(x, y)$ call the *left adjoint*, and $r \in B(y, x)$ call the *right adjoint*
- A pair of 2-cells $\eta : I_x \Rightarrow r * l$ called the *unit*, and $\epsilon : l * r \Rightarrow I_y$ called the *counit*.

AXIOMS: (Zigzag identity) In the following diagrams, the left and middle triangles together give $\eta * \epsilon = \mathrm{id}_r$, while the middle and right triangles together give $\epsilon * \eta = \mathrm{id}_l$.





If ϵ and η are invertible then the adjunction is called an *adjoint equivalence*, in which case we also call l and r *invertible*, and write $x \cong y$.

Remark 27. It is important to note that this notion is of an adjunction *within* a bicategory, rather than one *between* bicategories. That is a different concept known as a *biadjuction* which will not be discussed here.

As a variant of the above definition, if ϵ and η as defined above are invertible but do not satisfy the zigzag identities we still call l and r invertible and write $x \cong y$, but we will only call (l, r, η, ϵ) an *internal equivalence* in B. It is shown in [19] that every internal equivalence gives rise to an adjoint equivalence.

Example 28. An adjunction in the bicategory **Cat** is just an ordinary adjunction between categories, while an equivalence of categories in **Cat** is an adjoint equivalence.

Similarly weakened versions for other familiar categorical constructions such as limits, nerves, Kan extensions etc also exist in bicategories. We will not elaborate on these here but instead refer interested readers to [15] for further information.

Next, recall that for ordinary categories, a property is said to hold *locally* if it holds for every hom-set. Similarly, we have the following terminology for bicategories.

Definition 29. A property of bicategories is said to hold *locally* if it holds for every hom-category.

Recall the notions of subcategories of an ordinary category, and that they are called

- essentially wide when the inclusion functor is essentially surjective.
- full when the inclusion functor is surjective on morphisms.

We extend these notions to bicategories with the following definition.

Definition 30 (sub-bicategory)**.** Let B be a bicategory. Then a bicategory A is a *sub-bicategory* of B if and only if

- $A_0 \subset B_0$
- For every $x, y \in A$, $A(x, y)$ is a subcategory of $B(x, y)$
- The horizontal composition of B restricted to the hom-categories of A is the horizontal composition of A.

Such a sub-bicategory is called

- *essentially wide* if the inclusion function on 0-cells $f : A_0 \to B_0$ is *essentially surjective*, meaning that for all $y \in B_0$ there exists a $x \in A_0$ such that $f(x) \cong y$.
- *locally essentially wide* if $A(x, y)$ is an essentially wide subcategory of $B(x, y)$ for all $x, y \in A_0$
- *locally full* if $A(x, y)$ is a full subcategory of $B(x, y)$ for all $x, y \in A_0$
- *full* if $A(x, y) = B(x, y)$ is an equivalence of categories for all $x, y \in A_0$

1.5 Examples of Bicategories

We conclude this chapter with a few more interesting examples of bicategories.

Example 31. There is a bicategory **Module** whose 0-cells are rings, 1-cells $M : S \to T$ are (S, T)-bimodules for rings S, T, and 2-cells are bimodule homomorphisms, which compose vertically via composition of bimodule homomorphisms. Horizontal composition is given on 1-cells $S \xrightarrow{M} T \xrightarrow{N} U$ by $M \otimes_T N$, and on 2-cells by the tensor product of bimodule homomorphisms over T, which is indeed functorial. The associators and unitors in **Module** are given by the canonical bimodule isomorphisms.

Given a commutative ring R, the bicategory **Module** has a sub-bicategory $R - \mathbf{Alg}$ whose 0-cells are commutative algebras over R, which will themselves be commutative rings. The hom-categories of $R-\mathbf{Alg}$ are given similarly, as is horizontal composition. The category $R-\mathbf{Mod}$ of left R-modules is the sub-bicategory produced by removing all 0-cells not equal to R and setting S to be the trivial ring, and as such this is a monoidal category. In particular, taking

- $R = \mathbb{Z}$ we see that $\mathbb{Z} - \mathbf{Mod} = \mathbf{Ab}$, the category of abelian groups is monoidal with the tensor product of abelian groups as the monoidal product and $\mathbb{Z}$ as the unit.
- R to be a field, then the category of finite dimensional vector spaces over R is a symmetric monoidal category with braiding $\beta_{V,W} : V \otimes W \to W \otimes V$ given by $(v \otimes_R w) \mapsto w \otimes_R v$ for all $v \in V$, $w \in W$.

Example 32. Let X be a topological space. Then the *fundamental bigroupoid* $\pi_2(X)$ has:

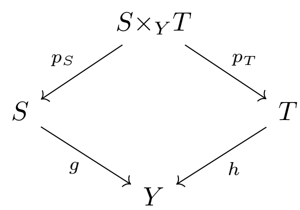
- 0-cells given by points $x \in X$,
- 1-cells given by continuous paths $p : x \to y$, defined as continuous functions $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$. For 1-cells $p : x \to y$ and $q : y \to z$, their composition $q * p : x \to z$ is given head to tail composition of paths, defined by the scaling
  - $p(2s)$ for $s \in \left[0, \frac{1}{2}\right]$, and
  - $q(2s-1)$ for $s \in \left[\frac{1}{2}, 1\right]$
- 2-cells given by equivalence classes of path homotopies $h : p \Rightarrow p' : x \to y$, defined as continuous functions $h : [0,1] \times [0,1] \to X$ such that for all $t \in [0,1]$ we have $h(0,t) = x$ and $h(1,t) = y$, and for all $s \in [0,1]$ we have $h(s,0) = p(s)$ and $h(s,1) = p'(a)$. Two path homotopies h, h' are equivalent if there is a 2-homotopy $h_2 : [0,1]^3 \to X$ between them, with $h_2(s,t,0) = h(s,t)$ and $h_2(s,t,1) = h'(s,t)$. Path homotopies compose horizontally by similarly scaling s, and vertically by scaling t.
- Associators and unitors which will have components given by path homotopies which correct for differences in parameterisation.

Note that every path from x to y has can be reversed to form a path form y to x, and any path homotopy from p to p' can similarly be reversed to form a path homotopy from p' to p, so that we do indeed get that every 1 and 2-cell is invertible, hence this is a bigroupoid.

Next, we see how some natural constructions from ordinary category theory give rise to bicategories.

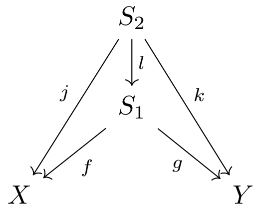
Example 33. Given a diagram of a particular shape, there is a 2-category whose 0-cells are categories with limits for diagrams of that shape, 1-cells are functors that preserve these limits, and 2-cells are natural transformations between those functors. This will be a sub-2-category of **Cat**.

Example 34. Recall that in any category C, a span of objects X and Y, written $(f,g) : X \leftarrow S \to Y$, is a diagram of the form $X \xleftarrow{f} S \xrightarrow{g} Y$ where S is some other object of the category, and if C has pullbacks then the spans $X \xleftarrow{f} S \xrightarrow{g} Y \xleftarrow{h} T \xrightarrow{i} Z$ compose to give the span $X \xleftarrow{f \circ p_S} S \times_Y T \xrightarrow{i \circ p_T} Z$ where the square

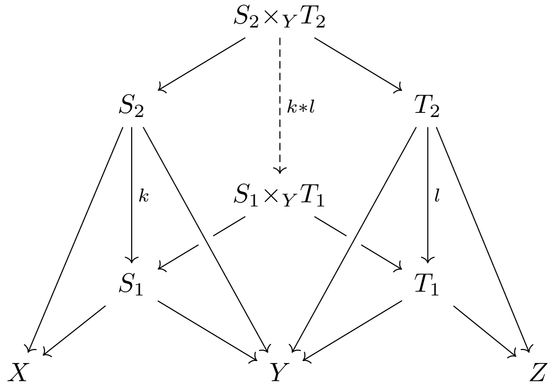


is a pullback.

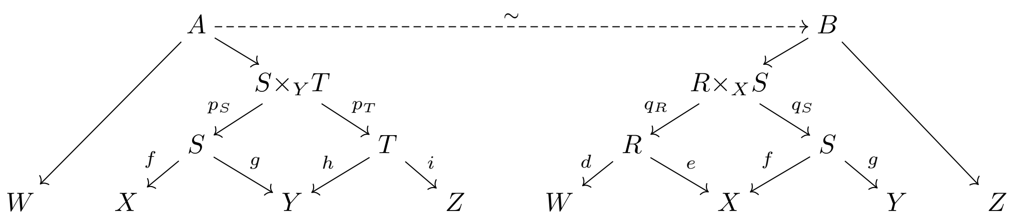
Given such a category C, there is a bicategory $\mathbf{Span}(C)$ whose 0-cells are objects of C, 1-cells are spans with composition as above, and 2-cells are morphisms of spans, given by morphisms l such that



which compose vertically via their ordinary composition in C, and horizontally to give $k * l$ as in the following diagram, in which we have suppressed labels for all other morphisms.



By the universal property of pullbacks, there is a unique isomorphism $\sim: A \to B$ in C between the pullbacks over S, as shown in the diagram below. This will be the component for the associator $\alpha_{W,X,Y,Z}$ in $\mathbf{Span}(C)$. The uniqueness of $\sim$ assures that the pentagon axiom holds.



Taking the unit I_X to be the identity on spans $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$, we see that the triangle axioms also hold.

Example 35. Recall that for any two sets X, Y a (X, Y)-*matrix* in a category C is a function $f : X \times Y \to \mathrm{Ob}(C)$, whose image on $(x, y) \in X \times Y$ we will write as $f_{x,y}$. If C is cartesian and cocartesian with products distributing over coproducts, then we can form the *bicategory of matrices in* C, $\mathbf{Mat}(C)$ whose

- 0-cells are sets
- 1-cells $f \in \mathbf{Mat}(C)(x, y)$ are (X, Y)-matrices in C.
- 2-cells $\phi : f \Rightarrow g$ for $f, g \in \mathbf{Mat}(C)(X, Y)$ are (X, Y)-indexed families of morphisms $\phi_{x,y} : f_{x,y} \to g_{x,y}$ in C, which compose vertically by point-wise composition in C.

Horizontal composition $*_{X,Y,Z}$ is given by formal matrix multiplication, where $\coprod_{y \in Y}$ denotes the coproduct in C of all objects indexed by $y \in Y$:

- $(f * g)_{x,z} := \coprod_{y \in Y} (f_{x,y} \times g_{y,z})$ on 1-cells
- $(\phi * \psi)_{x,z} := \coprod_{y \in Y} (\phi_{x,y} \times \psi_{y,z})$ on 2-cells

If $C = \mathbf{2}$, the category with two objects $\{0, 1\}$ and one non-identity morphism $f : 0 \to 1$, then $\mathbf{Mat}(C) = \mathbf{Rel}$, the bicategory whose 0-cells are sets, 1-cells are relations and 2-cells are inclusions between relations. Each matrix $X \times Y \to \mathbf{2}$ will be a bit-matrix indicating whether or not (x, y) is in the relation. Note that **Rel** is a locally posetal bicategory: every hom-category is a poset in the way described in the introduction.

2 Transfors of a Bicategory

One of the main insights of category theory is to look at how mathematical structures relate to one another via he structure preserving maps between them. This way of understanding mathematical structures has become known as the *relative point of view*, and was championed by Grothendieck. A general notion of an (n, k)-transfor was introduced in [7] to give a formalism to the notion of structure preserving maps in the context of higher category theory. For our purposes here, we consider the cases for $n \leq 2$. Then, (n, k)-transfors are given in the following table:

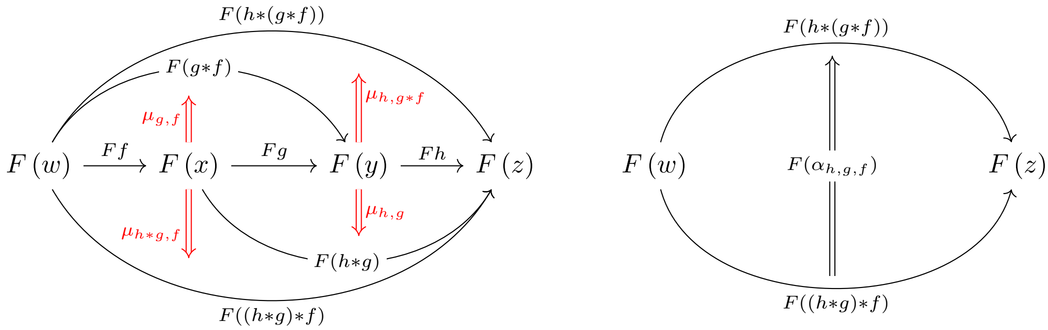
$n =$	-1	0	1	2
$k = 1$	implications	functions	functors	morphism of bicategories
$k = 2$	–	equalities of functions	natural transformations	transformations
$k = 3$	–	–	equalities of natural transformations	modifications
$k = 4$	–	–	–	equalities of modifications

We will see in chapter four how implications can be viewed as $(-1, 1)$-transfors. For now, we note that $(n, 1)$-transfors go between n-categories, while for $k > 1$, an (n, k)-transfor goes between $(n, k-1)$-transfors. The aim of this chapter will be to define the entries in the fourth column of this table, and to investigate their compositional structure.

2.1 Morphisms of bicategories

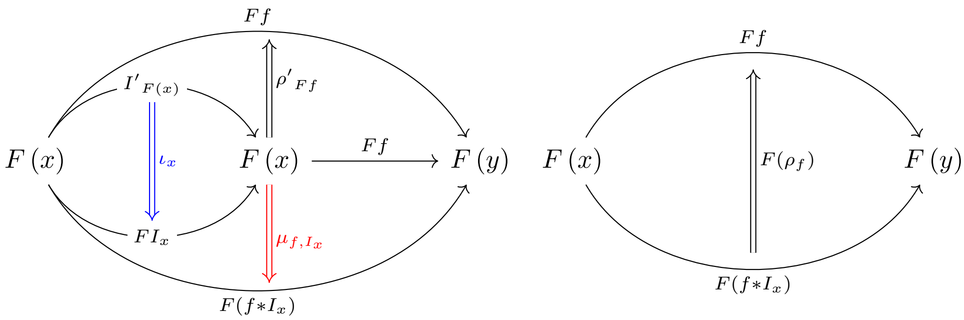
After considering the notion of a bicategory it is natural to ask what are the structure preserving maps between them. Functors between categories are given by a function on 0-cells (objects), and functions between hom-sets which preserve sources and targets and respect composition. Morphisms of bicategories should analogously be functions on 0-cells, and *functors* between hom-categories which respect *horizontal composition*. However, since there is another dimension available in bicategories, we may ask horizontal composition to only be respected up to a coherent enough 2-cell. The following 2-diagrams describe this notion for each of the coherences.

- For associator, consider the sphere formed by gluing together the following disks on their common 1-cells $F(h * (g * f))$ and $F((h * g) * f)$. The directed faces in red are what we need to define.

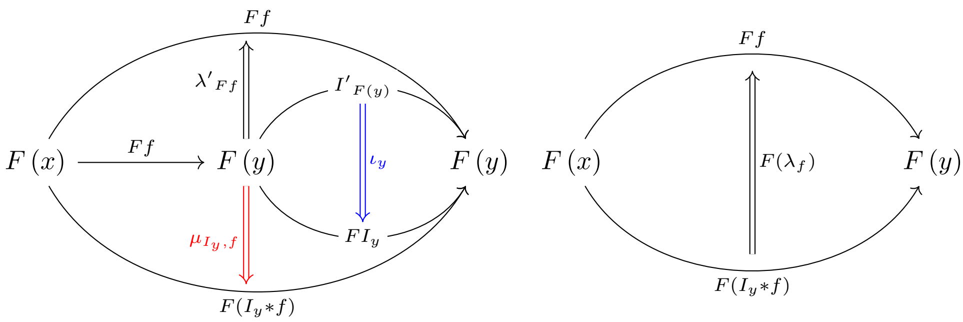


Note that there is a 'hidden' associator $\alpha_{Fh,Fg,Ff}$ going between the path of length 3 from 'head first' composition to 'tail first' composition. This appears because we are dealing with bicategories rather that strict 2-categories.

- For the right unitor, consider the sphere formed by gluing together the following disks on their common 1-cells Ff and $F(f * I_x)$. This time the directed face in blue is what we need to define, while the directed face in red is as we discussed above.



- For the left unitor, consider the sphere formed by gluing together the following disks on their common 1-cells Ff and $F(I_x * f)$. Note that the directed face in blue has the same source and target directed edges as that from the 2-diagrams considered for the right unitor.



To 'derive' the coherence axioms of a morphism of bicategories, one forms diagrams whose

- objects are given by paths along edges in the 2-diagrams above
- arrows are given by directed faces in the 2-diagrams above, including any 'hidden associators'.

Definition 36 (morphism of bicategories)**.** Given bicategories B, B', a *morphism of bicategories* $(F, \mu, \iota) : B \to B'$ consists of the following data subject to the following axioms.

DATA

- A function $F : B_0 \to B'_0$
- For every hom-category $B(x,y)$ in B, a functor $F_{x,y} : B(x,y) \to B(F(x), F(y))$ whose image on $f \in B(x,y)$ will be denoted by Ff
- For all $x, y, z \in B_0$, a natural transformation $\mu_{x,y,z}$ called the *multiplication*:

$$\begin{array}{ccc} B(y,z) \times B(x,y) & \xrightarrow{\quad * \quad} & B(x,z) \\ {\scriptstyle F_{y,z} \times F_{x,y}} \downarrow & \overset{\mu_{x,y,z}}{\Nearrow} & \downarrow {\scriptstyle F_{x,z}} \\ B'(F(y), F(z)) \times B'(F(x), F(y)) & \xrightarrow[*']{} & B'(F(x), F(z)) \end{array}$$

  with components $(\mu_{x,y,z})_{g \times f} := \mu_{g,f} : Fg * Ff \Rightarrow F(g * f)$ for $g \in B(y,z)$, $f \in B(x,y)$
- For all $x, \in B_0$, a natural transformation and ι_x called the *unit*:

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{\quad I_x \quad} & B(x,x) \\ & \searrow^{\iota_x}_{I'_{F(x)}} & \downarrow {\scriptstyle F_{x,x}} \\ & & B'(F(x), F(x)) \end{array}$$

  with a component $\iota_x : I_{F(x)} \Rightarrow F(I_x)$

AXIOMS:

For all 1-cells $w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$ in B, the following diagrams commute in B':

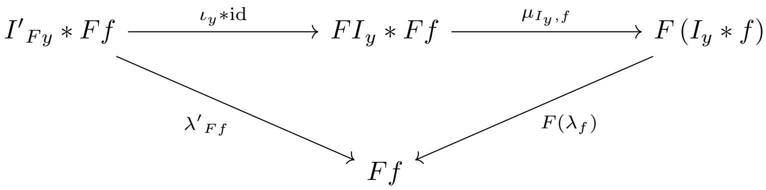
(hexagon axiom)

$$\begin{array}{ccccc} (Fh * Fg) * Ff & \xrightarrow{\mu_{h,g} * \mathrm{id}} & F(h * g) * Ff & \xrightarrow{\mu_{h*g,f}} & F((h * g) * f) \\ {\scriptstyle \alpha'_{Fh,Fg,Ff}} \downarrow & & & & \downarrow {\scriptstyle F(\alpha_{h,g,f})} \\ Fh * (Fg * Ff) & \xrightarrow[\mathrm{id} * \mu_{g,f}]{} & Fh * F(g * f) & \xrightarrow[\mu_{h,g*f}]{} & F(h * (g * f)) \end{array}$$

(right triangle axiom)

$$\begin{array}{ccccc} Ff * I'_{Fx} & \xrightarrow{\mathrm{id} * \iota_x} & Ff * FI_x & \xrightarrow{\mu_{f,I_x}} & F(f * I_x) \\ & {\scriptstyle \rho'_{Ff}} \searrow & & \swarrow {\scriptstyle F(\rho_f)} & \\ & & Ff & & \end{array}$$

(left triangle axiom)



If all $\iota_x, \mu_{x,y,z}$ are all natural isomorphisms then (F, μ, ι) is called a *homomorphism* of bicategories. If all $\iota_x, \mu_{x,y,z}$ are identity natural transformations, then (F, μ, ι) is called a *strict 2-functor*. If additionally to being strict we have that $B = B'$, $F(x) = x$ and $F_{x,y}$ is the identity functor on hom-categories for all $x, y \in B$ then F is called the *identity homomorphism* of B.

Indeed, the 2-diagrams discussed beforehand correspond exactly to the axioms in this definition.

Definition 37 (op-morphism)**.** An *op-morphism* (F, μ, ι) is defined similarly to morphisms of bicategories, except with the directions of the multiplication and unit reversed, and hence any arrows in the axioms involving them also reversed. These natural transformations will then be called *comultiplication* and *counit*.

Remark 38. Elsewhere in the literature, homomorphisms of bicategories are called *pseudofunctors*, morphisms of bicategories are called *lax-functors*, and op-morphisms are called *oplax functors*. Many authors denote the multiplication and unit of a morphism of bicategories with the same symbol despite their distinction as readers may distinguish between them based on their number of arguments. Note that the direction chosen for the multiplication and unit natural transformations are arbitrary. Finally, observe that if B and B' have one 0-cell each, then a morphism from B to B' whose function on 0-cells is invertible is just a monoidal functor.

Example 39. Let R be a commutative ring. There is a 2-functor from $F : R - \mathbf{Alg} \to \mathbf{Cat}$ that behaves in the following way:

- On 0-cells, every commutative R-algebra A is sent to the category $\mathbf{Mod} - A$ of right A modules
- The functor $F_{A,B} : (A, B) - \mathbf{Bimod} \to [\mathbf{Mod} - A, \mathbf{Mod} - B]$ sends
  - Every (A, B)-bimodule M to the functor $(-) \otimes_A M : \mathbf{Mod} - A \to \mathbf{Mod} - B$
  - Every (A, B)-bimodule homomorphism $f : M \to N$ to the natural transformation $(-) \otimes_A M \Rightarrow (-) \otimes_A N$ whose components on $L \in \mathbf{Mod} - A$ are given by the B-module homomorphism $L \otimes_A f : L \otimes_A M \to L \otimes_A N$.

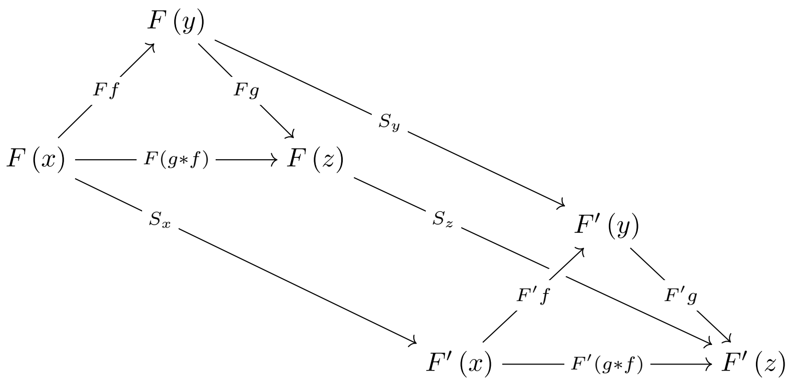
.

Example 40. Consider the adjunction of the forgetful functor from $U : \mathbf{Ab} \to \mathbf{Set}$ and the free functor $F : \mathbf{Set} \to \mathbf{Ab}$. Viewing $\mathbf{Ab}$ and $\mathbf{Set}$ as monoidal categories, there are canonical natural transformations given by components $\phi_{G,H} : U(G \otimes H) \to U(G) \times U(H)$ and $\psi_{X,Y} : F(X) \otimes F(Y) \to F(X \times Y)$ so that U is a lax-monoidal functor with multiplication ϕ and F is an oplax monoidal functor with multiplication ψ. Hence viewing these monoidal categories as bicategories with only one 0-cell, these constitute examples of a morphism and a op-morphism of bicategories respectively.

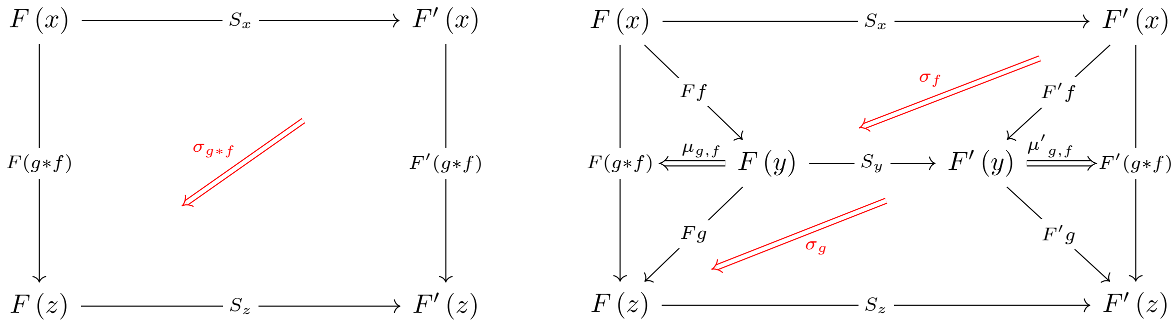
2.2 Transformations

Recall that for functors $F, G : C \to D$, we may have natural transformations $\phi : F \Rightarrow G$ given by component morphisms $\phi_X : F(X) \to G(X)$ which satisfy a commutativity axiom in D for the images under F and G of morphisms in C. A similar construction between morphisms of bicategories should involve component 1-cells between $F(x)$ and $G(x)$ such that the analogous naturality square commutes, but only up to a coherent 2-cell. Before giving a formal definition, we once again look at the 2-diagrams that exhibit what ‘coherent’ should mean in this context.

- For the associator, consider a triangular prism:

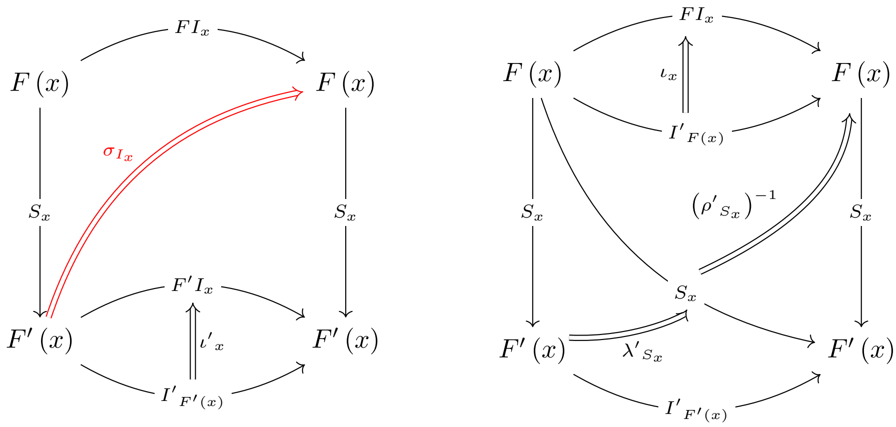


whose faces are given by the following two squares:

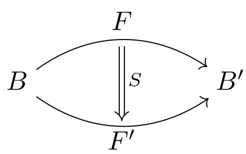


where the left square gives the bottom face and the right square gives all other faces. Note that similarly to the 2-diagram for the hexagon axiom for morphisms of bicategories, for every path of length 3 in the triangular prism there will also be a degenerate face given by an associator.

- For the unitors, consider the cylinder formed by gluing together the boundary edges in the following 2-diagrams:



Definition 41 (Transformation).



A *transformation* $(S, \sigma) : (F, \mu, \iota), (F', \mu', \iota') : B \to B'$ between morphisms of bicategories as shown above is given by the following data subject to the following axioms:

DATA

- For every 0-cell $x \in B_0$, a component 1-cell $S_x : F(x) \to F'(x)$ in B'
- For every pair of 0-cells $x, y \in B_0$, component natural transformations $\sigma_{x,y}$

$$\begin{array}{ccccc} & & B(x,y) & & \\ & \swarrow^{F'_{x,y}} & & \searrow^{F_{x,y}} & \\ B'(F'(x), F'(y)) & & \overset{\sigma_{x,y}}{\Longrightarrow} & & B'(F(x), F(y)) \\ & \searrow_{\mathrm{pre}(S_x)} & & \swarrow_{\mathrm{post}(S_y)} & \\ & & B'(F(x), F'(y)) & & \end{array}$$

where we write $\mathrm{pre}(S_x)$ for the functor induced by pre-composition by S_x, and similarly $\mathrm{post}(S_y)$ for the functor induced by post-composition by S_y. The components of $\sigma_{x,y}$ at $f \in B(x,y)$ are given by 2-cells $(\sigma_{x,y})_f := \sigma_f : F'f * S_x \Rightarrow S_y * Ff$ like so:

$$\begin{array}{ccc} F(x) & \xrightarrow{S_x} & F'(x) \\ {\scriptstyle Ff}\downarrow & \overset{\sigma_f}{\Swarrow} & \downarrow{\scriptstyle F'f} \\ F(y) & \xrightarrow[S_y]{} & F'(y) \end{array}$$

AXIOMS

(octagon axiom)

$$\begin{array}{ccccccc} F'g * (F'f * S_x) & \xrightarrow{\mathrm{id}*\sigma_f} & F'g * (S_y * Ff) & \xrightarrow{\left(\alpha'_{F'g,S_y,Ff}\right)^{-1}} & (F'g * S_y) * Ff & \xrightarrow{\sigma_g * \mathrm{id}} & (S_z * Fg) * Ff \\ {\scriptstyle \alpha'_{F'g,F'f,S_x}}\uparrow & & & & & & \downarrow{\scriptstyle \alpha'_{S_z,Fg,Ff}} \\ (F'g * F'f) * S_x & & & & & & S_z * (Fg * Ff) \\ {\scriptstyle \mu'_{g,f}*\mathrm{id}}\downarrow & & & & & & \downarrow{\scriptstyle \mathrm{id}*\mu_{g,f}} \\ F'(g*f) * S_x & & & \xrightarrow[\sigma_{g*f}]{} & & & S_z * F(g*f) \end{array}$$

(pentagon axiom)

$$\begin{array}{ccccc} I'_{F'x} * S_x & \xrightarrow{\lambda'_{S_x}} & S_x & \xrightarrow{\left(\rho'_{S_x}\right)^{-1}} & S_x * I'_{Fx} \\ {\scriptstyle \iota'_x*\mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}*\iota_x} \\ F'I_x * S_x & & \xrightarrow[\sigma_{I_x}]{} & & S_x * FI_x \end{array}$$

If all component natural transformations are natural isomorphisms then (S, σ) is called a *strong transformation*, while if they are all identities then (S, σ) is called a *strict transformation*. Lastly, if on top of this we have that $(F, \mu, \iota) = (F', \mu', \iota')$ and $S_x = I'_{F(x)}$ for all $x \in B$ then (S, σ) is called the *identity transformation* of (F, μ, ι)

Definition 42 (op-transformation)**.** An *op-transformation* is defined similarly to the above, except with the direction of the component natural transformation σ reversed. As with op-morphisms, any arrows in the axioms listed above involving the components of the transformation are also reversed. Additionally, the arrows given above by coherence isomorphisms α, λ and ρ are also inverted.

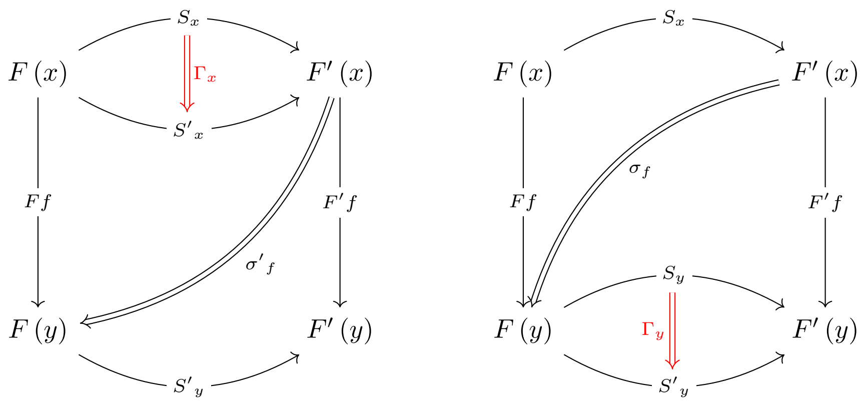
Remark 43. One can also define transformations between op-morphisms of bicategories, or op-transformations between morphisms, or indeed op-transformations between op-morphisms of bicategories by appropriately reversing arrows. We include the axioms in full in the appendix.

Remark 44. Elsewhere in the literature transformations as defined above are called *lax-natural transformations*, and strong transformations as above are called *pseudonatural transformations*. Strict transformations are sometimes called 2-natural transformations, although that term is usually reserved for the case when all bicategories and morphisms of bicategories involved are also strict.

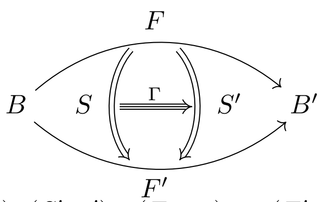
The choice of direction for component natural transformations, and hence the choice of which versions should be called 'op', is arbitrary for both morphisms of bicategories and for transformations between them. It may be argued that the choice has been made incorrectly for transformations, since they specialise to oplax monoidal transformations in the case of bicategories with one 0-cell. The opposite choice of direction is sometimes made for this reason, most notably in [10].

2.3 Modifications

Remark 45. Taking component 2-cells that go between the component 1-cells of two transformations between the same morphisms of bicategories, we can form structure preserving maps between those transformations as well. This type of structure preserving map is unavailable between ordinary natural transformations, as categories do not have any such 2-cells. Once again, to motivate the axiom in the upcoming formalism, we first consider the 2-diagram formed by attaching the boundary edges of the 2-diagrams below.



Definition 46 (modification).



Given transformations $(S,\sigma),(S',\sigma') : (F,\mu,\iota) \Rightarrow (F',\mu',\iota')$ between morphisms of bicategories as above, a *modification* $\Gamma : (S,\sigma) \Rrightarrow (S',\sigma')$ is given by the following data subject to the following axioms.

- DATA: For all $x \in B_0$, a component 2-cell $\Gamma_x : S_x \Rightarrow S'_x$
- AXIOM: The following diagram, which we call the *naturality square for modifications*, commutes:

$$\begin{array}{ccc} F'f * S_x & \xrightarrow{\mathrm{id}*\Gamma_x} & F'f * S'_x \\ \Big\downarrow{\scriptstyle \sigma_f} & & \Big\downarrow{\scriptstyle \sigma'_f} \\ S_y * Ff & \xrightarrow[\Gamma_y * \mathrm{id}]{} & S'_y * Ff \end{array}$$

If all Γ_x are invertible 2-cells then Γ is itself called invertible, and if $(S, \sigma) = (S', \sigma')$ and $\Gamma_x = \mathrm{id}_{S_x}$ then Γ is called the *identity modification* on (S, σ), which we will write as id_S.

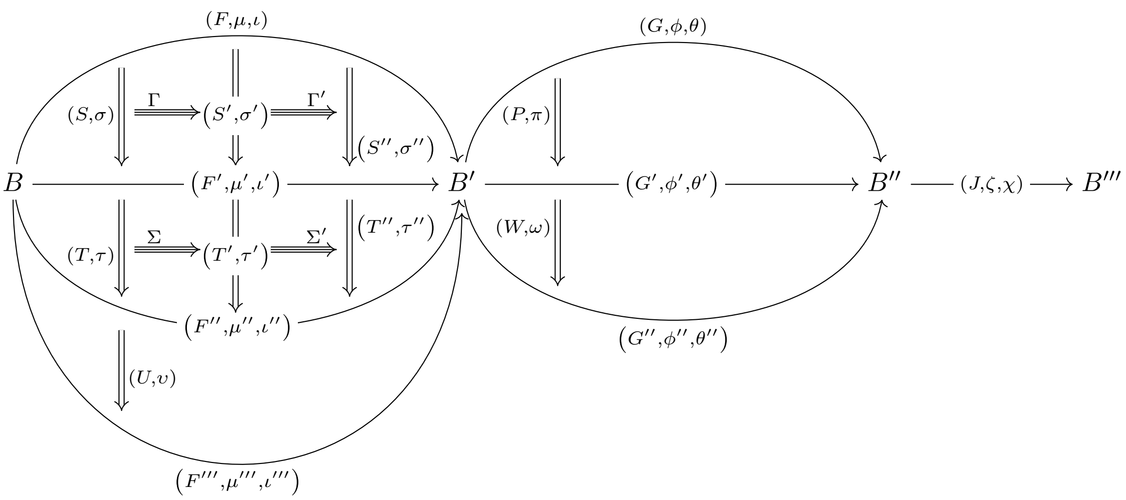
The 2-diagrams shown in the preceding remark will be referred to as a *naturality cylinder* for the modification Γ.

2.4 The Compositional Structure of (2, k)-Transfors

We now give an account of the compositional structure on the transfors of a bicategory. References are available for morphisms of bicategories, and for modifications, but diagrams involved in the proof of the result for transformations are quite large and difficult to find in the literature.

Notation 47. *For the next few pages we refer to the situation below in which*

- *B, B', B'', B''' are bicategories.*
- *Arrows are morphisms between respective bicategories.*
- *Double arrows are transformations between respective morphisms of bicategories.*
- *Triple arrows are modifications between respective transformations.*
- *All of the above are assumed to be arbitrary unless otherwise specified.*

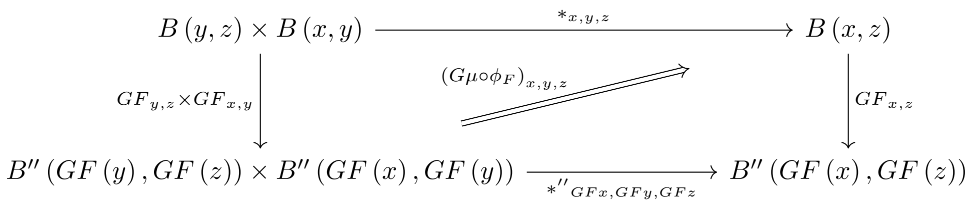


Definition 48 (composite of morphisms of bicategories)**.** Let the *composite* of (F, μ, ι) and (G, ϕ, θ), denoted $(GF, G\mu \circ \phi_F, G\iota \circ \theta_F) : B \to B''$ be given by

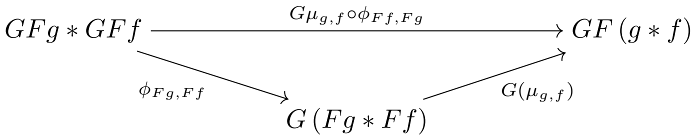
- $GF = G \circ F : B_0 \to B''_0$ the function on 0-cells
- $(GF)_{x,y} = G_{Fx,Fy} \circ F_{x,y}$ for the functors:

$$\begin{array}{ccccc} & & B'(F(x), F(y)) & & \\ & \nearrow^{F_{x,y}} & & \searrow^{G_{Fx,Fy}} & \\ B(x,y) & & \xrightarrow[(GF)_{x,y}]{\hspace{6em}} & & B''(GF(x), GF(y)) \end{array}$$

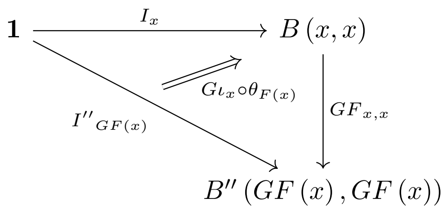
- For every $x, y, z \in B_0$, the natural transformation $G\mu \circ \phi_F$.



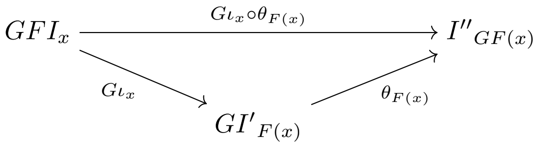
given by components $G\mu_{g,f} \circ \phi_{Fg,Ff}$



- For every $x \in B_0$, the natural transformation $G\iota \circ \theta_F$:



given by $G\iota_x \circ \theta_{F(x)}$:



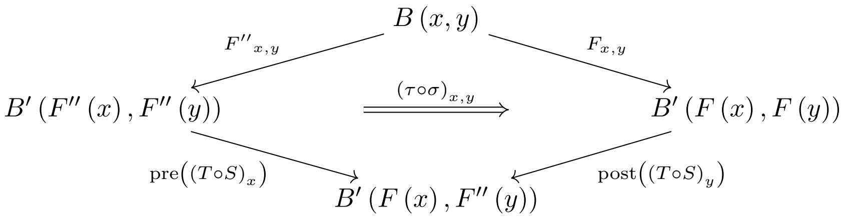
Proposition 49. *The composite of morphisms as defined above is itself a morphism of bicategories.*

Corollary 50. *There is a category* $\mathbf{Bicat_1}$ *whose objects are bicategories and morphisms are morphisms of bicategories.*

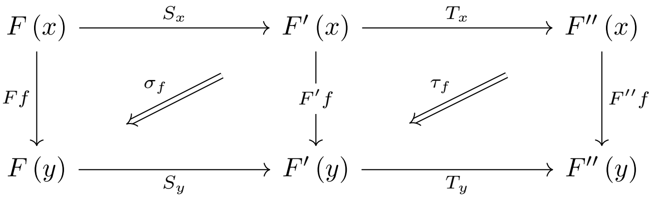
The proofs of the above results were originally given in [6]. We include them in the appendix for the sake of completeness. The intuition behind these proofs can be seen by considering the 2-diagrams: for example, the 2-diagrams for the right triangle axiom for (G, ϕ, θ) shares the common face $G(\rho_f)$ with the image under (G, ϕ, θ) of the 2-diagram for the right unitor axiom for (F, μ, ι). Note that if both the multiplications μ and ϕ are either natural isomorphisms or identities respectively, then so is the multiplication of the composite morphism of bicategories. The same holds for units, and hence the composition of homomorphisms of bicategories is also a homomorphism of bicategories, and the composition of strict 2-functors is also a strict 2-functor. Alternatively, if (F, μ, ι) and (G, ϕ, θ) were op-morphisms of bicategories instead, then the above construction can be changed to $\phi_F \circ G(\mu)$ for the multiplications and $\theta_F \circ G(\iota)$ for the units. Then the proofs for the above results with the direction of these arrows changed remains valid, and we also have a category $\mathbf{OpBicat_1}$ whose morphisms are op-morphisms of bicategories.

Definition 51 (vertical composite of transformations)**.** The *vertical composite* of (T, τ) and (S, σ), which we denote $(T \circ S, \tau \circ \sigma)$ is given by the following data:

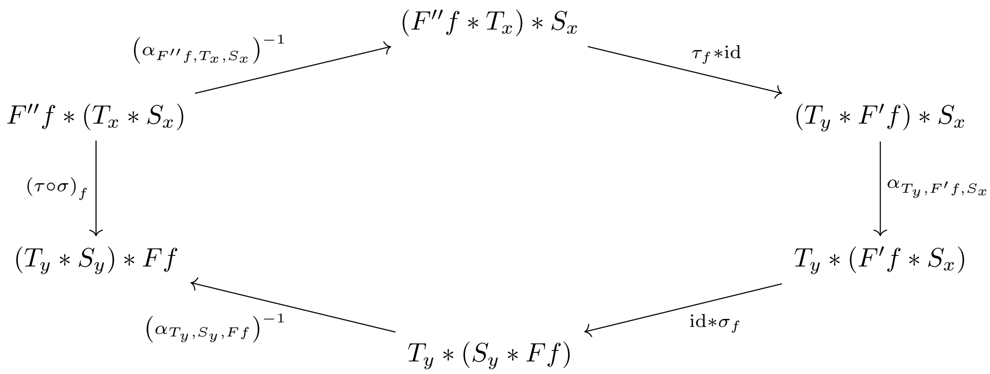
- For any $x \in B$, $(T \circ S)_x := T_x * S_x$
- For any $x, y \in B$, the natural transformation



whose components $(\tau \circ \sigma)_f$ on $f \in B(x,y)$ are given by the pasting of 2-diagrams:

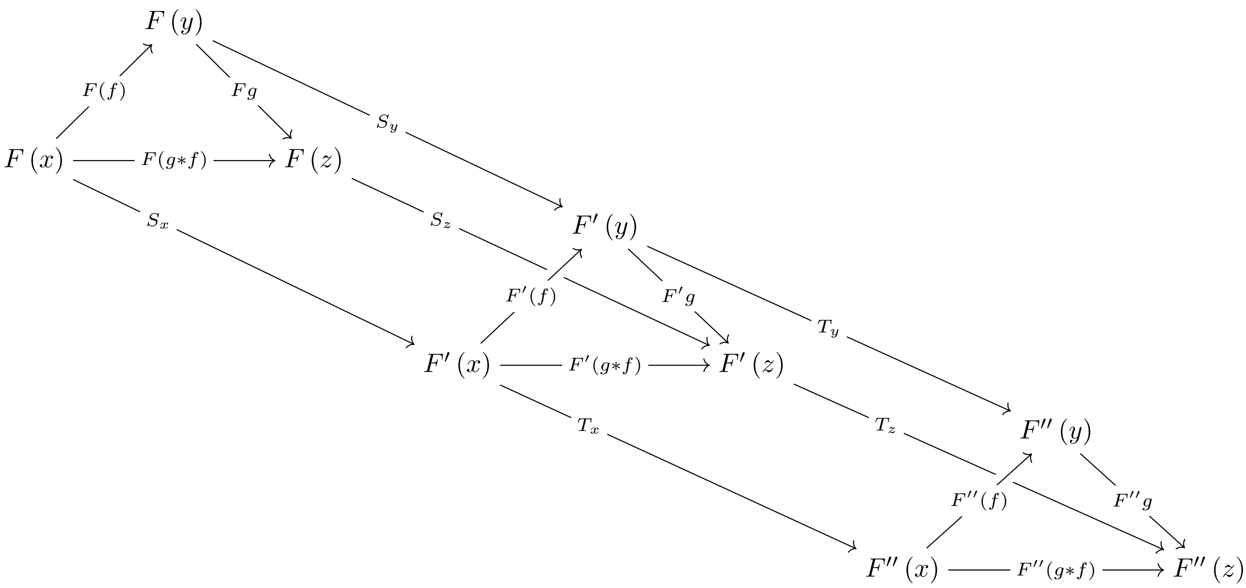


That is, they are given by the following composition of 2-cells in B'.

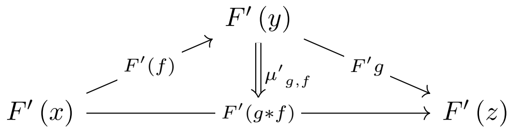


Proposition 52. *The vertical composite of transformations as defined above is itself a transformation.*

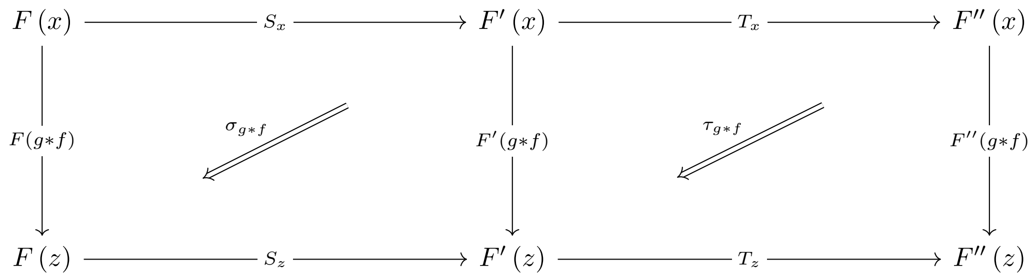
Remark 53. Before proving Proposition 52, we develop some intuition with the help of 2-diagrams similar to those provided in [2], which deals with the case of strong transformations between homomorphisms of bicategories from B^{op} to **Cat**. Indeed, the triangular prism for $T \circ S$ will be given by:



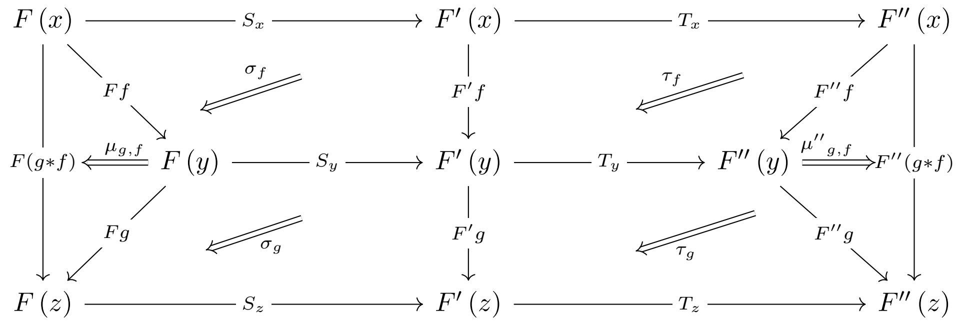
It may be formed by gluing together the corresponding triangular prisms for (S, σ) and (T, τ) on the common face:



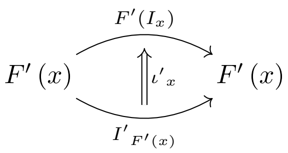
Its bottom face is given by the 2-diagram:



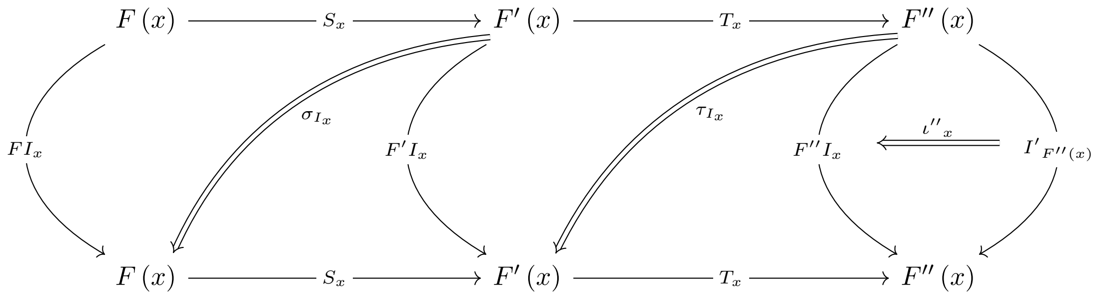
and all other faces are given by the 2-diagrams:

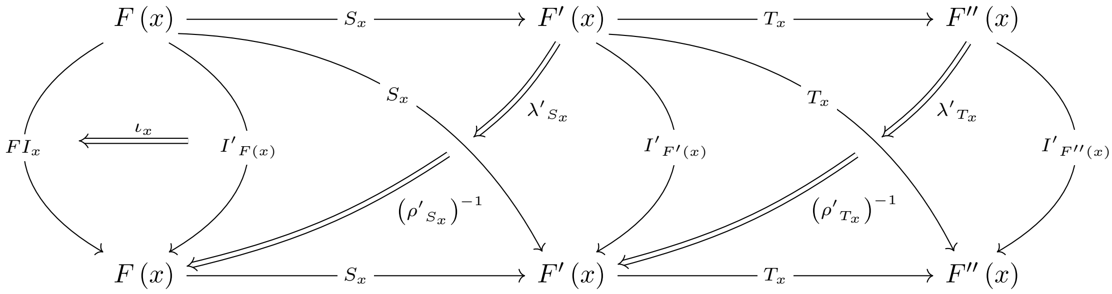


Similarly, the cylinder for the pentagon axiom for $T \circ S$ is given by gluing together the cylinders for (S, σ) and (T, τ) on their common face:



That is, it is formed by attaching the following 2-diagrams along their boundary edges:

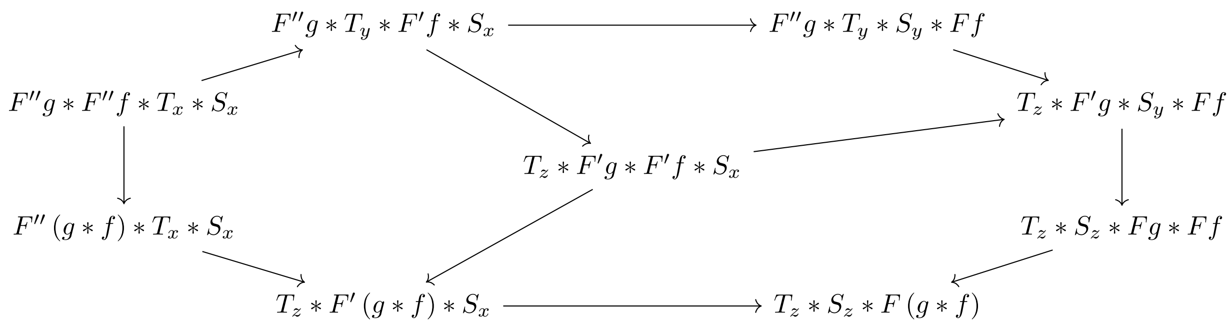




As previously mentioned, this constitutes a valid proof only if B' is strict. However, it is useful for intuition as we may apply a procedure similar to how we constructed the axioms above to translate this 2-diagram construction into a proof:

1. Consider a directed graph whose
   - vertices are non-parenthesised expressions denoting paths along edges in the above 2-diagrams
   - edges are given by the directed faces in the above 2-diagrams.
2. For each vertex in this graph of length k, insert the associahedron K_k. Note that the only non-trivial associahedra this will produce will be a path of length 3 for $k = 3$, and a pentagon for $k = 4$,
3. Join the edges going into or coming out of the vertices which have been expanded in the previous step to the appropriate new vertices in the associahedra.
4. Fill in any edges given by definitions, and reverse the directions of any isomorphisms as necessary.

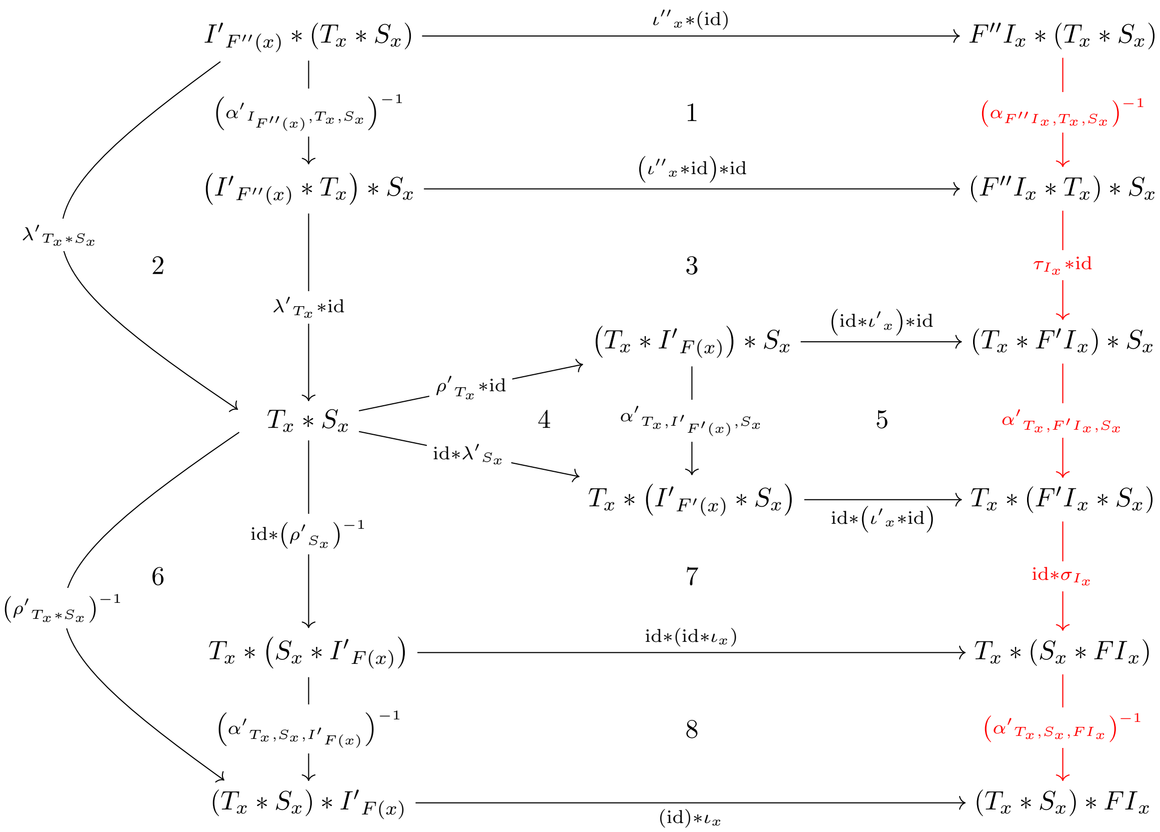
Proof. For the octagon axiom, we form the following directed graph as described in the remark above.



Into each vertex we may substitute a path of length 3, or an associativity pentagon, with arrows directed appropriately so as to commute with the rest of the diagram. Each edge not out of or into an expression of length 3 in this graph will correspond to at most two 2-cells between one of the possible ways of parenthesising the expressions in its source and target, and these will expand to give naturality squares of associators. The two pentagonal faces of the graph above will expand to give octagon axioms for (S, σ) and (T, τ), while the square above will commute due to naturality of σ, or equivalently, of τ. We give the full diagram and a proof that it commutes in the appendix.

Playing the same game with the pentagon axiom, we eventually see that it is given by the diagram below. In it, we see that

- The path in red defines $(\tau \circ \sigma)_{I_x}$,
- 1, 5 and 8 commute by naturality of α'.
- 4 is the coherence axiom for unitors in B' on the 1-cell S_x.
- 3 is the pentagon axiom for (T, τ) horizontally pre-composed with S_x.
- 7 is the pentagon axiom for (S, σ) horizontally post-composed with T_x.
- 2 and 6 commute by Proposition 6.



With the proof for the octagon axiom in the appendix, this completes the proof that $(T \circ S, \tau \circ \sigma)$ is a transformation, as required.

□

Corollary 54. *There is an invertible modification*
$A_{U,T,S} : ((U, \upsilon) \circ (T, \tau)) \circ (S, \sigma) \Rrightarrow (U, \upsilon) \circ ((T, \tau) \circ (S, \sigma))$
whose component on $x \in B$ is given by α'_{U_x, T_x, S_x}. This modification is an identity modification if and only if B' is strict.

Proof. The naturality square for the $A_{U,T,S}$ commutes as it is just the naturality square for α'. The modification is invertible because α is invertible. The last part of the corollary is just the definition of strictness in a bicategory. □

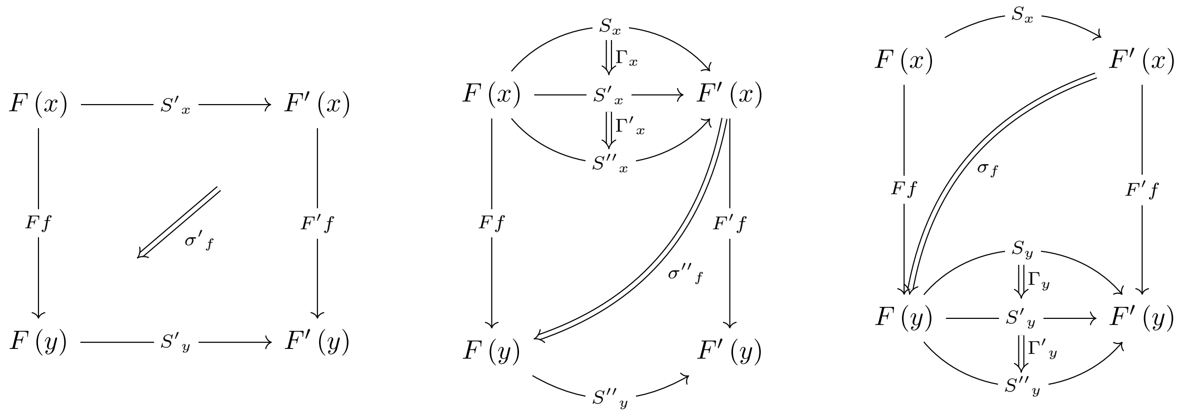
The next step in an investigation of the compositional structure of the transfors of a bicategory would be to describe a 'horizontal composition' of (P, π) and (S, σ) which one may suspect would behave similarly to horizontal composition of natural transformations between functors that go between ordinary categories. For reasons that will become clear later, we postpone this to the end of chapter four.

Definition 55 (vertical and horizontal composites of modifications)**.** For $x \in B$

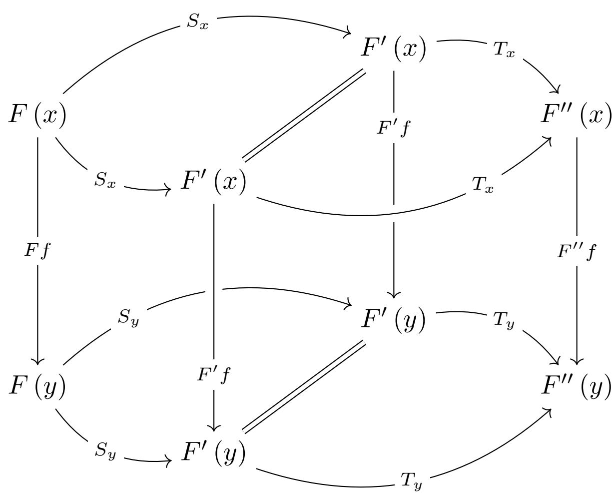
- The *vertical composite of modifications* $\Gamma' \circ \Gamma : (S, \sigma) \Rrightarrow (S'', \sigma'')$ is given by vertical composition of component 2-cells $(\Gamma' \circ \Gamma)_x = \Gamma'_x \circ \Gamma_x$
- The *horizontal composite of modifications* $\Sigma * \Gamma : (T, \tau) \circ (S, \sigma) \Rrightarrow (T', \tau) \circ (S', \sigma')$ is given by horizontal composition of component 2-cells $(\Sigma * \Gamma)_x = \Sigma_x * \Gamma_x$

Proposition 56. *The vertical and horizontal composites of modifications as defined above are themselves modifications.*

Remark 57. Before giving the proof, we develop some intuition by looking at pastings of the 2-diagrams of the naturality cylinders for the modifications. For the vertical composition, we may paste the naturality cylinders for Γ and Γ' along their common face shown on the left below, to obtain the naturality cylinder for $\Gamma' \circ \Gamma$ whose 'faces' are shown on the middle and right below.



Similarly, for the horizontal composition, adjoining the naturality cylinders for Γ and Σ along their common edge $F'f$, and then pulling the cylinders apart to obtain the naturality cylinder for $\Sigma * \Gamma$



Of course, once again, these constructions are not themselves proofs, but can be unfolded using the same method as for the vertical composite of transformations.

Proof. That the vertical composite of modifications is itself a modification follows from the commutativity of the following diagram of component 2-cells.

$$
\begin{array}{ccccc}
F'f * S_x & \xrightarrow{\mathrm{id}*\Gamma_x} & F'f * S'_x & \xrightarrow{\mathrm{id}*\Gamma'_x} & F'f * S''_x \\
\downarrow{\scriptstyle \sigma_f} & & \downarrow{\scriptstyle \sigma'_f} & & \downarrow{\scriptstyle \sigma''_f} \\
S_y * Ff & \xrightarrow[\Gamma_y*\mathrm{id}]{} & S'_y * Ff & \xrightarrow[\Gamma'_y*\mathrm{id}]{} & S''_y * Ff
\end{array}
$$

where the paths in red define $\Gamma' \circ \Gamma$, the left square is the naturality square for Γ', the right square is the naturality square for Γ, and the boundary square is the naturality square for $\Gamma' \circ \Gamma$.

For horizontal composition, we consider the following diagram.

$$
\begin{array}{ccccc}
F''f * (T_x * S_x) & \xrightarrow{\mathrm{id}*(\Sigma_x*\mathrm{id})} & F''f * (T'_x * S_x) & \xrightarrow{\mathrm{id}*(\mathrm{id}*\Gamma_x)} & F''f * (T'_x * S'_x) \\
\downarrow{\scriptstyle (\alpha'_{F''f,T_x,S_x})^{-1}} & & \downarrow{\scriptstyle (\alpha'_{F''f,T'_x,S_x})^{-1}} & & \downarrow{\scriptstyle (\alpha'_{F''f,T'_x,S'_x})^{-1}} \\
(F''f * T_x) * S_x & \xrightarrow{(\mathrm{id}*\Sigma_x)*\mathrm{id}} & (F''f * T'_x) * S_x & \xrightarrow{(\mathrm{id}*\mathrm{id})*\Gamma_x} & (F''f * T'_x) * S'_x \\
\downarrow{\scriptstyle \tau_f*\mathrm{id}} & & \downarrow{\scriptstyle \tau'_f*\mathrm{id}} & & \downarrow{\scriptstyle \tau'_f*\mathrm{id}} \\
(T_y * F'f) * S_x & \xrightarrow{(\Sigma_y*\mathrm{id})*\mathrm{id}} & (T'_y * F'f) * S_x & \xrightarrow{(\mathrm{id}*\mathrm{id})*\Gamma_x} & (T'_y * F'f) * S'_x \\
\downarrow{\scriptstyle \alpha'_{T_y,F'f,S_x}} & & \downarrow{\scriptstyle \alpha'_{T'_y,F'f,S_x}} & & \downarrow{\scriptstyle \alpha'_{T'_y,F'f,S'_x}} \\
T_y * (F'f * S_x) & \xrightarrow{\Sigma_y*(\mathrm{id}*\mathrm{id})} & T'_y * (F'f * S_x) & \xrightarrow{\mathrm{id}*(\mathrm{id}*\Gamma_x)} & T'_y * (F'f * S'_x) \\
\downarrow{\scriptstyle \mathrm{id}*\sigma_f} & & \downarrow{\scriptstyle \mathrm{id}*\sigma_f} & & \downarrow{\scriptstyle \mathrm{id}*\sigma'_f} \\
T_y * (S_y * Ff) & \xrightarrow{\Sigma_y*(\mathrm{id}*\mathrm{id})} & T'_y * (S_y * Ff) & \xrightarrow{\mathrm{id}*(\Gamma_y*\mathrm{id})} & T'_y * (S'_y * Ff) \\
\downarrow{\scriptstyle (\alpha'_{T_y,S_y,Ff})^{-1}} & & \downarrow{\scriptstyle (\alpha'_{F''f,S_y,T'_y})^{-1}} & & \downarrow{\scriptstyle (\alpha'_{T'_y,S'_y,Ff})^{-1}} \\
(T_y * S_y) * Ff & \xrightarrow[(\Sigma_y*\mathrm{id})*\mathrm{id}]{} & (T'_y * S_y) * Ff & \xrightarrow[(\mathrm{id}*\Gamma_y)*\mathrm{id}]{} & F''f * (T'_y * S'_y) * Ff
\end{array}
$$

To complete the proof, we make the following observations about this diagram

- The paths in red define the component of $\Sigma * \Gamma$ on x and y, and the paths in blue define $\tau \circ \sigma$ and $\tau'' \circ \sigma''$, so that the boundary gives the naturality square for modifications.
- The first, third, and fifth rows of squares commute by naturality of α'.
- The first square in the second row is the naturality square for modifications for Σ, horizontally pre-composed with S_x.
- The second square in the second row defines ${\tau'}_f * \Gamma_x$.
- The first square in the fourth row defines $\Sigma_y * \sigma_f$.
- The second square in the fourth row is the naturality square for modifications for Γ, horizontally post-composed with T'_x.

$\square$

Corollary 58. *The vertical and horizontal compositions of modifications as described above obey the binary interchange law* $(\Sigma' \circ \Sigma) * (\Gamma' \circ \Gamma) = (\Sigma' * \Gamma') \circ (\Sigma * \Gamma)$.

Proof. Compositions of modifications are defined component-wise on their respective 2-cells, so this follows from the binary interchange law for 2-cells in a bicategory. ☐

3 Bicategorical Yoneda Lemma and the Coherence Theorem

The Yoneda Lemma is a central result of category theory with many diverse applications across mathematics. In this chapter we will follow a similar route to proving the Yoneda Lemma for Bicategories as in [18] and [23]. We will then apply this to the proof of the strictness and coherence theorems, and show how they greatly simplify the theory of bicategories.

3.1 Morphism Bicategories

Recall that for categories C and D there is a category $[C, D]$ whose objects are functors $F : C \to D$ and morphisms are natural transformations $\phi : F \Rightarrow G : C \to D$. An analogous result for bicategories is essentially a summary of the results of the previous chapter. To prove it, we will need the following lemma.

Lemma 59. *For* $(F, \mu, \iota), (G, \nu, \theta) : B \to B'$ *morphisms of bicategories, there is a category which we denote* $[B, B'](F, G)$ *with*

- *objects given by transformations* $(F, \mu, \iota) \Rightarrow (G, \nu, \theta)$
- *morphisms given by modifications between these transformations.*
- *composition of morphisms is given by vertical composition of modifications*

Proof. For a transformation $(S, \sigma) : (F, \mu, \iota) \Rightarrow (G, \nu, \theta)$ its identity will be its corresponding identity modification. Indeed, the vertical composition of modifications has these modifications as its identities and is associative as it is simply given by composition of component morphisms in their respective hom-categories. ☐

Theorem 60. *Given bicategories* B, B' *there is a bicategory* $[B, B']$ *called the* morphism bicategory*, which has*

- 0*-cells morphisms from* B *to* B'
- *Hom-categories are* $[B, B']((F, \mu, \iota), (G, \nu, \theta))$
- *Horizontal composition is given by*
  - *Vertical composition of transformations on objects*
  - *Horizontal composition of modifications on morphisms*
- *Identities are given by identity transformations and their identity modifications.*
- *Unitors are given by unitors in* B'
- *Associator is given by the associator modification from corollary 54.*

Proof. The horizontal composition as defined above is a bifunctor as modifications obey the binary interchange law.

The triangle and pentagon axioms for the unitors and associators in $[B, B']$ follow respectively from those in B'. This completes the proof. ☐

Remark 61. Note that $[B, B']$ has a locally full sub-bicategory whose 0-cells are homomorphisms and 1-cells are strong transformations. We call this the *homomorphism bicategory* and denote it by $[B, B']_h$. Note further that if B' is strict then $[B, B']_h$ is also strict, since the coherence isomorphisms of B' are identities.

For the remainder of this chapter, all morphisms of bicategories will be homomorphisms unless otherwise specified. We relax our notation on the transfors of bicategories, and only notate components as necessary.

Definition 62 (Biequivalence)**.** Bicategories, B, B' are *biequivalent* if $F : B \to B'$, $G : B' \to B$ are homomorphisms with $\mathrm{id}_B \to GF$ an equivalence inside $[B, B]$ and $\mathrm{id}_{B'} \to GF$ an equivalence inside $[B', B']$.

Remark 63. Recall that a functor is part of an equivalence of categories if and only if it is full, faithful and essentially surjective. Analogously, a morphism $(F, \mu, \iota) : B \to B'$ is part of a biequivalence if and only if it is a homomorphism which is surjective up to equivalence on 0-cells, and $F_{x,y}$ is an equivalence of categories for all x, $y \in B_0$.

Recall that a presheaf of a category C is a functor $F : C^{op} \to \mathbf{Set}$. We have the following analogous definition for bicategories.

Definition 64 (2-presheaf)**.** Let B be a bicategory. A 2-*presheaf* on B is a homomorphism $F : B^{op} \to \mathbf{Cat}$ which maps

- 0-cells $x \in B$ to categories $F(x)$
- 1-cells $f : x \to y$ to functors $Ff : F(x) \to F(y)$
- 2-cells $\phi : f \Rightarrow g$ to natural transformations $F(\phi) : Ff \Rightarrow Fg$

Call $[B^{op}, \mathbf{Cat}]_h$ the 2-category of 2-presheaves of B.

Recall that every category has a presheaf given by the contravariant hom-functor. Similarly, we have the following example of a 2-presheaf.

Example 65. Given a bicategory B, for any 0-cell $z \in B_0$ there is a 2-presheaf $B(-, z) : B^{op} \to \mathbf{Cat}$ defined by

- A map that sends 0-cells $x \in B_0$ to the category $B(x, z)$
- For all 0-cells $x, y \in B_0$, a functor $B_{x,y}(-, z) : B(x, y) \to \mathbf{Cat}(B(y, z), B(x, z))$ such that
  - For 1-cells $f \in B(x, y)$, $B(-, z)(f) = B(f, z) : B(y, z) \to B(x, z)$ is a functor which maps $(g : y \to z) \mapsto (g * f : x \to z)$ and $(\phi : g \Rightarrow g' : y \to z) \mapsto \phi * \mathrm{id}_f$.
  - For 2-cells $\psi : f \Rightarrow f' : x \to y$, $B(-, z)(\psi) = B(\psi, z) : B(f, z) \Rightarrow B(f', z)$ a natural transformation with components at $g : y \to z$ given by 2-cells $B(\psi, z)_g = g * \psi$

Note that the naturality squares for $B(\psi, z)$

$$\begin{array}{ccc} B(f,z)(g) & \xrightarrow{B(f,z)(\phi)} & B(f,z)(g') \\ {\scriptstyle g*\psi}\downarrow & & \downarrow{\scriptstyle g'*\psi} \\ B(f',z)(g) & \xrightarrow[B(f',z)(\phi)]{} & B(f',z)(g') \end{array}$$

commute as a result of the binary interchange law. Such a presheaf is called *representable.* Note that every representable $B(-, y) : B^{op} \to \mathbf{Cat}$ has an opposite representable $B(y, -) : B \to \mathbf{Cat}$.

Notation 66. *Given a homomorphism $F : B \to \mathbf{Cat}$, we write $[B, \mathbf{Cat}](B(x, -), F)$ for the category of transformations from $B(x, -)$ to F, and similarly $[B^{op}, \mathbf{Cat}](B(-, x), F)$.*

Definition 67. Given two representable homomorphisms $B(-,x)$ and $B(-,y)$, for any 1-cell $g : x \to y$ in B_1 denote by $g * (-)$ the collection of

- functors $B(v,-)$ indexed by $v \in B_0$, and
- natural isomorphisms $\phi_f : B(w,-)\, B(f,x) \Rightarrow B(f,y)\, B(v,-)$, indexed by $f : v \to w$, where we have written composition of functors by juxtaposition as usual.

Remark 68. The construction for $g*(-)$ above constitutes a strong transformation from $B(-,x)$ to $B(-,y)$.

3.2 Bicategorical Yoneda Lemma

We recall the Yoneda embedding theorem before stating and proving the analogous result for bicategories.

Theorem 69 (Yoneda Embedding)**.** *Let C be a locally small category. Define the functor $H : C \to [C^{op}, \mathbf{Set}]$ so that*

- $H(y) = Hom(-,y)$ *on objects* $y \in C$
- $H(f) = Hom(f,y)$ *on morphisms* $f \in Hom_C(w,x)$

where

- $Hom(-,y) : Ob(C) \to \mathbf{Set}$ *a function sending* $x \mapsto Hom_C(x,y)$
- $Hom(f,y) : Hom_C(x,y) \to Hom_C(w,y)$*, a function on hom-sets sends* $g \mapsto g \circ f$

Then H is fully faithful.

Definition 70 (Yoneda Embedding)**.** For B a locally small bicategory, define its *Yoneda Embedding* (Y, ϕ, ι) as

- A function on 0-cells $Y : B_0 \to ([B^{op}, \mathbf{Cat}]_h)_0$ which maps $y \mapsto B(-,y)$
- For every $x, y \in B_0$, a functor $Y_{x,y} : B(x,y) \to [B^{op}, \mathbf{Cat}]_h (B(-,x), B(-,y))$ which maps $(j : x \to y) \mapsto j * (-) : B(-,x) \Rightarrow B(-,y)$.
- For every $g : w \to x$, the natural isomorphism $(\rho_g)^{-1} : I_{Y(x)} \to Y I_x$
- Given 1-cells $v \xrightarrow{f} w \xrightarrow{g} x \xrightarrow{j} y \xrightarrow{k} z$, the natural isomorphism $(\alpha_{k,j,g})^{-1} * \mathrm{id}_f$.

Theorem 71 (Yoneda Embedding Theorem for bicategories)**.** *The Yoneda Embedding as defined above is a homomorphism of bicategories.*

Proof. We construct for all $x, y, z \in B_0$ a natural isomorphism $\phi_{x,y,z}$ as in the 2-diagram

$$\begin{array}{ccc} B(y,z) \times B(x,y) & \xrightarrow{\quad *_{x,y,z} \quad} & B(x,z) \\ \Big\downarrow {\scriptstyle Y_{y,z} \times Y_{x,y}} & \overset{\phi_{x,y,z}}{\Rrightarrow} & \Big\downarrow {\scriptstyle Y_{x,z}} \\ [B^{op}, \mathbf{Cat}]_h (B(-,y), B(-,z)) \times [B^{op}, \mathbf{Cat}]_h (B(-,x), B(-,y)) & \xrightarrow[{*'}_{x,y,z}]{} & [B^{op}, \mathbf{Cat}]_h (B(-,x), B(-,z)) \end{array}$$

whose components $(\phi_{x,y,x})_{h,g} : (P,\pi) \Rrightarrow (Q,\varrho)$ are modifications between transformations formed by traversing the square anticlockwise and clockwise, respectively.

$$\begin{array}{ccc} k \times j & & \\ \downarrow & & \\ (k*(-)) \times (j*(-)) & \longmapsto & k*(j*(-)) := (P,\pi) \end{array} \qquad \begin{array}{ccc} k \times j & \longmapsto & k*j \\ & & \downarrow \\ & & (k*j)*(-) := (Q,\varrho) \end{array}$$

Indeed, we first observe that the component 1-cells of these transformations on $w \in B_0$ are functors $P_w, Q_w : B(w,x) \to B(w,z)$, defined by

- $P_w(g') = k*(j*g')$ on objects and $P_w(\psi) = (\mathrm{id}_k * (\mathrm{id}_j * \psi))$ on morphisms.
- $Q_w(g') = (k*j)*g'$ on objects and $Q_w(\psi) = \mathrm{id}_{k*j} * \psi$ on morphisms.

Next, we see that the component 2-cells of these transformations on 1-cells $f : v \to w$ may be given by associators $\pi_f := \mathrm{id}_k * (\alpha_{j,g,f})^{-1} : k*(j*(g*f)) \Rightarrow k*((j*g)*f)$ and $\varrho_f : (\alpha_{k*j,g,f})^{-1} : (k*j)*(g*f) \to ((k*j)*g)*f$.

So the components $\phi_w := (\phi_{k,j})_w$ of the modifications are given by natural transformations between P_w and Q_w, whose components on $g : w \to x$ are given by the associators $(\phi_w)_g = (\alpha_{k,j,g})^{-1} * \mathrm{id}_f$ from the definition of the Yoneda embedding.

As $[B, \mathbf{Cat}]$ inherits composition from B we see that ϕ_w is a natural isomorphism. The octagon and pentagon axioms for the strong transformations commute by associativity coherence, as do the naturality squares for the modifications and the hexagon axiom for Y. The left and right triangle axioms for Y are similarly automatic from the triangle axiom in B. Hence Y is a homomorphism as required. $\square$

Next, we recall the Yoneda Lemma in full generality for ordinary categories, and prove an analogous result for bicategories.

Theorem 72 (Yoneda's Lemma). *Let C be a locally small category, and $F : C \to \mathbf{Set}$ a functor. Then there is a bijection $E : Nat(Hom(x,-), F) \to F(x)$ given by evaluation at the identity of each object $E(\phi) = \phi_x(id_x)$, and this bijection is natural in F and x.*

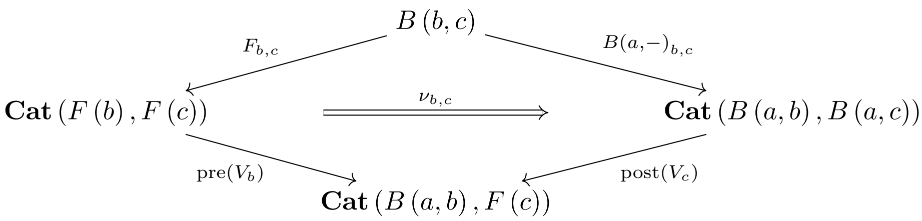
Notation 73. *Let $B(?,-) : B \to [B, \mathbf{Cat}]_h$ be the homomorphism which sends $x \in B$ to $B(x,-)$*

Definition 74 (Yoneda Equivalence)**.** Let B be a locally small bicategory, and $(F,\mu,\iota) : B \to \mathbf{Cat}$ a homomorphism. We define the *Yoneda Equivalence* to be the pair (S,T) which for $a \in B_0$ is given by the following table:

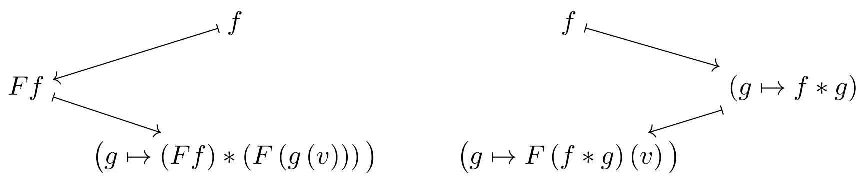
$S_a : [B, \mathbf{Cat}]_h (B(a,-), F) \to F(a)$	$T_a : F(a) \to [B, \mathbf{Cat}]_h (B(a,-), F)$
For $(P, \pi) : B(a,-) \Rightarrow (F, \mu, \iota)$ $$P \longmapsto \hat{P} = P_a(I_a)$$	For a 0-cell v in the image of (F, μ, ι) $$v \longmapsto \left(B(a,-) \overset{(V,\nu)}{\Longrightarrow} F \right)$$ with $V_b : B(a,b) \to b$ $f \mapsto Ff(v)$
For $\Gamma : (P,\pi) \Rrightarrow (Q,\varrho) : B(a,-) \Rightarrow (F,\mu,\iota)$ $$\left((P,\pi) \overset{\Gamma}{\Rrightarrow} (Q,\varrho) \right) \longmapsto \left(P_a(I_a) \overset{(\Gamma_a)_{I_a}}{\Rrightarrow} Q_a(I_a) \right)$$	For a 1-cell $h : v \to w$ where v and w are in the image of (F, μ, ι) $$\left(v \overset{h}{\to} w \right) \longmapsto \left((V,\nu) \overset{\Omega}{\Rrightarrow} (W,\omega) \right)$$ with $\Omega_b : V_b \Rightarrow W_b$ $(\Omega_b)_f = (Ff) * h$

Lemma 75 (Bicategorical Yoneda Lemma)**.** *The Yoneda Equivalence* $S : [B, \mathbf{Cat}](B(?,-), F) \to F$ *and* $T : F \to [B, \mathbf{Cat}](B(?,-), F)$ *as defined above is an equivalence internal to* $[B, \mathbf{Cat}]_h$.

Proof. It is clear form the definition that S is a strong transformation. To see that T is also well-defined as a strong transformation, we want natural isomorphisms $\nu_{a,b,c}$



Following $f \in B(b,c)$ around the diagram, we see



So each route sends $f \in B(b,c)$ to a functor from $B(a,b)$ to $F(c)$ whose image on $g \in B(a,b)$ is given by $(Ff)*(Fg(v))$ and $F(f*g)(v)$. The component of $\nu_{b,c}$ at f is a natural transformation between these two functors. For $\phi_{g*f} := (\phi_{b,c,d})_{g \times f}$, let $(\nu_{b,c})_g = (\phi_{g*f})_v$. Note that $\phi_{b,c,d}$ is a natural isomorphism since (F, ϕ, ι) is a homomorphism. So $\nu_{b,c}$ is an isomorphism, and hence T

is a strong transformation.

Then for all $a \in B_0$ to show S_a, T_a give an equivalence, we need

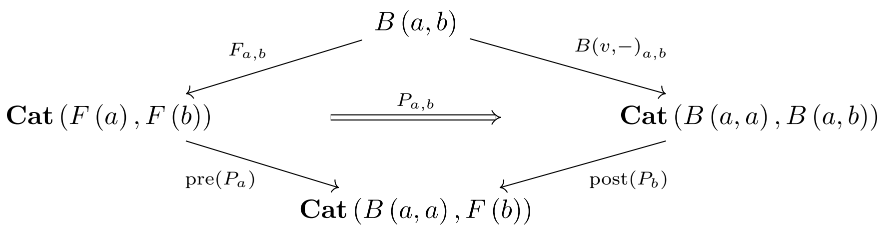
- $\eta : 1_{\mathbf{Cat}(B(a,-),F)} \to T_a * S_a$
- $\epsilon : S_a * T_a \to 1_{F(a)}$

For ϵ, note that $S_a * T_a$ is an endofunctor on $F(a)$ which maps $v \in F(a)$ to $FI_a(v)$. Since F is a homomorphism, there is a natural isomorphism ${\phi_a}^{-1} : FI_a \to I_{F(b)}$. But $I_{F(a)}$ is a unit on $F(a) \in \mathbf{Cat}$, which is a strict 2-category. So $I_{F(a)}$ is simply the identity functor and hence ${\phi_a}^{-1} : S_a * T_a \to 1_{F(a)}$ as required.

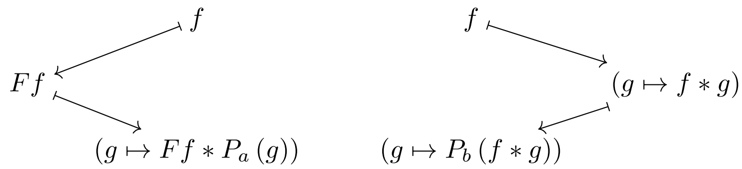
For η, note that $T_b * S_b : [B, \mathbf{Cat}]_h (B(a,-), F) \to [B, \mathbf{Cat}]_h (B(a,-), F)$ acts in the following way:

$$\begin{aligned} P : B(a,-) \Rightarrow F \quad & \mapsto \left(\hat{\hat{P}} : B(a,-) \Rightarrow F\right) \\ P_b : B(a,b) \to F(b) \quad & \mapsto \left(\hat{\hat{P}}_c : B(a,b) \to F(b)\right) \\ \left(f \mapsto P_b(f)\right) \quad & \mapsto \left(f \mapsto Ff\left(P_a(I_a)\right)\right) \end{aligned}$$

Since P is a strong transformation, we have a natural isomorphism $P_{a,b}$:



Then, following an arbitrary 1-cell f around this diagram, we see



In particular, for $g = I_a$ there is an isomorphism $P_b(f * I_a) \cong I_a \mapsto F(f) * P_a I_a$. Also note $f * I_a \cong f$ by definition. Hence we have that $(P_f)_{I_a} * P_b\left(\rho_f^{-1}\right) : P_b(f) \to Ff * P_a(I_a)$ is an isomorphism, as required. $\square$

3.3 Strictness and Coherence

Much like the Yoneda Lemma for ordinary categories, the Yoneda Lemma for bicategories has many applications across mathematics. We present one such application which allows bicategories to be 'strictified' via a biequivalence to give a strict 2-category.

Theorem 76 (Bicategorical Strictness Theorem)**.** *Every bicategory is biequivalent to a strict 2-category.*

Proof. We claim that B is biequivalent to the full image under the Yoneda embedding $Y(B) = S \subset [B^{op}, \mathbf{Cat}]_h$ with

- 0-cells $\{Y(x) \,|x \in B\}$
- For all $y, z \in B$, hom-categories $S(Y(y), Y(z)) = [B^{op}, \mathbf{Cat}]_h (Y(y), Y(z))$

Thus we have to prove that $Y' := Y|_{\text{codomain}(Y)=S}$ is a biequivalence.

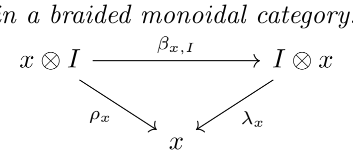
Note that Y' is precisely surjective on objects, by construction. Further, for $x, y \in B$, we see that ${Y'}_{x,y} = Y_{x,y}$ is the equivalence given by the Yoneda Lemma for bicategories applied to the 2-presheaf $B(-, y)$. As above the components of this equivalence are equivalences of categories. Hence we have the biequivalence as desired. □

Theorem 77. Bicategorical Coherence Theorem *Let B be a bicategory with $f_1, ..., f_n \in B_1$, let g and h be arbitrarily parenthesised horizontal compositions of $f_1, ..., f_n$ in that order, possibly with arbitrary insertions identity 1-cells I_-. Let $\phi, \varphi \in B(p, q)$ be isomorphisms formed entirely by compositions of constraint isomorphisms α, λ and ρ. Then $\phi = \varphi$.*

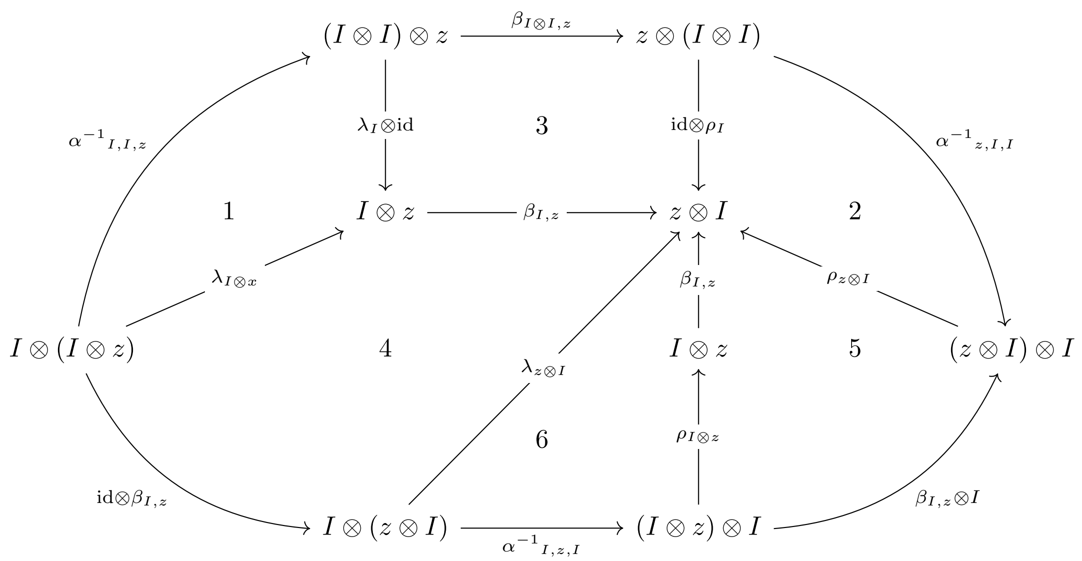
Proof. From the bicategorical strictness theorem, we may consider a strict bicategory B' such that there exists a biequivalence $(F, \mu, \iota) : B \to B'$. Then $F(\phi)$ and $L(\varphi)$ will be some composition of associators and unitors in B'. But as B' is strict, this means that $F(\phi) = F(\varphi)$, and hence $\phi = \varphi$ as F is a biequivalence and hence has faithful component functors going between hom-categories. □

Viewing a monoidal category as a bicategory with one 0-cell, the above also gives analogous result for monoidal categories. The upshot of these results is that any diagram composed of coherence constraints in bicaegories is 'contractable' in the sense that its morphisms are just identities, and in particular it commutes. An illustration of this is in the proof for the following theorem.

Theorem 78 (Unit Coherence For Braiding)**.** *The following diagram commutes for all objects x in a braided monoidal category:*



Proof. In the following diagram:



- The boundary hexagon commutes by the second hexagon axiom for braided monoidal categories, with $x = y = I$
- 3, 4 and 5 commute by naturality of β, λ, and ρ respectively

- 1 and 2 commute by the coherence theorem for monoidal categories

Hence 6 commutes. Using the coherence theorem again to remove the associator, the required result is obtained. □

Thanks to the coherence theorem, we will herein omit coherences for the convenience of dealing with smaller diagrams.

Interestingly, the unit coherence for braiding was originally taken as an axiom until it was later realised that the result follows from the coherence theorem for monoidal categories. One may at this point be tempted to conjecture that an even larger class of diagrams which would include braidings are also contractable. Unfortunately however, braidings can not be similarly strictified away. Indeed, there exist braided monoidal categories which are **not** equivalent to one in which the braiding is symmetric, let alone the identity natural isomorphism. **Braid**, the category of braids discussed in Example 14 is an example of such a category. We recall the definition of a braided monoidal functor, before stating the coherence theorem for braided monoidal categories due to Joyal and Street [13].

Definition 79 (Braided Monoidal Functor)**.** Let $(M, \otimes, I, \lambda, \rho, \alpha, \beta)$ and $(M', \otimes', I', \lambda', \rho', \alpha', \beta')$ be braided monoidal categories. A *braided monoidal functor* $(F, \mu, i) : M \to M'$ is a monoidal functor between the two underlying monoidal categories in which the following diagram commutes.

$$\begin{array}{ccc} F(x) \otimes F(y) & \xrightarrow{\beta_{F(x),F(y)}} & F(y) \otimes F(x) \\ {\scriptstyle \mu_{x,y}} \downarrow & & \downarrow {\scriptstyle \mu_{y \otimes x}} \\ F(x \otimes y) & \xrightarrow[\beta_{y,x}]{} & F(y \otimes x) \end{array}$$

Theorem 80 (Coherence Theorem for Braided Monoidal Categories)**.** *For every braided monoidal category* $(M, \otimes, I, \lambda, \rho, \alpha, \beta)$*, the underlying category* M *is equivalent to the category of braided monoidal functors* $(F, \mu, i) : \mathbf{Braid} \to (M, \otimes, I, \lambda, \rho, \alpha, \beta)$*.*

4 Further Models of Higher Categories

In this chapter we look towards models of some higher categorical structures. The study of weak higher categories, where structure on k-cells holds up to a $k+1$-cell, is a topic of ongoing research with many interesting unanswered questions. Strict models of higher categories have fewer examples, but are much easier to grasp.

Weak or strict, models of higher categories usually come in two different types: algebraic and geometric. Algebraic models equip a class with some operations that interact according to some laws, such as the definition of a bicategory provided in chapter one. On the other hand, geometric models encode higher categorical structures into an appropriately structured topological space. We give an overview of two distinct iteratively defined models of strict higher categories. Our aim will not be to provide a complete and rigorous treatment, but rather to give the reader a feel for how bicategories both inform and foreshadow a more general theory of higher categories.

4.1 Enrichment

The concept of an enriched category has far reaching applications. It allows one to talk about categories whose hom-classes have the extra structure of themselves being objects of some other category. This phenomenon is widespread in mathematics.

Definition 81 (Enriched Category)**.** Let B be any bicategory and let IX be the indiscrete bicategory on a class X, in which each hom-category is the terminal category $\mathbf{1}$. Then a morphism of bicategories $(F, \mu, \iota) : IX \to B$ is a called a *B-enriched category on X*.

Remark 82. Unpacking this definition, we see that a category enriched in B is equivalently given by

- A class X
- A function $F_0 : X \to B_0$
- A function $F : X \times X \to B_1$ given by the functor component of the morphism of bicategories. In the case of enriched categories, we denote this function as 'hom'. Functoriality is precisely the statement that $\hom(x, y) \in B(F_0(x), F_0(y))$ for all $x, y \in X$
- A function $\circ : X \times X \times X \to B_2$ given by the multiplication. In the case of an enriched category, we call this function *composition*. Naturality of the multiplication is precisely the statement that $\circ_{x,y,z} : \hom(y, z) * \hom(x, y) \to \hom(x, z)$ for all $x, y, z \in X$.
- A function $i : X \to B_2$ given by the unit of the morphism of bicategories. In the case of an enriched category, we call this function the *identity*. Naturality of the unit is precisely the statement that $i_x : I_{p(x)} \to \hom(x, x)$ for all $x \in X$.

Additionally, we see that the following diagrams commute in B

$$\begin{array}{ccc}
(\hom(y,z) * \hom(x,y)) * \hom(w,x) & \xrightarrow{\circ_{x,y,z} * I} & \hom(x,z) * \hom(w,x) \\
\downarrow{\scriptstyle \alpha_{\hom(y,z),\hom(x,y),\hom(w,x)}} & & \\
\hom(y,z) * (\hom(x,y) * \hom(w,x)) & & \downarrow{\scriptstyle \circ_{w,x,z}} \\
\downarrow{\scriptstyle I * \circ_{w,x,y}} & & \\
\hom(y,z) * \hom(w,y) & \xrightarrow[\circ_{w,y,z}]{} & \hom(w,z)
\end{array}$$

$$\begin{array}{ccc}
I * \hom(w,x) & \xrightarrow{i_x * I} & \hom(x,x) * \hom(w,x) \\
& \searrow{\scriptstyle \lambda_{\hom(w,x)}} & \downarrow{\scriptstyle \circ_{w,x,x}} \\
& & \hom(w,x)
\end{array}
\qquad
\begin{array}{ccc}
\hom(w,x) * I & \xrightarrow{I * i_w} & \hom(w,x) * \hom(w,w) \\
& \searrow{\scriptstyle \rho_{\hom(w,x)}} & \downarrow{\scriptstyle \circ_{w,w,x}} \\
& & \hom(w,x)
\end{array}$$

We call B the *base* of enrichment. If B has one object and is therefore a monoidal category, the X is called a *monoidally enriched B-category*. We will often write $X(-,-)$ for $\hom(-,-)$.

Definition 83 (Enriched Functor)**.** Let $(C, \circ, i), (D, \circ', j)$ be B-enriched categories for a bicategory B. A *B-enriched functor* $F : (C, \circ, i) \to (D, \circ', j)$ consists of the following data subject to the following axioms.

DATA

- A function $F_0 : C_0 \to D_0$
- For all $x, y \in C_0$, a collection of 2-cells in B, $F_{x,y} : \mathrm{Hom}_C(x, y) \to \mathrm{Hom}_D(F(x), F(y))$

AXIOMS

The following diagrams commute in B for all $x, y, z \in C_0$

$$\begin{array}{ccc}
C(y,z) * C(x,y) & \xrightarrow{\circ_{x,y,z}} & C(x,z) \\
\downarrow{\scriptstyle F_{y,z} * F_{x,y}} & & \downarrow{\scriptstyle F_{x,z}} \\
D(F(y),F(z)) * D(F(x),F(y)) & \xrightarrow[\circ'_{F(x),F(y),F(z)}]{} & D(F(x),F(z))
\end{array}
\qquad
\begin{array}{ccc}
& I_x & \\
\swarrow{\scriptstyle i_x} & & \searrow{\scriptstyle j_{F(x)}} \\
C(x,x) & \xrightarrow[F_{x,x}]{} & D(F(x),F(x))
\end{array}$$

Proposition 84. *For every bicategory B, there is a category $B - \mathbf{Cat}$ of B-enriched categories and B-enriched functors.*

Proof. As functor composition is associative, and the identity functor on a B-enriched category is clearly B-enriched, we need only show that the usual composition of B-enriched functors is also a B-enriched functor. Consider the following diagram

$$\begin{array}{ccc}
C(y,z) * C(x,y) & \xrightarrow{\circ_{x,y,z}} & C(x,z) \\
\Big\downarrow{\scriptstyle F_{y,z} * F_{x,y}} & & \Big\downarrow{\scriptstyle F_{x,z}} \\
D(F(y), F(z)) * D(F(x), F(y)) & \xrightarrow{\circ'_{F(x),F(y),F(z)}} & D(F(x), F(z)) \\
\Big\downarrow{\scriptstyle G_{F(y),F(z)} * G_{F(x),F(y)}} & & \Big\downarrow{\scriptstyle G_{F(x),F(z)}} \\
E(GF(y) GF(z)) * E(GF(x), GF(y)) & \xrightarrow[\circ''_{GF(x),GF(y),GF(z)}]{} & E(GF(x), GF(z))
\end{array}$$

The top square commutes as F is a B-enriched functor, while the bottom square commutes as G is also a B-enriched functor, hence the outer square commutes.

Further, consider the following diagram

$$\begin{array}{ccccc}
 & & I & & \\
 & \swarrow{\scriptstyle i_x} & \Big\downarrow{\scriptstyle j_{F(x)}} & \searrow{\scriptstyle k_{GF(x)}} & \\
C(x,x) & \xrightarrow[F_{x,x}]{} & D(F(x), F(x)) & \xrightarrow[G_{Fx,Fx}]{} & E(GF(x), GF(x))
\end{array}$$

The left triangle commutes as F is a B-enriched functor, while the right triangle commutes as G is a B-enriched functor, hence the outer triangle commutes. So GF is also a B-enriched functor as required. ☐

Notation 85. *For B a bicategory, we write $B - \mathbf{Cat}$ for the category of B-enriched categories and B-enriched functors between them.*

Herein we will restrict ourselves to considering enrichment over monoidal categories. For an important class of monoidal categories, the hom-classes $\mathrm{Hom}(x,y)$ are themselves objects in the category.

Definition 86 (Closed Monoidal Categories)**.** A *(right) closed monoidal category* is a monoidal category $(C, \otimes, I, \lambda, \rho, \alpha)$ in which the functor R_y induced by right multiplication by $y \in C$, $R_y : x \mapsto x \otimes y$ has a right adjoint which we write as $[y, -]$. That is, the isomorphism $\mathrm{Hom}_C(x \otimes y, z) \cong \mathrm{Hom}_C(x, [y,z])$ is natural in all $x, z \in C$. The functor $[-,-] : C \times C^{op} \to C$ given by $(y,z) \mapsto [y,z]$ is called the *internal hom-functor.*

A *(left) closed monoidal category* is as above, but requires a right adjoint to the functor L_x induced by left multiplication by $L_x(y) = x \otimes y$. A category is called a *biclosed monoidal category* if it is both a left closed and right closed monoidal category,

Note that any symmetric monoidal category is left closed if and only if it is right closed.

Definition 87 (Cartesian Closed Category)**.** A cartesian category which is closed as a monoidal category with respect to the monoidal structure induced by categorical products is called *cartesian closed.*

Note that for a cartesian closed category, the internal homs $[y, -]$ are given by exponentials $(-)^y$.

Proposition 88. *Let C be a category with a zero object. Then C is cartesian closed if and only if it is* **1***, the category with one object and only its identity morphism.*

Proof. Let C be a category with $I \in C$ an initial object and $T \in C$ a terminal object, with $I \cong T$. If C is cartesian closed then for all $x, y \in C$ we have.

$$\begin{aligned} |\mathrm{Hom}_C(x, y)| &= |\mathrm{Hom}_C(T \otimes x, y)| \\ &= |\mathrm{Hom}_C(T, y^x)| \\ &= |\mathrm{Hom}_C(I, y^x)| \\ &= T \end{aligned}$$

The first equality uses the fact that I is a monoidal unit since it is also terminal, the second uses the adjucntion between products and exponentials, the third uses the fact that $I \cong T$, and finally the fourth uses universal property of I as an initial object.

Hence I is the only object in C, and its only morphism is its identity. □

Definition 89 (Cosmos)**.** A symmetric monoidal closed category which is complete and cocomplete is called a *cosmos*.

Remark 90. Most popular categories to work in are cosmoses. Recalling the existence theorems for limits, one need only prove the existence of products and equalisers to show that all limits exist, and dually for colimits. The right adjoint to the tensor product functor can either be constructed directly, or proven to exist using the adjoint functor theorem.

The following result from [14] essentially says that cosmoses provide the best environment for monoidal enrichment.

Theorem 91. *If C is a cosmos then $C -$ **Cat** is also a cosmos.*

Remark 92. Note that the category **1** can be considered as a category enriched over any arbitrary base monoidal category, as hom-objects are all given by the unit of the monoidal category, and its identity morphism is the identity. Note also that if C is a closed monoidal category, then C is itself a C-enriched category as its hom-objects are internal.

We now give some examples of enriched categories.

Example 93. The poset $([0, \infty], \geq)$ can be viewed as a category similarly as in the above construction, and this category has a monoidal product given by $x \otimes y = x + y$, with the monoidal unit being 0. A category M enriched over can be viewed as an type of generalised metric space. The distance function $d : M \times M \to ([0, \infty])$ is given by the hom-objects, composition yields the triangle inequality, and the identity at an object x, $i_x : 0 \to d(x, x)$ assures that $0 \geq \mathrm{d}(x, x) \Rightarrow \mathrm{d}(x, x) = 0$. The triangle axioms hold by transitivity of $\geq$, while the pentagon amounts to the triangle inequality giving the same result whether applied to a series of distances from the left or the right. Note however that the distance function is not necessarily symmetric, and nor does it separate points.

An enriched functor $f : C \to D$ between categories enriched over $([0, \infty], \geq)$ corresponds to a contraction map with $d(f(x), f(y)) \leq d(x, y)$.

Example 94. A category is called *preadditive* if it is enriched over the monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$. A ring is just a preadditive category with one object: its underlying additive group is the unique hom-object, and its multiplication is given by composition. The enrichment axioms amount to associativity and the existence of a multiplicative identity, as required. Considering rings in this way, enriched functors $F : R^{op} \to \mathbf{Ab}$ are then just R- modules.

Example 95. The category **2** may be viewed as the category of truth values, with a unique non-identity morphism FALSE → TRUE being interpreted as implication. This category has a monoidal product usually interpreted as integer multiplication, but viewing it this way it may be seen as logical conjunction. All morphisms in this category are determined by source and target, with $|\mathrm{Hom}_{\mathbf{2}}(x,y)| \leq 1$ for $x, y \in \mathbf{2}$. Categories C enriched over $\mathbf{2}$ are precisely the preordered sets $(P, \leq)$, with hom-objects given by $x \leq y \iff |\mathrm{Hom}_C(x,y)| = 1$ for all $x, y \in P$.

To see this, first note that reflexivity $x \leq x$ in P corresponds precisely to the identity $i_x : I \to C(x,x)$, while transitivity $(x \leq y) \wedge (y \leq z) \Rightarrow x \leq z$ corresponds precisely to the composition $\circ_{x,y,z} : C(y,z) \otimes C(x,y) \to C(x,z)$. The pentagon and triangle axioms are immediately seen to hold as all diagrams commute in $\mathbf{2}$.

Enriched functors between categories enriched over $\mathbf{2}$ are monotonically increasing functions.

Example 96. Additionally, if we restrict C from the example above to be skeletal then we get a partially ordered set, while if C is instead a groupoid then it will uniquely determine an equivalence relation. Interestingly, if we ask for C to be both a groupoid and skeletal, then each equivalence class collapses to a unique element and what we are left with is just a set with no additional structure. In this sense, sets can be viewed as skeletal groupoids enriched over $\mathbf{2}$, the category of truth values.

Example 97. A category enriched over $(\mathbf{Set}, \times, 1)$ where 1 is a singleton set is a locally small category, and enriched functors correspond to ordinary functors between categories.

The last example motivates the following inductive definition.

Definition 98 (strict n-category)**.** Let $n \in \mathbb{N}$. Then define:

- A 0-*category* as a set, and a 0-functor as a function.
- For all $n \in \mathbb{N}$, an $(n+1)$-*category* as a category enriched over the category $\mathbf{n} - \mathbf{Cat}$ of n-*categories* and n-*functors*.

Note that by the corollary above, this is indeed a well defined construction.

Example 99. A strict 2-category is just a bicategory whose associators and unitors are all identity natural isomorphisms.

Remark 100. Generalising backwards, we see that $\mathbf{2}$ is almost what a '(-1)-category' should be. Sets are enriched skeletal groupoids over $\mathbf{2}$, rather than just ordinary categories.

This definition of a strict n-category has the advantage of being succinct, however it does not give much insight into what higher strict n-categories should look like or how they should behave, and it does not allow one to easily check whether a given structure is an n-category. We therefore present an equivalent, though more explicit, geometric model which was also discussed in [17]. Indeed, the topological properties of diagrams, or indeed 2-diagrams in 2-categories, suggests that spheres would be a good geometric model for higher categorical structures.

Definition 101 (Globe category)**.** The *globe category*, which we denote $\mathbb{G}$, has as its objects natural numbers $n \in \mathbb{N}$ and has morphisms generated by $s_n : n \to n+1$ and $t_n : n \to n+1$, such that $s_{n+1} \circ t_n = t_{n+1} \circ t_n$ and $t_{n+1} \circ s_n = s_{n+1} \circ s_n$. That is, the following squares must commute:

$$\begin{array}{ccc} n & \xrightarrow{t_n} & n+1 \\ {\scriptstyle t_n}\downarrow & & \downarrow{\scriptstyle s_{n+1}} \\ n+1 & \xrightarrow[t_{n+1}]{} & n+2 \end{array} \qquad \begin{array}{ccc} n & \xrightarrow{s_n} & n+1 \\ {\scriptstyle s_n}\downarrow & & \downarrow{\scriptstyle t_{n+1}} \\ n+1 & \xrightarrow[s_{n+1}]{} & n+2 \end{array}$$

Definition 102 (Globular Sets)**.** A *globular set* F is a presheaf on the globe category, with $F : \mathbb{G}^{op} \to \mathbf{Set}$.

Definition 103 (directed n-graphs and m-cells)**.** A presheaf X on the full subcategory G_n containing as objects only the integers $k \leq n$ is called a *directed n-graph*, and we call an element of $X(m)$ for $m \leq n$ an *m-cell*.

Notation 104. *Let $0 \leq p \leq m \leq n$ and let A be an n-graph. We then write $A_m \times_{A_p} A_m = \{(x, y) \in A_m \times A_m \mid t^{m-p}(x) = s^{m-p}(y)\}$ for the sets of m-cells that can be joined along p-cells.*

Remark 105. Note that a directed 0-graph is just a set, and a directed 1-graph is just a directed graph. A directed n-graph has

- A directed graph whose vertices are 0-cells and directed edges are 1-cells
- For each pair of zero cells x, y, a directed (k)-graph for some $k < n$ whose vertices are the 1-cells $p : x \to y$

So, for $1 < j \leq n$, j-cells are directed edges that go between $j - 1$ cells with the same source and target. For $n = 2$ we certainly have an equivalent construction to the 2-diagrams discussed in chapter one. In fact, the structure modelled by directed n-graphs is precisely what is need to model strict n-categories. We are now ready to give the following explicit definition for an *n-category.*

Definition 106 ((globular) strict n-category)**.** Let $n \in \mathbb{N}$. A globular strict n-category is a directed n-graph A equipped with

DATA

- A function $\circ_p : A_m \times_{A_p} A_m \to A_m$ called *p-composition* for every $0 \leq p \leq m \leq n$.
- A function $\mathrm{id}_p : A_p \to A_{p+1}$ called *p-identity.*

AXIOMS

For all $0 \leq q \leq p \leq m \leq n$, we have

- (Sources and targets of composites) For $(x, y) \in A_m \times_{A_p} A_m$, if $p = m - 1$ then $s(x \circ_p y) = s(y)$ and $t(x \circ_p y) = t(x)$. Otherwise, $s(x \circ_p y) = s(x) \circ_p s(y)$ and $t(x \circ_p y) = t(x) \circ_p t(y)$,
- (Sources and targets of identities) For $x \in A_p$, we have that $s(1_x) = x = t(1_x)$,
- (Associativity) For $x, y, z \in A_m$, if $(z, y), (y, x) \in A_m \times_{A_p} A_m$ then $(z \circ_p y) \circ_p x = z \circ_p (y \circ_p x)$,
- (Identities) For $x \in A_m$, we have that $\mathrm{id}^{m-p}(t^{m-p}(x)) \circ_p x = x = x \circ_p \mathrm{id}^{m-p}(s^{m-p}(x))$,
- (Binary interchange) For $w, x, y, z \in A_m$, if $(z, y), (x, w) \in A_m \times_{A_p} A_m$ then $(z \circ_p y) \circ_q (x \circ_p w) = (z \circ_q x) \circ_p (y \circ_q w)$, and
- (Nullary interchange) If $(x, y) \in A_p \times_{A_q} A_p$ then $1_y \circ_q 1_x = 1_{y \circ_q x}$.

A globular strict n-functor can then also be defined as a function between globular strict n-categories which commutes with compositional and identity structure. One can then show that globular strict n-categories and globular strict n-functors form a globular strict $(n + 1)$-category, and then by induction on n that the globular and enriched definitions of strict n-categories are equivalent.

Remark 107. We observe the following pattern.

	In a 1-category A	In a 2-category B	In an n-category C
Between 0-cells x, y	A hom-*set* $\mathrm{Hom}_A(x,y)$ of morphisms, which we will call 1-cells	A hom-*category* $\mathrm{Hom}_B(x,y)$ whose 0-cells are called 1-cells of B, and 1-cells are called 2-cells of B	A hom-*(n-1)-category* $\mathrm{Hom}_C(x,y)$ whose k-cells will be called the $(k+1)$-cells of C for every $0 \leq k \leq n-1$
For every 0-cell	A *function* $\mathrm{id}_x : \{*\} \to \mathrm{Hom}_A(x,x)$ from the singleton set	a *functor* $I_x : \mathbf{1} \to \mathrm{Hom}_B(x,x)$ whose domain is the terminal category, with one 0-cell and one 1-cell	An *(n-1)-functor* $I_x : \mathbf{1}_{n-1} \to \mathrm{Hom}_C(x,x)$, where $\mathbf{1}_{(n-1)}$ is an *(n-1)-category* with a unique k-cell for every $0 \leq k \leq n-1$
For every triple x,y,z of 0-cells	A composition *function* $\circ_{x,y,z} : \mathrm{Hom}_A(y,z) \times \mathrm{Hom}_A(x,y) \to \mathrm{Hom}_A(x,z)$ that is associative and for which every 0-cell x has an identity 1-cell in $\mathrm{Hom}_A(x,x)$ given by the image of id_x	a composition *functor* $\circ_{x,y,z} : \mathrm{Hom}_B(y,z) \times \mathrm{Hom}_B(x,y) \to \mathrm{Hom}_B(x,z)$ that is associative and for which every 0-cell x has an identity in $\mathrm{Hom}_B(x,x)$ given by the image of I_x	a composition *(n-1)-functor* $\circ_{x,y,z} : \mathrm{Hom}_C(y,z) \times \mathrm{Hom}_C(x,y) \to \mathrm{Hom}_C(x,z)$ that is associative and for which every 0-cell x has an identity in $\mathrm{Hom}_C(x,x)$ given by the image of I_x

4.2 Internalisation

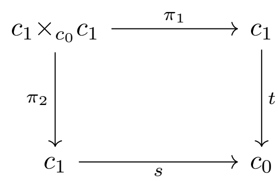
We now present an alternative algebraic model of a higher categorical structure with a different geometric model.

Definition 108 (Internal Category)**.** Let C be a category with pullbacks. Then, a category *internal* to C is given by the following data subject to the following axioms

DATA

- An object of *objects* $c_0 \in C$
- An object of *morphisms* $c_1 \in C$
- Morphisms $s, t : c_1 \to c_0$ called *source* and *target*
- A morphism $i : c_0 \to c_1$ called the *identity assigner*
- A morphism $\circ : c_1 \times_{c_0} c_1 \to c_1$ called *composition*

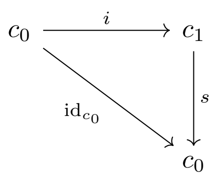
where $c_1 \times_{c_0} c_1$ is given by the pullback square

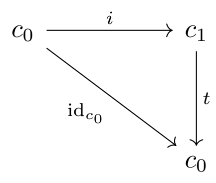


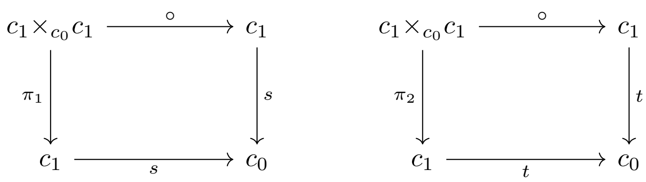
AXIOMS

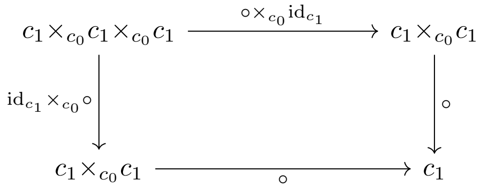
The following diagrams commute in C

- Sources and targets for identities

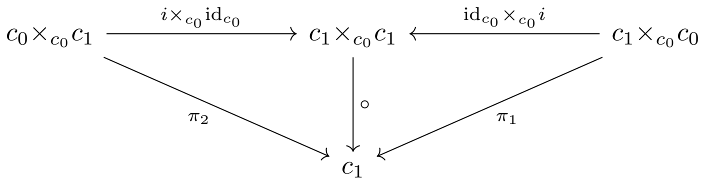




- Sources and targets for composites 

- The associativity law of composition 

- Left and right unit laws for composition of morphisms



Example 109. A category internal to **Set** is just a small category.

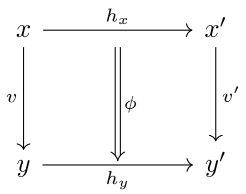
Example 110. Recall **Diff**, the category of paracompact smooth manifolds and smooth maps. A groupoid internal to **Diff** with source and target morphisms that are submersions so that the required pullbacks exist is called a *lie groupoid*. These will precisely be the groupoids G for which $\text{Ob}\,(G)$ and $\text{Mor}\,(G)$ are both smooth manifolds, and composition identity, source and target maps are all smooth, with the last two having surjective differentials between tangent spaces $d\,(s_x) : T_x\text{Mor}\,(G) \to T_{s(x)}\text{Ob}\,(G)$ and $d\,(t_x) : T_x\text{Mor}\,(G) \to T_{t(x)}\text{Ob}\,(G)$. The morphisms of lie groupoids with only one object are just lie groups.

Definition 111 (double category)**.** A *double category* is a category internal to **Cat**

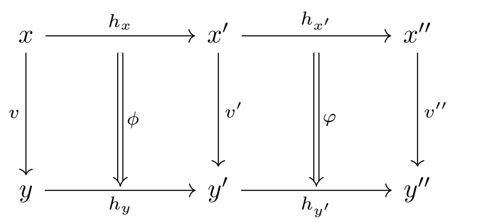
Remark 112. Unpacking this definition, we see that a the morphisms which constitute the data of a double category are functors. So a double category (C_0, C_1) will be given by a pair of categories C_0, C_1. Call

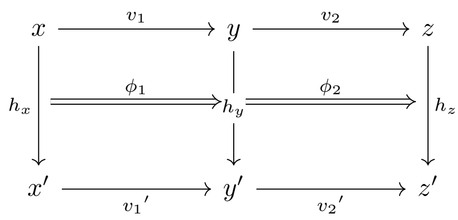
- The objects of C_0 *objects* of (C_0, C_1)
- The morphisms of C_0 *vertical arrows*
- The objects of C_1 *horizontal arrows*
- The morphisms of C_1 2-*cells*

where a 2-cell is given as



for $x, x', y, y' \in C_0$, $h_x \in C_0\,(x, x')\,, h_y \in C_0\,(y, y')$, $v, v' \in C_1$, $\phi \in C_1\,(v, v')$. The horizontal and vertical arrows compose as in C_0 and C_1, while the 2-cells compose horizontally and vertically as shown, respectfully, in the two diagrams below:





by the composition $\circ$ in the data of (C_0, C_1) as an internal category. Equivalently, a double category is

- A set of objects C
- For every $x, y \in C$, two sets of morphisms $\mathrm{Hor}_C(x, y)$ called horizontal arrows and $\mathrm{Ver}_C(x, y)$ called vertical arrows which both turn C into a category
- For every diagram of the following form in C, a collection of 2-cells ${\mathrm{Ver}'}_C(h_x, h_y)$ of 2-cells such that these collections form the hom-sets of a category whose objects are the horizontal arrows

Remark 113. A double categories whose horizontal arrows are all identities are just 2-categories with 0-cells objects of C, 1-cells the vertical arrows, and 2-cells the 2-cells of the double category.

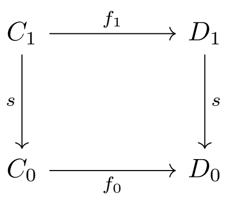
Definition 114 (Internal Functor)**.** Let A be a category with pullbacks and let C, D be categories internal to A. An *internal functor* $(f_0) : C \to D$ is given by the following data subject to the following axioms.

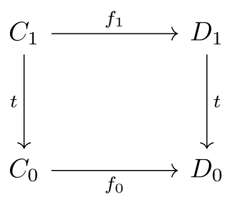
DATA

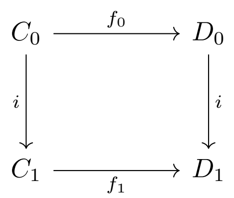
- Two morphisms in A, $f_0 : C_0 \to D_0$ and $f_1 : C_1 \to D_1$

AXIOMS: The following diagrams commute in A

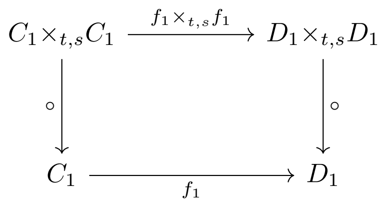
- Respect for sources, targets, and identity assigners







- Respect for composition



Proposition 115. *For C a category with pullbacks, there is a category* $\mathbf{Cat}(C)$ *whose objects are categories internal to C and morphisms are internal functors.*

Proof. Let $(f_0, f_1) : C \to D$, $(g_0, g_1) : D \to E$ be internal functors for C, D, E categories internal to some category A which has pullbacks. Each of the diagrams for the axioms in the definition of an internal functor are squares which can be joined like so

$$\begin{array}{ccccc} C_1 & \xrightarrow{f_1} & D_1 & \xrightarrow{g_1} & E_1 \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ C_0 & \xrightarrow[f_0]{} & D_0 & \xrightarrow[g_0]{} & E_0 \end{array}$$

to yield the corresponding axiom for an internal functor $(g_0, g_1) \circ (f_0, f_1) : C \to E$. Note that the axiom for composition also requires the pullback lemma. As composition of morphisms in A is well defined, associative, and has units, this completes the proof. □

The following result from [4] and [5] says that the process of internalisation can also be iterated.

Theorem 116. *Let C be finitely complete and cartesian closed. Then* $\mathbf{Cat}\,(C)$ *is also finitely complete and cartesian closed.*

Hence we have the following algebraic model of a higher categorical structure.

Definition 117 (Strict Cubical n-tuple Category)**.** Define inductively,

- 0-tuple-**Cat** := **Set**
- n-tuple-**Cat** := **Cat** ($n-1$-tuple-**Cat**)

Remark 118. The reason this model of a higher categorical structure is called 'cubical' is that its n-th case is indeed geometrically modelled by the n-cube in a similar way to what we described for strict n-categories defined via enrichment. Another name for this construction is a multiple category. Of course, the n-cube is homeomorphic to the n-sphere, but in fact these models are not equivalent. Indeed, one can see that a strict n-category is a degenerate case of a double category in one of two equivalent ways: by asking for vertical morphisms to be identities or by asking the same from horizontal morphisms.

We now use a special kind of morphism of bicategories to compare the notions of enrichment and internalisation.

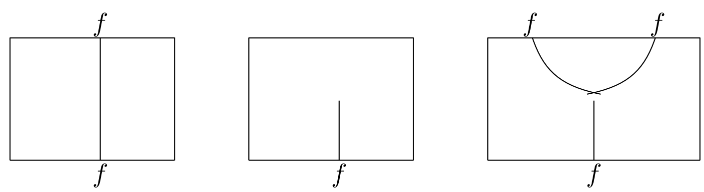
Definition 119. The data of a morphism of bicategories (F, μ, ι) from $\underline{\mathbf{1}}$, the bicategory with only one k-cell for $k \in \{0, 1, 2\}$ which we denote by k, to a bicategory B is called a *monad.*

Remark 120. In the case of a monad, we write the multiplication and unit without indexation. The axioms of a morphism of bicategories reduce to

$$\begin{array}{ccccc} I_0 * F1 & \xrightarrow{\iota * \mathrm{id}} & F1 * F1 & \xleftarrow{\mathrm{id} * \iota} & F1 * I_0 \\ & {\scriptstyle \lambda_{F1}} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle \rho_{F1}} & \\ & & F1 & & \end{array} \qquad \begin{array}{ccc} (F1 * F1) * F1 & \xrightarrow{\mu * \mathrm{id}} & F1 * F1 \\ {\scriptstyle \alpha_{F1,F1,F1}} \downarrow & & \\ F1 * (F1 * F1) & & \downarrow{\scriptstyle \mu} \\ {\scriptstyle \mathrm{id} * \mu} \downarrow & & \\ F1 * F1 & \xrightarrow[\mu]{} & F1 \end{array}$$

while for a comonad the axioms involving the multiplication and unit are reversed. This is historically a more traditional definition of a (co)monad in a bicategory. Note the if $B' = \mathbf{Cat}$ then this gives the familiar notion of a monad as a (co)monoid object in an endofunctor category.

The data of monads can be represented by the string diagrams, in which each region is the 0-cell x.



Their axioms are represented by the equivalences on process network diagrams given in Remark 5.

Definition 121 (Bicategory of Monads)**.** For B a bicategory, define the *bicategory of monads* in B by $[\underline{\mathbf{1}}, B]$.

Theorem 122. *For C a category with finite limits, a monad in the bicategory of* $\mathbf{Span}\,(C)$ *is a category internal to C.*

Proof. Let the 0-cell of the monad be x, and the 1-cell of the monad be the span $(s, t) : x \leftarrow y \rightarrow x$. Multiplication $\mu : y \times_x y \rightarrow y$ will be given by the composite spans. Then,

- The source and target axioms for the identity assigner and composition in a category internal to C are equivalent to the assertion that they are morphisms of spans in C.
- Associativity of μ is equivalent to the associativity of the composition.
- Unit laws of a monad correspond to identity laws for an internal category.

□

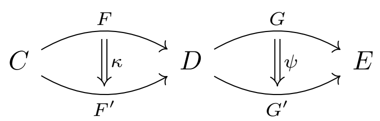
Theorem 123. *For C a category with finite limits and colimits in which products distribute over coproducts, a monad in the bicategory* $\mathbf{Mat}\,(C)$ *is a category enriched over C.*

Proof. Let the 0-cell of the monad be set X and the 1-cell be the matrix $F : X \times X \rightarrow \mathrm{Ob}\,(C)$, which we identify with the hom-object of the C category. The multiplication map evaluates via formal matrix multiplication and hence coincides with composition. Associativity and unit conditions for the monad and the enriched category match, once again. □

Indeed, it can be shown that when $C = \mathbf{Set}$, $\mathbf{Mat}\,(C)$ and $\mathbf{Span}\,(C)$ are biequivalent. This explains why enrichment and internalisation agree on $\mathbf{Set}$ to both give small categories.

4.3 Horizontal composite of transformations

Recall that for natural transformations between functors, as in



we may define their left whiskering to have components $\psi_{F'(x)}$, and their right-whiskering to have components $G\kappa_x$ for all $x \in C$ We can then vertically compose these whiskerings by the usual composition of components in E to define the horizonatal composition of κ and ψ. Furthermore, we have that $\phi_{F'} \circ G\kappa = G'\kappa \circ \phi_F$, and indeed this is the definition of horizontal composition in $\mathbf{Cat}$.

We recall the notation developed in chapter two and return to the issue of describing a horizontal composition of (P, π) and (S, σ). This should be a transformation from $(GF, G\mu \circ \phi_F, G\iota \circ \theta_F)$ to $(G'F', G'\mu' \circ \phi_{F'}, G'\iota' \circ \theta'_{F'})$, the respective composites of morphisms of bicategories. Unfortunately this is not possible at this level of generality, and can only work if the morphisms of bicategories involved are homomorphisms.

To see the difficulty with transformations between morphisms of bicategories, consider just an attempt at a 'right-whiskering' of (S, σ) with (G, ϕ, θ), which ought to be a transformation from $(GF, G\mu \circ \phi_F, G\iota \circ \theta_F)$ to $(GF', G\mu \circ \phi_{F'}, G\iota' \circ \theta_{F'})$.
A component 1-cell may be defined as $G\,(S_x)$ without any issue, however a problem arises when we try to assign the component 2-cell of the component natural transformation $G\sigma$ at some 1-cell $f : x \rightarrow y$ in B. This would need to go from $GF'f * GS_x$ to $GS_y * GFf$. The usual way to define this would use the 2-cell $G\,(\sigma_f) : G\,(F'f * S_x) \Rightarrow G\,(S_y * Ff)$ and apply the multiplications of G. As we can see however, one of multiplications is pointing in the wrong direction:

$$GF'f * GS_x \xrightarrow{\phi_{F'f,S_x}} G(F'f * S_x) \xrightarrow{G(\sigma_f)} G(S_y * Ff) \xleftarrow{\phi_{S_y,Ff}} GS_y * GFf$$

Note that switching to op-morphisms will not fix this issue as it would also reverse $\phi_{F'f,S_x}$, while considering op-transformations is also unhelpful as this would still leave $\phi_{F'f,S_x}$ pointing in the wrong direction. However, if we consider homomorphisms instead, then the multiplication ϕ being invertible allows us to define component 2-cells in B''.

Definition 124 (Whiskerings of transformations with homomorphisms)**.** The *right-whiskering* $(GS, G\sigma)$ of (S,σ) with (G,ϕ,θ) consists of

- For every $x \in B$, the 1-cell $G(S_x) \in B''$.
- For every $f : x \to y$ in B, the 2-cell $(G\sigma)_f = \left(\phi_{S_y,Ff}\right)^{-1} * G(\sigma_f) * \phi_{F'f,S_x}$.

The *left-whiskering* (P_F, π_F) of (P,π) with (F,μ,ι) consists of

- For every $x \in B$, the 1-cell $P_{F(x)}$,
- For every $f : x \to y$ in B, the 2-cell $\pi_{Ff} : G'Ff * P_{F(x)} \Rightarrow P_{F(y)} * GFf$.

Theorem 125. *The left and right-whiskerings as defined above are both themselves transformations.*

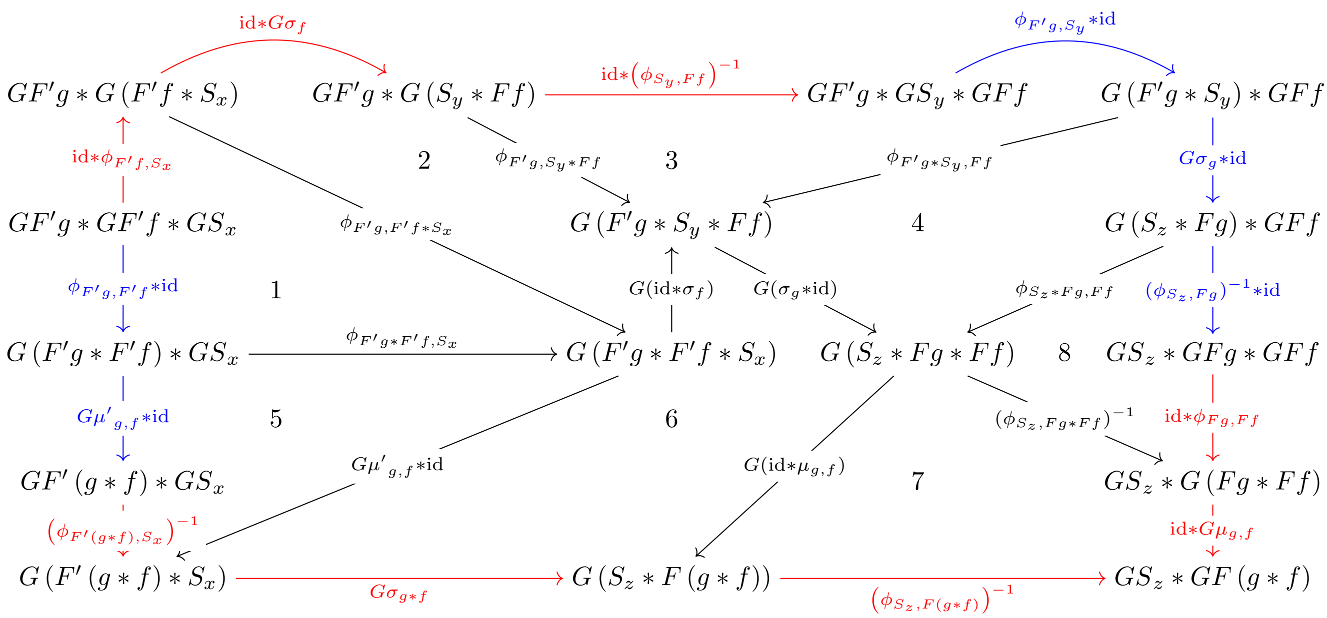
Proof. The octagon axiom for the right-whiskering is given by the following diagram, in which we use the coherence theorem to contract associators. Going clockwise from $GF'g * GF'f * GS_x$ to $GS_x * GF(g*f)$,

- The first red path is the component 2-cell of the left-whiskering of f horizontally post-composed with $GF'g$.
- The blue path is the same on g horizontally pre-composed with GFf
- The last two red arrows are the component of the multiplication on $g \times f$ horizontally post-composed with GS_z.

Going anticlockwise,

- The blue path defines the component of the multiplication on $g \times f$ horizontally pre-composed with GS_x.
- The red path defines the component 2-cell of the right-whiskering on $g \times f$

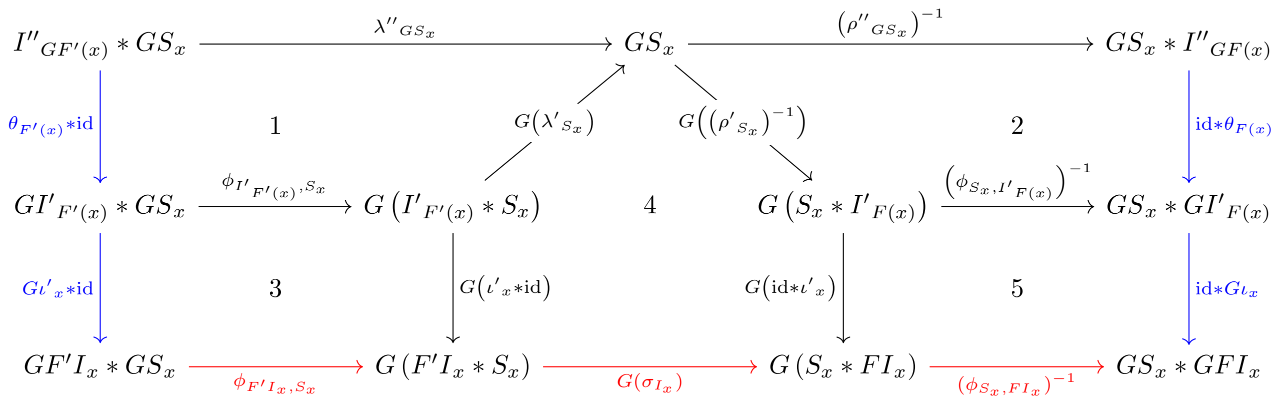
So the boundary is indeed the required octagon, with associators suppressed.



We observe:

- 1, 3 and 8 are hexagon axioms for ϕ with associators suppressed.
- 2 and 4 commute by naturality of ϕ
- 5 and 7 commute by naturality of μ and functoriality of horizontal composition.
- 6 is the octagon axiom for (S, σ) with associators suppressed.

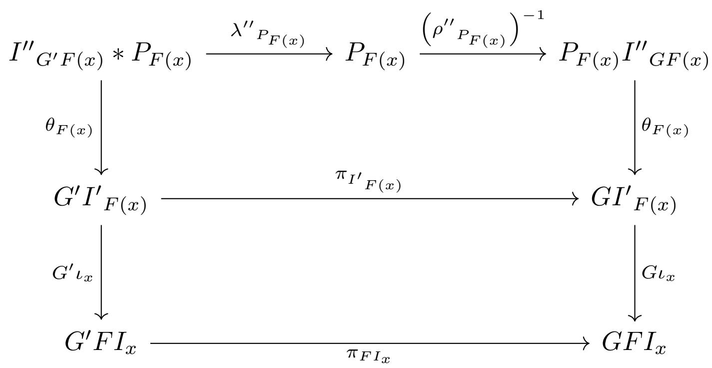
The pentagon axiom for the right-whiskering is given by the following diagram:



We observe that

- The blue paths define the component identities of the composite homomorphisms of bicategories and the red path defines the component 2-cell of the right-whiskering.
- 1 and 2 commute by the triangle axiom for θ.
- 4 commutes by the pentagon axiom of (S, σ).
- 3 and 5 are naturality squares of ϕ.

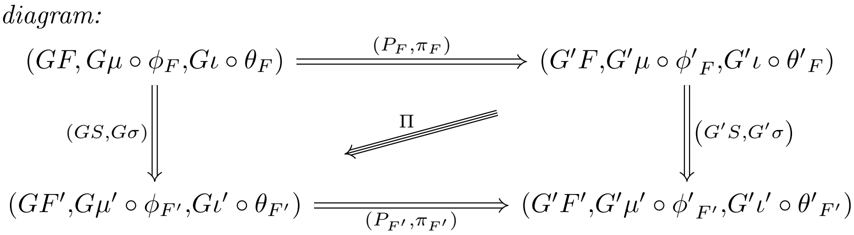
For the left-whiskering, the pentagon axiom is given by



We see that the vertical paths define the component units for the composite morphism of bicategories, so that the boundary is the pentagon axiom for the left-whiskering. Furthermore, 1 is the pentagon axiom for (P, π), while 2 is the naturality square for (P, π), so the diagram commutes. A similar proof with vertical paths instead defining the component multiplication of the composite morphism of bicategories will show that the octagon axiom is also satisfied by the left-whiskering. □

Note that we can just as easily define the right-whiskering $(G'S, G'\sigma) : (G'F, G'\mu * \phi'_F, G'\iota * \theta'_F) \to (G'S, G'\sigma) : (G'F', G'\mu' * \phi'_{F'}, G'\iota * \theta'_{F'})$ and the left-whiskering $(P_{F'}, \pi_{F'}) : (GF', G\mu' * \phi_{F'}, G\iota' * \theta_{F'}) \to (G'F', G'\iota'_{F'} * \phi'_{F'}, G'\iota' * \theta'_{F'})$, and go on to deduce that they will both also themselves be transformations in a completely analogous way. Taking vertical composites of transformations, we arrive at two possible candidates for a horizontal composite of transformations.

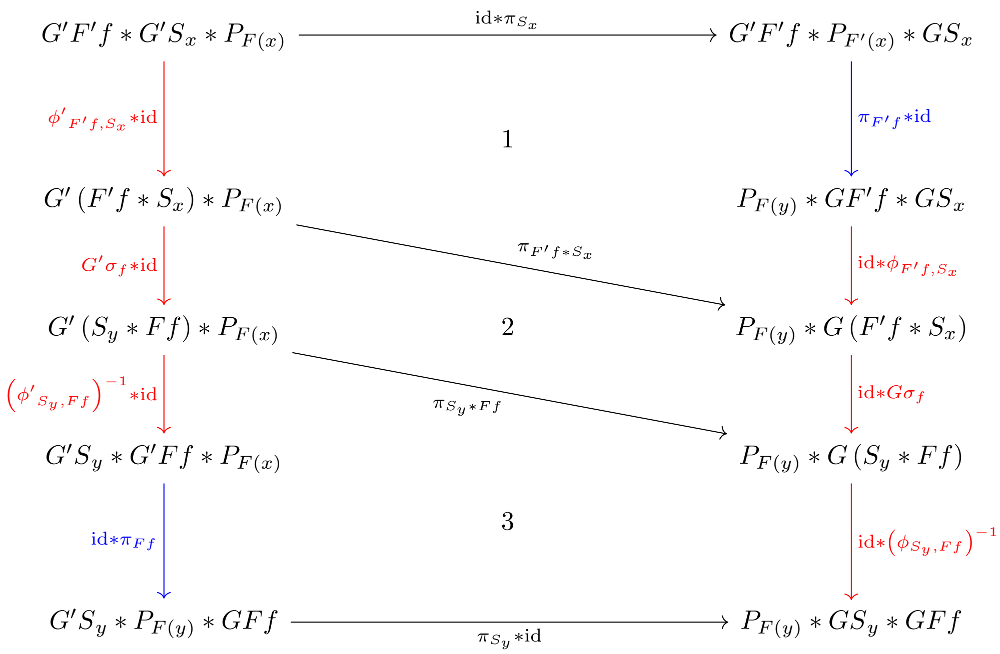
Theorem 126. *There is a modification* $\Pi : (G'S, G'\sigma) \circ (P_F, \pi_F) \Rrightarrow (P_{F'}, \pi_{F'}) \circ (GS, G\sigma)$ *as in the diagram:*



whose component on $x \in B_0$ *is given by* π_{S_x}*. This modification is*

- *invertible if and only if* (P, π) *is a strong transformation.*
- *the identity if and only if* (P, π) *is a strict transformation.*

Proof. The naturality square for the modification is given by the following diagram.



To complete the proof, we observe

- The paths in red define the right-whiskerings and the paths in blue define left-whiskerings, so that the left and right columns define vertical composites of these transformations and hence the boundary is the required naturality square for Π.
- 1 and 3 commute by octagon axioms of (P, π).
- 2 commutes by naturality of π.

□

So the modification goes form the horizontal composite whiskered 'left-first' to horizontal composition whiskered 'right-first'. We now investigate the relationship between horizontal and vertical composition of transformations. We denote by $(S * P, \sigma * \pi)$ and $(P * S, \pi * \sigma)$ the respective horizontal composites of transformations, so that $\Pi : (S * P, \sigma * \pi) \Rrightarrow (P * S, \pi * \sigma)$ as described in the theorem above has the component 2-cell π_{S_x} for every $x \in B_0$. Similarly, there will be a modification $\Omega : (T * W, \tau * \omega) \Rrightarrow (W * T, \omega * \tau)$, whose component 2-cell is given by ω_{T_x}. Keeping the notation developed on page 21, we further introduce the following notation for the four possible transformations from $(GF, G\mu \circ \phi_F, G\iota \circ \theta_F)$ to $(G''F'', G''\mu'' \circ \pi''_{F''}, G''\iota'' \circ \theta''_{F''})$.

	Left First	Right First
Horizontal First	$('H, '\xi)$	(H', ξ')
Vertical First	$('V, '\nu)$	(V', ν')

We compute the 1-cell components of these transformations:

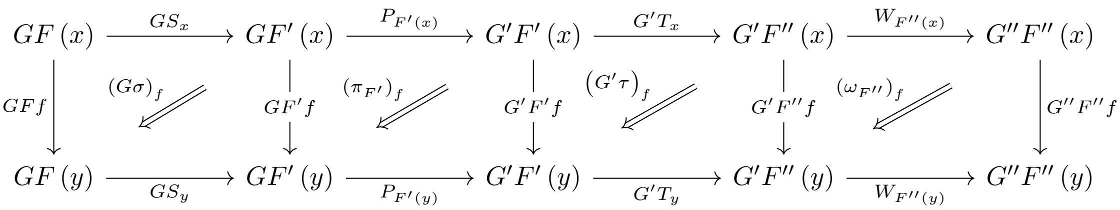
$$H'_x = ((W * T) \circ (P * S))_x = ((W_{F''} * G'T) * (P_{F'} * GS))_x = \left(W_{F''(x)} * G'T_x\right) * \left(P_{F'(x)} * GS_x\right)$$

$$'H_x = ((T * W) \circ (S * P))_x = ((G''T * W_{F'}) \circ (G'S * P_F))_x = \left(G''T_x * W_{F'(x)}\right) * \left(G'S_x * P_{F(x)}\right)$$

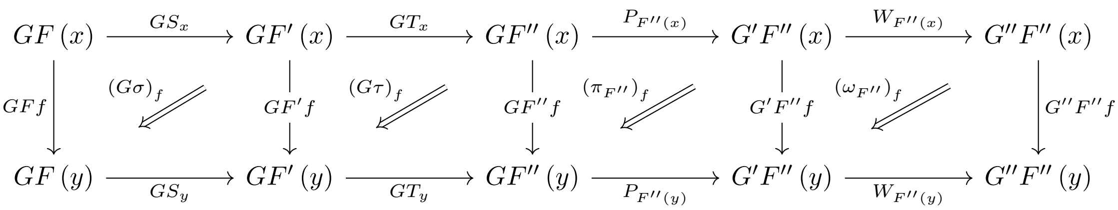
$$V'_x = ((W \circ P) * (T \circ S))_x = ((W \circ P)_{F''} * G(T \circ S))_x = \left(W_{F''(x)} * P_{F''(x)}\right) * G\left(T_x * S_x\right)$$

$$'V_x = ((T \circ S) * (W \circ P))_x = (G''\left(T \circ S\right) * (W \circ P)_F)_x = G''\left(T_x * S_x\right) * \left(W_{F(x)} * P_{F(x)}\right)$$

Meanwhile, for all $f : x \to y$ in B_1, we have 2-cell components such as $\xi'_f = ((\omega * \tau) \circ (\pi * \sigma))_f :$ $G''F''f * H'_x \Rightarrow H'_y * GFf$ given by pasting schemes of whiskerings such as

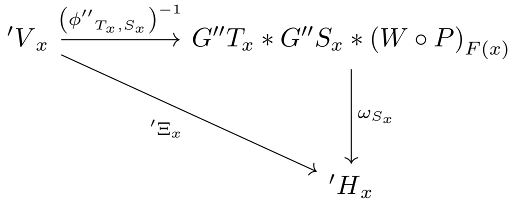


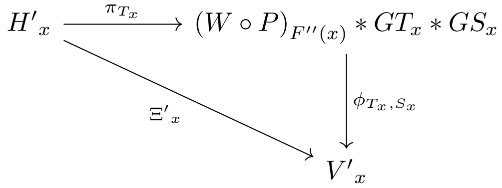
For vertical-first transformations, we have 2-cell components such as $\nu' = (\omega \circ \pi) * (\tau \circ \sigma)$ given by pasting schemes which, modulo the isomorphism ϕ_{T_x,S_x}, looks like



We observe that by taking horizontal compositions of modifications $\Omega * \Pi : (H, \xi) \Rrightarrow (H', \xi')$ is a well-defined modification with the component $(\Omega * \Pi)_x = (\omega_{T_x}) * (\pi_{S_x})$. Similarly, by instead taking vertical compositions of transformations, there is a modification $(\Omega * \Pi)' : (V, \nu) \Rrightarrow (V', \nu')$ with component $(\Omega * \Pi)'_x = (\omega \circ \pi)_{(T_x * S_x)}$.

Definition 127 (Left and Right Interchangers)**.** The *left interchanger* $'\Xi$ and the *right interchanger* Ξ' of transformations (S, σ), (T, τ), (P, π) and (W, ω), are classes of 2-cells in B'' indexed by $x \in B$. They are defined respectively by the following compositions, where we have omitted identity 2-cells.





Theorem 128. *The interchangers defined above constitute modifications* $'\Xi : ('V, '\nu) \Rrightarrow ('H, '\xi)$, *and* $\Xi' : (H', \xi') \Rrightarrow (V', \nu')$ *such that the following diagram commutes in* $[B, B'']$

$$\begin{array}{ccc} ('H, '\xi) & \xLeftarrow{\quad '\Xi \quad} & ('V, '\nu) \\ {\scriptstyle \Omega * \Pi} \Big\Downarrow & & \Big\Downarrow {\scriptstyle (\Omega * \Pi)'} \\ (H', \xi') & \xRightarrow[\quad \Xi' \quad]{} & (V', \nu') \end{array}$$

The naturality squares can be constructed in a similar way to the proof for Π, although the diagrams involved are much larger.

Corollary 129. *The following conditions hold for the interchangers:*

- Ξ' *is invertible if and only if* (P, π) *is strong.*
- Ξ' *is the identity if* B'' *and* (P, π) *are strict, and* ϕ *is an identity natural isomorphism.*
- $'\Xi$ *is invertible if and only if* (W, ω) *is strong.*
- $'\Xi$ *is the identity if* B'' *is strict and* ϕ' *is the identity natural isomorphism.*

So both notions of horizontal composition of transformations fail the binary interchange law whenever the modification above has a non-identity component. This means that transformations fail to be the 2-cells in a bicategory whose 0-cells are bicategories and 1-cells are morphisms of bicategories, as horizontal composition in such a bicategory would fail to be functorial. This motivates the following definition, originally introduced in [16].

Definition 130 (Icon)**.** An op-transformation $(N, \eta) : (F, \mu, \iota) \Rightarrow (F', \mu', \iota')$ between morphisms of bicategories is called an *icon* if:

- $F(x) = F'(x)$ for all $x \in B_0$.
- $N_x = {I'}_{F(x)}$ for all $x \in B_0$.
- $\eta_f : Ff \Rightarrow Gf$ for every $f \in B_1$
- For every 2-cell $\gamma : f \Rightarrow g$ in B_2, $\eta_g * F(\gamma) = F'(\gamma) * \eta_f$
- η_{I_x} modulo ι, ι' is an identity
- For $x \xrightarrow{f} y \xrightarrow{g} z$ in B, $\eta_g * \eta_f = \eta_{g*f}$ modulo μ, μ'.

Corollary 131. *There is a* 2*-category* **Icon** *whose* 0*-cells are bicategories,* 1*-cells are morphisms of bicategories which agree on objects,* 2*-cells are icons which compose vertically and horizontally by vertical and horizontal composition of transformations respectively.*

Remark 132. If B, B' above have one object each, then icons between them are precisely monoidal natural transformations and the 2-category **MonCat** is a full sub-2-category of **Icon**

4.4 Weak Higher Categories

The phenomena we witnessed with the interchanges motivates the definition of a three dimensional categorical structure, much in the same way as horizontal composition of natural transformations between functors in ordinary category theory motivates the definition of bicategories.

We now give a brief discussion of how bicategorical phenomena generalises, or indeed fails to generalise, to trcategories. A detailed definition which we will not reproduce here can be found in [9]. It is then straightforward to check that

- Bicategories and their transfors as defined in chapter two *do not* form a tricategory as the whiskerings are not well defined.
- Bicategories, homomorphisms, transformations and modifications do not form a tricategory either as the interchangers are not invertible.
- There is a tricategory whose 0-cells are bicategories, 1-cells are homomorphisms, 2-cells are strong transformations and 3-cells are modifications. The interchangers discussed above form an important part of the underlying data of this tricategory, in a similar way to how associators and unitors are part of the data for a bicategory. Being invertible is indeed an essential part of the structure for an interchanger, just as it is for associators and unitors in a bicategory, and this is why we require strong transformations to form a tricategory. In fact, the left-first and right-first definitions give two distinct ways of defining this tricategory, but since the modification between them is invertible and there is a commuting square of modifications as mentioned in Theorem , one can show that these tricategories are in fact 'triequivalent'. Identifying them up to triequivalence, we will denote this tricategory by **Bicat**.
- **Bicat** has a sub-tricategory with only *strict* 2-categories, *strict* 2-functors, strong transformations and modifications which we will denote **Gray**.
- $2-\mathbf{Cat}$ as defined via enrichment may be considered as a 3-category, and as such forms a further sub-tricategory of **Gray**.

In the following definition, we consider $2-\mathbf{Cat}$ as just an ordinary category whose objects are 2-categories and morphisms are 2-functors.

Definition 133 (Gray Tensor Product)**.** The *Gray Tensor Product* $\otimes : 2-\mathbf{Cat} \times 2-\mathbf{Cat} \to 2-\mathbf{Cat}$ is defined so that $2-\mathbf{Cat}\,(B \otimes B', B'') \cong 2-\mathbf{Cat}\,(B, [B', B'']_s)$, where $[B', B'']_s$ is the category of strict 2-functors form B' to B'' and strong transformations between them.

We see that the gray tensor product automatically induces a symmetric monoidal closed structure on $2-\mathbf{Cat}$ and hence we have the following definition.

Definition 134 (Gray Categories)**.** A *Gray category* is a category enriched over the category of 2-categories equipped with the gray tensor product.

Gray categories are strict in every sense *except* for having a non-identity interchanger. There is the following strictness theorem for tricategories.

Theorem 135 (Tricategory Strictness Theorem)**.** *Every tricategory is triequivalent to a Gray Category.*

Theorem 136. *A tricategory with at most one* 0*-cell and at most one* 1*-cell is a braided monoidal category.*

The proof is via the Eckmann-Hilton Argument, and is similar to that of the proof that every bicategory with only one 0-cell and only one 1-cell is a commutative monoid. Indeed, the category of braids provides a degenerate example of tricategory that is not equivalent to a strict 3-category, as if it were it would be equivalent to one with an identity braiding.

The compositional structure of $(2, k)$-transfors as discussed in chapter two was seen to be richer than the compositional structure of functors and natural transformations, or $(1, k)$-transfors. The compositional structure of $(3, k)$-transfors is even richer. It is shown in [9] that tricategories and the weakest notion of morphisms between them do not form a category as composition of morphisms is not associative 'on the nose', but only up to a 'tritransformation'. To form a category, one needs to instead consider a stronger notion which has invertible components.

It is natural to seek a general theory for weak-n categories. Indeed, there are notions of 'weak enrichment' and 'weak internalisation', as well as many geometric and algebraic models for weak higher categories. An explicit description in elementary terms involving at most $(3,k)$-transfors has been formulated for a 'tetracategory', although it is fifty pages long and barely comprehensible. No coherence results are known for tetracategories, although degenerate examples are suspected to be given by 'monoidal tricategories' and 'braided monoidal bicategories', in a phenomenon known as the delooping hypothesis.

We conclude with the following 'example' of a weak higher category, which was historically the motivating example of much of higher category theory. It is sometimes taken as a litmus test for candidate definitions for higher categorical models. It strongly resembles Example 32. Indeed, from the perspective of homotopy theory there is no need to just stop at the fundamental bigroupoid, and we could similarly define analogous fundamental higher groupoids of a space where we take equivalence classes of n-th homotopies for a given $n \geq 2$.

Example 137. Let X be a topological space. For every $n \in \mathbb{N}$, there is a weak n-category whose 0-cells are points $x \in X$, 1-cells are paths $p : x \to y$, 2-cells are homotopies $h : p \Rightarrow q : x \to y$, and so on such that for all $k \leq n$ the k cells are homotopies between $k-1$ cells. Note that this is not a strict category, as associativity needs only to hold up to weak equivalence. It is, however, groupoidal.

It is conjectured that topological spaces can be viewed as '∞-groupoids'. This is known as the homotopy hyposthesis.

5 Appendix

5.1 Axioms of op versions of transformations

5.1.1 Op-transformations between morphisms of bicategories

For an op-transformation between morphisms of bicategories, we have the following axioms:

(octagon axiom)

$$\begin{tikzcd}
F'g*(F'f*S_x) \arrow[d, "(\alpha'_{F'g,F'f,S_x})^{-1}"'] & F'g*(S_y*Ff) \arrow[l, "\mathrm{id}*\sigma_f"'] & (F'g*S_y)*Ff \arrow[l, "\alpha'_{F'g,S_y,Ff}"'] & (S_z*Fg)*Ff \arrow[l, "\sigma_g*\mathrm{id}"'] \\
(F'g*F'f)*S_x \arrow[d, "\mu'_{g,f}*\mathrm{id}"'] & & & S_z*(Fg*Ff) \arrow[u, "(\alpha'_{S_z,Fg,Ff})^{-1}"'] \arrow[d, "\mathrm{id}*\mu_{g,f}"] \\
F'(g*f)*S_x & & & S_z*F(g*f) \arrow[lll, "\sigma_{g*f}"]
\end{tikzcd}$$

(pentagon axiom)

$$\begin{tikzcd}
I'_{F'x}*S_x \arrow[r, "\lambda'_{S_x}"] \arrow[d, "\iota'_x*\mathrm{id}"'] & S_x \arrow[r, "(\rho'_{S_x})^{-1}"] & S_x*I'_{Fx} \arrow[d, "\mathrm{id}*\iota_x"] \\
F'I_x*S_x & & S_x*FI_x \arrow[ll, "\sigma_{I_x}"]
\end{tikzcd}$$

5.1.2 Op-transformations between op-morphisms of bicategories

For an op-transformation between op-morphisms of bicategories, we have the following axioms:

(octagon axiom)

$$
\begin{array}{ccccccc}
F'g*(F'f*S_x) & \xleftarrow{\mathrm{id}*\sigma_f} & F'g*(S_y*Ff) & \xleftarrow{\alpha'_{F'g,S_y,Ff}} & (F'g*S_y)*Ff & \xleftarrow{\sigma_g*\mathrm{id}} & (S_z*Fg)*Ff \\
\Big\downarrow {\scriptstyle (\alpha'_{F'g,F'f,S_x})^{-1}} & & & & & & \Big\uparrow {\scriptstyle (\alpha'_{S_z,Fg,Ff})^{-1}} \\
(F'g*F'f)*S_x & & & & & & S_z*(Fg*Ff) \\
\Big\uparrow {\scriptstyle \mu'_{g,f}*\mathrm{id}} & & & & & & \Big\uparrow {\scriptstyle \mathrm{id}*\mu_{g,f}} \\
F'(g*f)*S_x & & & \xleftarrow{\qquad\sigma_{g*f}\qquad} & & & S_z*F(g*f)
\end{array}
$$

(pentagon axiom)

$$
\begin{array}{ccccc}
I'_{F'x}*S_x & \xrightarrow{\lambda'_{S_x}} & S_x & \xrightarrow{(\rho'_{S_x})^{-1}} & S_x*I'_{Fx} \\
\Big\uparrow {\scriptstyle \iota'_x*\mathrm{id}} & & & & \Big\uparrow {\scriptstyle \mathrm{id}*\iota_x} \\
F'I_x*S_x & & \xleftarrow{\qquad\sigma_{I_x}\qquad} & & S_x*FI_x
\end{array}
$$

5.1.3 Transformations between op-morphisms of bicategories

For a transformation between op-morphisms of bicategories, we have the following axioms:

(octagon axiom)

$$
\begin{array}{ccccccc}
F'g*(F'f*S_x) & \xrightarrow{\mathrm{id}*\sigma_f} & F'g*(S_y*Ff) & \xrightarrow{(\alpha'_{F'g,S_y,Ff})^{-1}} & (F'g*S_y)*Ff & \xrightarrow{\sigma_g*\mathrm{id}} & (S_z*Fg)*Ff \\
\Big\uparrow {\scriptstyle \alpha'_{F'g,F'f,S_x}} & & & & & & \Big\downarrow {\scriptstyle \alpha'_{S_z,Fg,Ff}} \\
(F'g*F'f)*S_x & & & & & & S_z*(Fg*Ff) \\
\Big\uparrow {\scriptstyle \mu'_{g,f}*\mathrm{id}} & & & & & & \Big\uparrow {\scriptstyle \mathrm{id}*\mu_{g,f}} \\
F'(g*f)*S_x & & & \xrightarrow{\qquad\sigma_{g*f}\qquad} & & & S_z*F(g*f)
\end{array}
$$

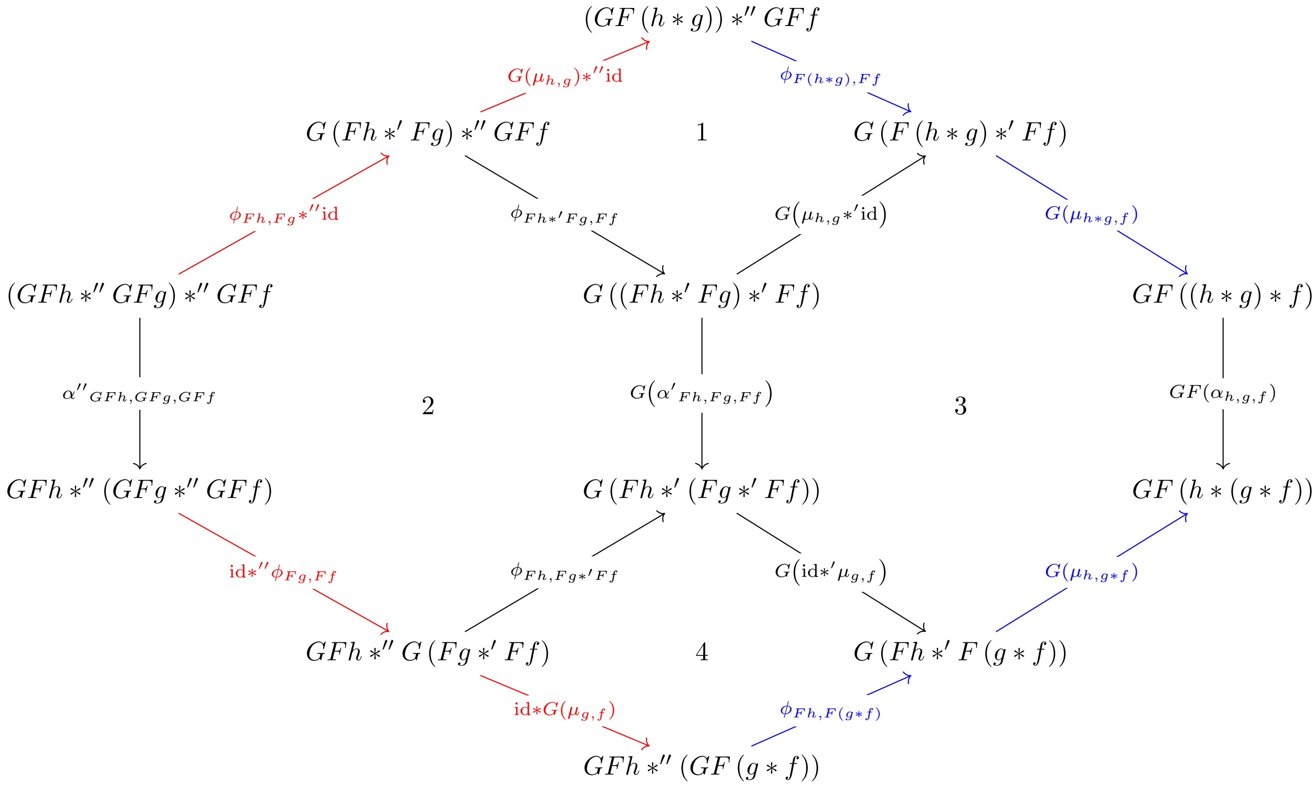
(pentagon axiom)

$$
\begin{array}{ccccc}
I'_{F'x}*S_x & \xrightarrow{\lambda'_{S_x}} & S_x & \xrightarrow{(\rho'_{S_x})^{-1}} & S_x*I'_{Fx} \\
\Big\uparrow {\scriptstyle \iota'_x*\mathrm{id}} & & & & \Big\uparrow {\scriptstyle \mathrm{id}*\iota_x} \\
F'I_x*S_x & & \xrightarrow{\qquad\sigma_{I_x}\qquad} & & S_x*FI_x
\end{array}
$$

5.2 Proof of Proposition 49

We give a proof of Proposition 49

Proof. Consider the following diagram for any 1-cells $w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$ in B.

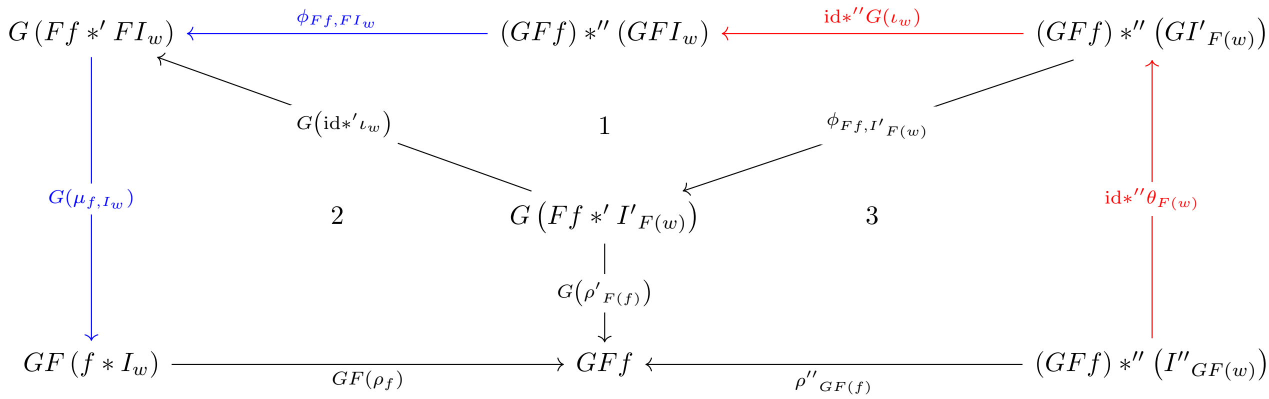


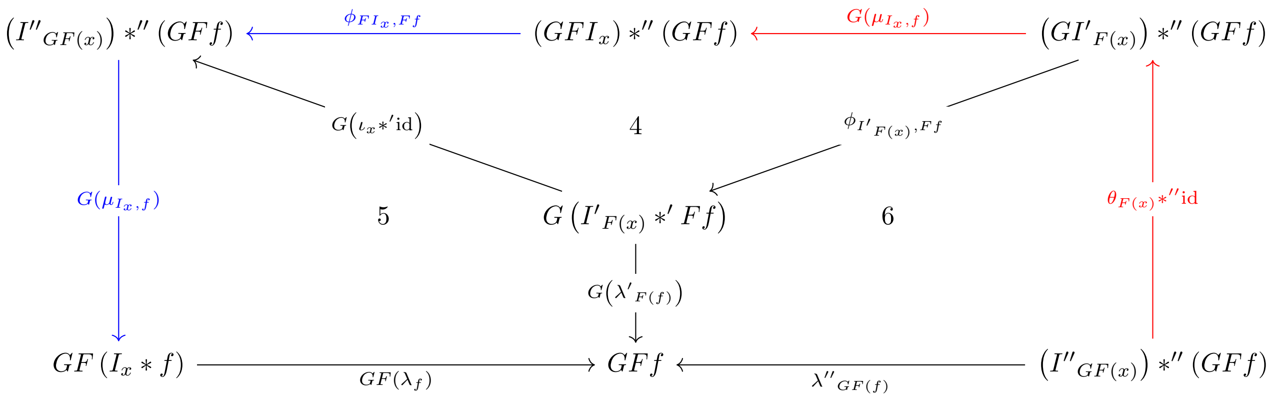
We observe that

- The paths in blue define the component multiplications of the composition, and the upper path in red is the component multiplication horizontally pre-composed with GFf, while the lower path in red is the same but horizontally pre-composed with GFh. Hence the boundary is the required hexagon axiom for the composite morphism of bicategories.
- 6 and 3 commute by the naturality of ϕ.
- 4 is the hexagon axiom for ϕ.
- 5 is the image of the hexagon axiom for μ under the functor $G_{F(w),F(z)}$.

and hence the boundary commutes so that the composite morphism of bicategories satisfies the hexagon axiom.

To show that the triangle axioms hold for the units of the composite GF, we consider the following diagrams:





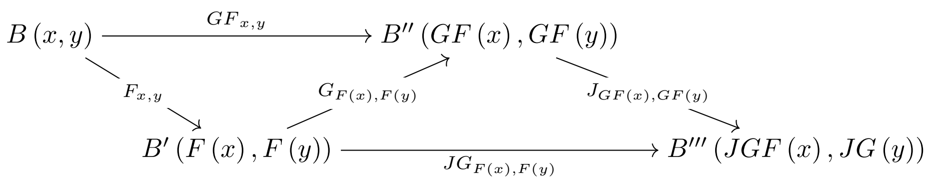
Indeed, we observe that

- The paths in red define the component identity of the composite morphism of bicategories, horizontally post-composed and pre-composed with GFf in the upper and lower diagrams respectively. Similarly, the paths in blue define the multiplication of the composite, so that the boundary of the upper and lower diagrams are the required right and left triangle axioms for $(GF, G\mu * \phi_F, G\iota * \theta_F)$ respectively.
- 3 commutes by the naturality of ϕ
- 4 is the image of the right triangle axiom for F under G
- 5 is the right triangle axiom for G on Ff
- 6, 7, 8, 9, and 10 commute as they are dual to 1, 2, 3 ,4 and 5 respectively.

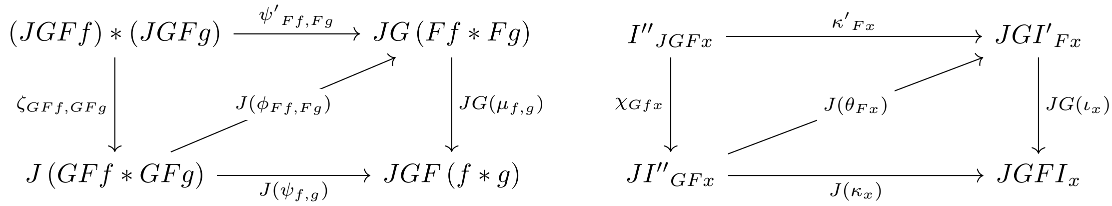
Hence the above diagrams commute. So the composite of morphisms as defined above satisfy the triangle identities, and hence is indeed also a morphism, of bicategories as required. □

5.3 Proof of Corollary 50

Proof. Denote $\psi := G\mu * \phi_F$, $\kappa := G\iota * \theta_F$, $\psi' := J(\phi) * \zeta_G$ and $\kappa' := J(\theta) * \chi_G$ for convenience. We first observe that the identity homomorphisms as defined above are indeed identities under composition of morphisms. Next, we show that composition of morphisms is associative. Indeed, the functions on 0-cells $(J \circ G) \circ F$ and $J \circ (G \circ F)$ are equal, as function composition is associative, and similarly the functors between hom-categories $J_{GF(x),GF(y)}\left(G_{F(x),F(y)}F_{x,y}\right)$ and $\left(J_{GF(x),GF(y)}G_{F(x),F(y)}\right)F_{x,y}$ as seen in the following diagram are also equal, as functor composition is also associative.



Then for component the multiplications and units, we consider the following diagrams.

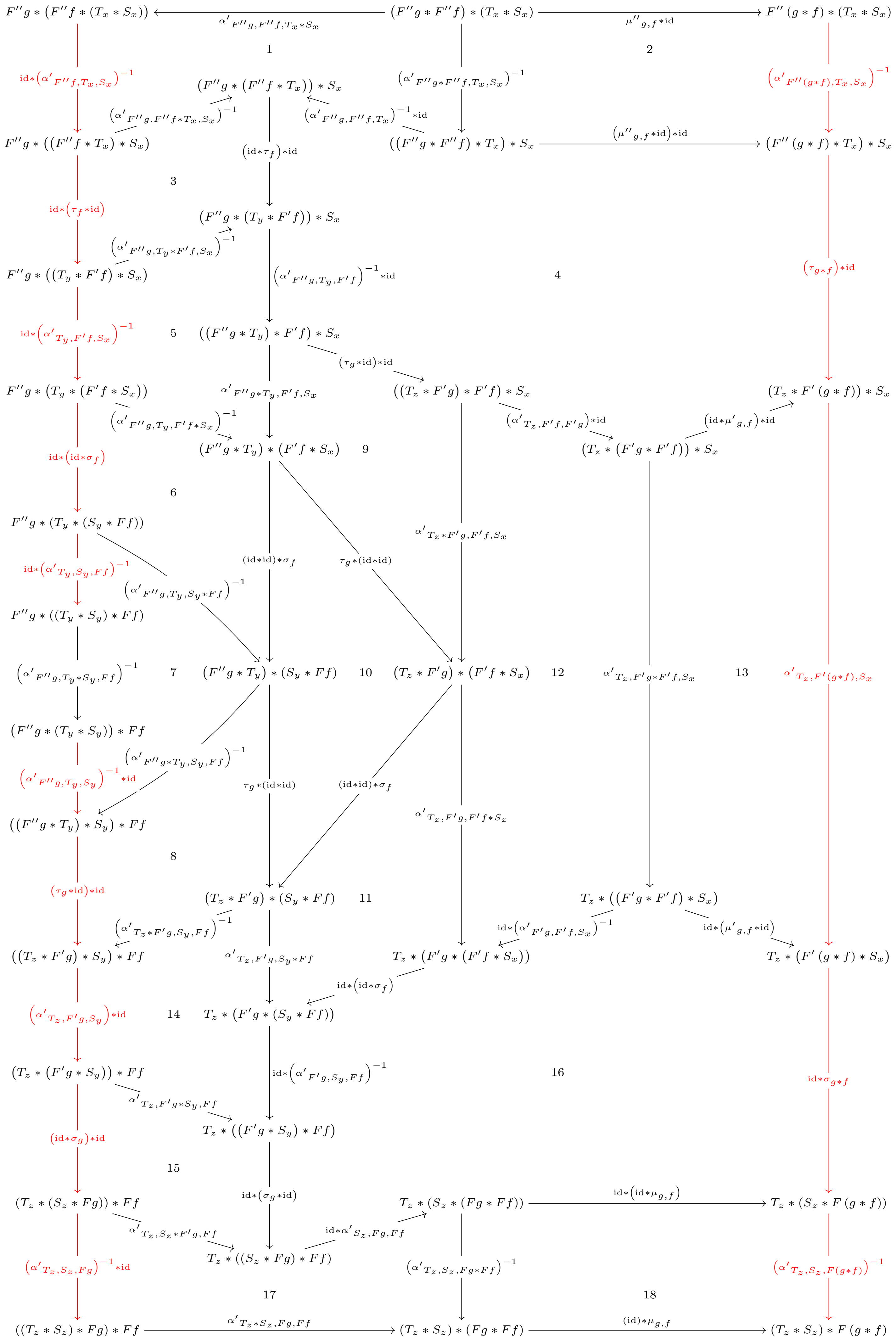


In both diagrams, the upper triangles commute by the definitions of κ' and ψ', while the bottom triangles commute as they are the images of the definitions of ψ and κ under J. The commutativity of these diagrams expresses the associativity of component multiplication and unit natural transformations under composition of morphisms of bicategories. Hence $\mathbf{Bicat_1}$ as defined above forms a category. □

5.4 Proof of the octagon axiom for the vertical composite of transformations

We complete the proof for the octagon axiom for the vertical composition of transformations in the diagram on the next page. In it, we observe that

- The path in red on he right defines $(\tau \circ \sigma)_{g*f}$, the path in red on the top left is the definition of $(\tau \circ \sigma)_f$ horizontally post-composed with $F''g$, and $(\tau \circ \sigma)_g$ horizontally pre-composed with Ff. Hence the boundary gives the required octagon axiom for $(S \circ T, \sigma \circ \tau)$.
- 4 and 16 are octagon axioms for (T, τ) and (S, σ) respectively.
- 1, 5, 7, 12, 14 and 17 are associativity pentagons, possibly with directions of some arrows reversed.
- 2, 3, 6, 9, 8, 11, 13 and 18 are naturality squares for α'
- 10 commutes by naturality of σ, or equivalently by naturality of τ.



6 Bibliography

References

[1] Baez J. C. (1997) *An Introduction to n-Categories*, 7th Conference on Category Theory and Computer Science, eds. E. Moggi and G. Rosolini, Lecture Notes in Computer Science vol. 1290, Springer, Berlin, 1997, pp. 1-33

[2] Balkovic I. *Bicategorical Yoneda Lemma* Available online at `http://www.irb.hr/korisnici/ibakovic/yoneda.pdf`

[3] Batanin M. (2002) The Eckmann Hilton Argument and Higher Operads. Advanced Mathematics 2008.

[4] Bastiani A. & Ehresmann C. *Catégories de foncteurs structurés*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 11 no. 3 (1969), p. 329-384

[5] Bastiani A. & Ehresmann C. *Categories of sketched structures*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 13 no. 2 (1972), p. 104-214

[6] Benabou J. (1967) *Introduction to Bicategories* In: Reports of the Midwest Category Seminar. Lecture Notes in Mathematics, vol 47. Springer, Berlin, Heidelberg

[7] (1998) Crans S. *Localization of Functors*. Available online at `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.26.506&rep=rep1&type=pdf`

[8] Feferman S., Kreisel G. (1969) *Set-Theoretical foundations of category theory.* In: Reports of the Midwest Category Seminar III. Lecture Notes in Mathematics, vol 106. Springer, Berlin, Heidelberg

[9] Gurski N. (2006) *An algebraic theory of tricategories*. PhD thesis, University of Chicago. Available online at `http://www.math.yale.edu/~mg622/tricats.pdf`

[10] Johnstone P. (2002) *Sketches of an Elephant: A Topos Theory Compendium.* Oxford Logic Guides.

[11] Joyal A. & Street R. (1991) *The Geometry of Tensor Calculus, I.* Advances in Mathematics, Vol 88, Issue 1.

[12] Joyal A. & Street R. (1991) *The Geometry of Tensor Calculus, II.* Advances in Mathematics, Vol 88, Issue 1.

[13] Joyal A. & Street R. (1993) *Braided Tensor Categories.* Advances in Mathematics, Vol 102, Issue 1.

[14] Kelly G. M. *Basic Concepts of Enriched Category Theory.* In: Reprints in Theory and Applications of Categories (2005).

[15] Lack S. *A 2-categories Companion.* In *Towards Higher Categories*, eds. John C. Baez and J. Peter May, Springer, (2009).

[16] Lack S. (2010) *Icons*, Applied Categorical Structures, 18(3):289-307.

[17] Leinter T. (2003) *Higher Operads, Higher Categories*

[18] Leinster T. (1998) *Basic Bicategories.* Available online at `http://arxiv.org/abs/math/9810017`

[19] MacLane S. (1978) *Categories for the Working Mathematician*, Springer-Verlag, New York

[20] Power A.J. *A 2-categorical pasting theorem.* Journal of Algebra, 129. 1988. p439-445

[21] Power A.J. (1991) *An n-categorical pasting theorem.* In: Carboni A., Pedicchio M.C., Rosolini G. (eds) Category Theory. Lecture Notes in Mathematics, vol 1488. Springer, Berlin, Heidelberg

[22] Verity D. (1992) *Enriched Categories, Internal Categories and Change of Base.* PhD thesis, Cambridge University. Reprints in Theory and Applications of Categories No. 20, (2001) pages 1-266

[23] *Coherence Theorems in Two-Dimensional Category Theory.* Available online at `http://www.math.utah.edu/~honigs/coherence_essay.pdf`