GEOMETRIC INVARIANT THEORY AND REPRESENTATIONS OF QUIVERS

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November 2022

Submitted in partial fulfillment of the requirements of the degree of
Bachelor of Science with Honours

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Acknowledgements

Firstly I would like to thank my supervisor Daniel Chan for his encouraging guidance throughout the year. I value his passion for the subject and eagerness to teach it.

Next, I am obliged to thank my friends and family, whom have all supported this endeavour, especially in the past few weeks before it's completion.

Finally, thanks also go to all the UNSW teaching staff I've been taught by in the past 5 years of study, as they have led me to this moment.

Abstract

Alastair D. King extends the notion of stability from Geometric Invariant Theory to the topic of quiver representations in his paper "Moduli of Representations of Finite Dimensional Algebras" [King94]. We summarise his contributions in the representation of quivers and explore two examples which each offer extensive analysis through the lens of affine invariant theory.

Introduction

A quiver is simply a directed graph. Representations of quivers are affine spaces susceptible to the group actions of matrix conjugation. Through Geometric Invariant Theory and the Mumford numerical criterion, Alastair D. King in his paper "Moduli of Representations of Finite Dimensional Algebras" [King94] presented a framework for studying the notions of stability in quiver representations. The goal of this thesis is to analyse King's notions of stability in quivers using tools from Geometric Invariant Theory, and furthermore prove a result with respect to a specific quiver. To achieve this, however, the readers require an introduction in algebraic geometry.

Algebraic geometry is the study of the zero locus of polynomials in affine or projective space. Where other fields in pure mathematics may also study the zeroes of sets of multivariate polynomials, algebraic geometry distinguishes itself by focusing on their intrinsic properties. Objects of importance in algebraic geometry are these zero loci, also known as varieties. One can study varieties using sheaves, although this is outside the scope of the thesis. Instead we choose to delve into the algebraic side of the field, which involves associating these geometric objects with an algebraic structure. We will first encounter this through the algebra-geometry duality, which invokes a correspondence between morphisms of varieties and morphisms of 'sufficiently nice' algebras over the same field.

Geometric Invariant Theory (GIT) uses algebraic geometry as a foundation to form quotients of group actions on affine and projective spaces. The goal is for these quotients to have a geometric structure, and furthermore parameterise the orbits. The first and most naive choice for a quotient is the set of all group orbits. This proves to be sufficient in some cases, but has no geometric structure. The next quotient we study is unremarkably called the GIT quotient and is denoted $X/\!\!/G$ for a group G acting on an affine space X. This quotient is again sufficient in some but not all cases, as orbits may sometimes coalesce to a single point. When a quotient is sufficient, we have criteria for this and may even denote it as a *geometric quotient*, implying that it has given us the geometric structure that we desire.

With this groundwork in place, we set our sights on quiver representations. Quivers are directed graphs with vector spaces for vertices and linear maps for edges, or arrows as we will call them. A representation of a quiver is the choice of such linear maps, and we denote the set of all possible choices as the representation space. We can act on this space by conjugation, that is, performing a change of basis on each vector space and respectively acting each map. But linear algebra tells us this does not change the maps at a fundamental level. Thus the goal in studying representations of quivers is to parameterise the isomorphism classes of the representations. This requires the use of GIT to select a quotient which will parameterise the classes and furthermore separate each of the orbits. Sometimes this can only be achieved by first restricting the domain of the quotient map, which requires determining the points that need to be removed. This is accomplished through the notion of stability, which the Hilbert-Mumford numerical criterion covers.

It is worth mentioning that the focus of King's paper is moduli spaces, which are not covered in the thesis. As a brief introduction out of interest; moduli spaces are geometric spaces which parameterise other objects of some fixed kind or their isomorphism classes. For example, the complex projective space $\mathbb{CP}^n$ is a moduli space which parameterises the space of all lines in $\mathbb{C}^{n+1}$ which pass through the origin.

The thesis assumes that readers are familiar with the tools and theorems in algebraic geometry up to some light modules and representation theory. This mainly includes having a grasp of algebras. The texts which this thesis relies on, primarily Hoskins' "Moduli Problems and Geometric Invariant Theory" [Hos16], use the notion of schemes. Rather for our purposes, we only care about varieties. Thus when referring to results in the text, the two terms may be freely interchanged.

The thesis covers the necessary definitions and results in algebraic geometry in Chapter 1. We then use this knowledge to explore quotients in Geometric Invariant Theory throughout Chapter 2. Finally, quivers and their representations are introduced in Chapter 3, which concludes by utilising these newfound techniques to extract results from a simple quiver and then a more well-known one.

Contents

Chapter 1

Algebraic Geometry

Algebraic geometry is the study of the zero locus of polynomials in affine and projective space, later known as varieties. We will introduce notation and common results of the theory, so proofs may be omitted. This chapter is based on Robin Hartshorne's book *Algebraic Geometry* [Hart10]. For further reading, see *Basic algebraic geometry* by Igor R. Shafarevich [Shaf94]. We begin with the affine case and generalise our concepts later to the projective world.

1.1 Affine Algebraic Geometry

In this section we deal with the zero locus of polynomials as a subset of affine space. It is classically denoted $\mathbb{A}^n_k$, where k is a field. We only work over the complex numbers and thus the n-dimensional affine space is just denoted $\mathbb{C}^n$ for our purposes. We begin by defining the zero locus of polynomials.

Definition 1.1.1. Given a subset $S \subseteq \mathbb{C}[x_1, \dots, x_n]$, we have the *vanishing set*

$$V(S) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\,,\, \forall f \in S\}.$$

If $I \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$ is an ideal, then we call $V(I)$ an *affine algebraic set.*

In fact, if $X \subseteq \mathbb{C}^n$ is an affine algebraic set if $X = V(I)$ for some ideal $I \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$. Note that $V(\langle f_1, \dots, f_r\rangle) = V(f_1, \dots, f_r)$ for $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_n]$. That is, the ideal generated by a set produces the same vanishing set as the vanishing set of all its generators.

Example 1.1.2. Consider the affine algebraic sets $V(x)$, $V(y) \subseteq \mathbb{C}^2$. Then $V(x) \cup V(y) = V(xy)$ is the union of the x and y axes and $V(x) \cap V(y) = V(x, y)$ is the single point $\{(0, 0)\}$.

This example motivates the following. Given two ideals $I, J \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$, we have

$$V(I) \cup V(J) = V(IJ), \qquad V(I) \cap V(J) = V(I + J).$$

Where $IJ = \{\sum fg : f \in I,\, g \in J\}$ only contains finite sums. We also have the two cases $V(\langle 0\rangle) = \mathbb{C}^n$ and $V(\mathbb{C}[x_1, \dots, x_n]) = \emptyset$. This lays the foundation for a topology.

Definition 1.1.3. The *Zariski topology* on $\mathbb{C}^n$ consists of Zariski-open/closed sets. A subset $X \subseteq \mathbb{C}^n$ is *Zariski-closed* if X is an algebraic set. The complement of a Zariski-closed set is *Zariski-open.*

The 'affine' in the definition was purposefully omitted as it is also applicable to projective algebraic sets, which will come later. The Zariski topology can be induced on any affine algebraic set X. Furthermore, a nonempty subset $Y \subseteq X$ of a topological space X is called *irreducible* if it cannot be expressed as the union of two closed subsets of $\mathbb{C}^n$. That is, there does not exist Y_1, $Y_2 \subseteq Y$ such that $Y = Y_1 \cup Y_2$.

For example, $\mathbb{C}$ is irreducible, as its Zariski-closed subsets consist of a finite set of points, yet $\mathbb{C}$ itself is infinite. This allows us to define the main object of study in algebraic geometry, varieties.

Definition 1.1.4. An *affine variety* is an irreducible affine algebraic set. They are closed in the Zariski topology. An open subset of an affine variety is called a *quasi-affine variety.*

The vanishing set maps ideals to algebraic sets, now we wish to define the reverse. Consider a subset $X \subseteq \mathbb{C}^n$. Then *the ideal of* X *in* $\mathbb{C}[x_1, \dots, x_n]$ is defined as

$$I(X) = \{f \in \mathbb{C}[x_1, \dots, x_n] \ : \ f(x) = 0, \, \forall x \in X\}.$$

We now have the means to study the relationship between subsets of $\mathbb{C}^n$ and ideals in $\mathbb{C}[x_1, \dots, x_n]$. Some properties of this relationship are as follows.

i) If $I_1 \subseteq I_2 \subseteq \mathbb{C}[x_1, \dots, x_n]$, then $V(I_1) \supseteq V(I_2)$.
ii) If $Y_1 \subseteq Y_2 \subseteq \mathbb{C}^n$, then $I(Y_1) \supseteq I(Y_2)$.
iii) If $I \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$ is an ideal, then $I(V(I)) = \sqrt{I}$ is the radical of I.
iv) If $Y \subseteq \mathbb{C}^n$ is a subset, then $V(I(Y)) = \overline{Y}$ is the Zariski-closure of Y.

Where $\sqrt{I} = \{f \in \mathbb{C}[x_1, \dots, x_n] : f^n \in I \text{ for some } n > 0\}$. Note (iii) is a consequence of the Hilbert's Nullstellensatz (theorem about zero loci). This theorem is stated in the proof of Hartshorne's text [Hart10] Proposition 1.2.

Note if $X \subseteq \mathbb{C}^n$ is an affine variety then $X \setminus V(f)$ is too for some $f \in \mathbb{C}[x_1, \dots, x_n]$. Thus a curious fact follows. The punctured affine line $\mathbb{C} \setminus \{0\}$ is affine. However $\mathbb{C}^2 \setminus \{0\}$ is not affine. The proof is tedious so we omit it.

This relationship between the algebraic sets and ideals can be more precisely described by restricting to radical ideals, which are ideals equal to their own radical.

Corollary 1.1.5. The correspondence between algebraic sets in $\mathbb{C}^n$ and radical ideals in $\mathbb{C}[x_1, \dots, x_n]$ given by $Y \mapsto I(Y)$ and $I \mapsto V(I)$ is one-to-one and inclusion-reversing. Furthermore, an algebraic set is irreducible if and only if its ideal is prime. ([Hart10] Corollary 1.4)

Note that prime ideals are their own radical. For example, $\mathbb{C}^n$ is irreducible as it corresponds to prime ideal $\langle 0\rangle$. This correspondence is an elementary version of the algebra-geometry duality which will come up later.

Recall for $X \subseteq \mathbb{C}^n$ that $I(X)$ is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$ so it is natural to take the quotient.

Definition 1.1.6. Given $X \subseteq \mathbb{C}^n$, the *coordinate ring* of X is

$$\mathbb{C}[X] := \frac{\mathbb{C}[x_1, \ldots, x_n]}{I(X)}.$$

If X is an affine variety, then $I(X)$ is prime and so $\mathbb{C}[X]$ is an integral domain. So $\mathbb{C}[X]$ is a non-zero commutative ring with no non-zero zero divisors. Furthermore the coordinate ring is a finitely-generated $\mathbb{C}$-algebra with no non-zero nilpotents (that is, $f^n = 0$ implies $f = 0$ for $f \in \mathbb{C}[X]$).

In fact, any nilpotent-free, finitely generated $\mathbb{C}$-algebra can be expressed in this form. Consider a $\mathbb{C}$-algebra A with generators $\{a_1, \ldots, a_m\}$. Then we have the $\mathbb{C}$-algebra homomorphism given by the evaluation map $\phi_a : \mathbb{C}[x_1, \ldots, x_m] \to \mathbb{C}$; $x_i \mapsto a_i$, where $a = (a_1, \ldots, a_m)$. Applying the first isomorphism theorem gives the desired result. This induces the following duality.

$$\begin{aligned} \{\text{Affine varieties}\} &\longleftrightarrow \{\text{Finitely-generated } \mathbb{C}\text{-algebras}\} \\ X &\longmapsto \mathbb{C}[X] \\ V(\ker \phi_a) &\longleftarrow\!\!\shortmid A \end{aligned}$$

Note the $\mathbb{C}$-algebras A are defined to be integral domains, as the coordinate ring $\mathbb{C}[X]$ of an affine variety X is an integral domain. We can denote $\operatorname{Spec} A := V(\ker \phi_a)$, which will act as a working definition and be precisely defined later in this section. We have the tools to interpret varieties, but the relationship between two varieties is still unknown. The following definition will aid in this.

Definition 1.1.7. An *affine morphism* between two varieties $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ is a map $\varphi : X \to Y$ of the form

$$\varphi = (\varphi_1, \ldots, \varphi_m).$$

Where $\varphi_i \in \mathbb{C}[X]$. If φ is bijective and it has an affine morphism for an inverse, then φ is an *affine isomorphism*.

Historically in algebraic geometry such morphisms are called *regular maps* as each $\varphi_i : \mathbb{C}[X] \to \mathbb{C}$ is known as a regular function, but we will not use this notation.

Example 1.1.8. Consider $X = V(y - x^2) \subseteq \mathbb{C}^2$. The polynomial $y - x^2 \in \mathbb{C}[x, y]$ is irreducible and thus so is X, so it is an affine variety. Then the map $\varphi : \mathbb{C} \to X;\ t \mapsto (t, t^2)$ is an affine isomorphism with inverse map $\varphi^{-1} : X \to \mathbb{C};\ (x, y) \mapsto x$.

Isomorphic varieties can be pictured as the same curve, but embedded into different spaces. That is, they are topologically equivalent, although we will not study this extensively. The notion of isomorphism can also be extended to $\mathbb{C}$-algebras through coordinate rings.

Definition 1.1.9. The *pullback* of an affine morphism $\varphi : X \to Y$ is the $\mathbb{C}$-algebra homomorphism φ^* defined by

$$\begin{aligned} \varphi^* : \mathbb{C}[Y] &\longrightarrow \mathbb{C}[X], \\ f &\longmapsto f \circ \varphi. \end{aligned}$$

The composition of affine morphisms is affine, so $f \circ \varphi$ is an affine morphism.

Observe the action of the pullback on the previous example.

Example 1.1.10. The coordinate ring of $X = V(y-x^2)$ is $\mathbb{C}[X] = \mathbb{C}[x,y]/\langle y - x^2\rangle$. The coordinate ring of $\mathbb{C}$ is just $\mathbb{C}[t]$. The homomorphism ϕ^* is determined uniquely by the generators of $\mathbb{C}[X]$ as it is $\mathbb{C}$-linear. Thus we have

$$\varphi^*(1) = 1, \qquad \varphi^*(x) = t, \qquad \varphi^*(y) = t^2.$$

But $\{1, t\}$ generate $\mathbb{C}[t]$ and thus φ^* is an isomorphism between $\mathbb{C}[X]$ and $\mathbb{C}[t]$.

As by the example, the two affine varieties and their respective coordinate rings are both isomorphic. This is no coincidence and is highlighted by the algebra-geometry duality.

Theorem 1.1.11 (Algebra-Geometry Duality). We know a correspondence exists between the category of affine varieties over $\mathbb{C}$ and the category of nilpotent-free, finitely generated $\mathbb{C}$-algebras. We may extend this correspondence to the category of affine variety morphisms over $\mathbb{C}$ and the category of $\mathbb{C}$-algebra homomorphisms.

$$\left\{\begin{matrix}\text{Affine variety morphisms} \\ X \to Y\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{C}\text{-algebra homomorphisms} \\ \mathbb{C}[Y] \to \mathbb{C}[X]\end{matrix}\right\}$$

$$\begin{aligned} \varphi &\longmapsto \varphi^* \\ \operatorname{Spec}\phi &\longleftarrow\!\shortmid\ \phi \end{aligned}$$

([Hart10] Proposition 3.5)

The forward map is naturally induced through the pullback. The backwards map is defined as follows. Given a $\mathbb{C}$-algebra homomorphism $\phi : B \to A$, then $\operatorname{Spec}\phi : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of affine varieties. Note that $\operatorname{Spec}\mathbb{C}[X] = X$.

Algebraically, $\operatorname{Spec} R$ of a commutative ring R is the set containing all prime ideals of R, called the *spectrum* of R. We depart from this algebraic notion, however. Instead, we view it analogous to taking the vanishing set of a coordinate ring, which should grant us an affine variety. Thus let A be a nilpotent-free, finitely generated $\mathbb{C}$-algebra. We can define the Zariski topology on $\operatorname{Spec} A$. For any $f \in A$, define the set $D_f := \operatorname{Spec} A \setminus V(f)$ which is open as $V(f)$ is closed. It has coordinate ring as follows.

$$\mathbb{C}[D_f] = \mathbb{C}[X][f^{-1}].$$

We call D_f a *principal open set* and the set of all of these forms a basis for the Zariski topology on $\operatorname{Spec} A$. Additionally, each D_f is isomorphic to an affine variety. For further reading on the spectrum of a ring, the text *Basic Algebraic Geometry 2* by Shafarevich [Shaf13] introduces the notion in order to study sheaves, but this is not required for the scope of the thesis.

It is worth mentioning this construction is based on the localisation of prime ideals. Let $f \in A$ be irreducible, so $\langle f \rangle \trianglelefteq A$ is a principal ideal and thus $A \setminus \langle f \rangle$ is multiplicatively closed as $\langle f \rangle$ is prime. Then we have the *localisation* of $\langle f \rangle$ in A defined as

$$A_{\langle f \rangle} = \left\{ \frac{a}{s} \;:\; a \in A,\, s \in A \setminus \langle f \rangle \right\}.$$

This is used to prove various corollaries regarding Spec. For more reading, seek Hartshorne's text [Hart10].

Finally, the duality poses an identification for isomorphisms in each category.

Corollary 1.1.12. An affine variety morphism $\varphi : X \to Y$ is an isomorphism if and only if $\varphi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ is an isomorphism.

1.2 Projective Algebraic Geometry

We will work over the complex projective space $\mathbb{CP}^n$ for the rest of the thesis. Thus we may omit the '$\mathbb{C}$' and just denote it $\mathbb{P}^n$. We wish to define the same notions of affine algebraic geometry but in projective space.

Definition 1.2.1. The n-dimensional complex *projective space* $\mathbb{P}^n$ is defined by

$$\mathbb{P}^n := \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\sim}.$$

Where $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if $(x_0, \ldots, x_n) = \lambda(y_0, \ldots, y_n)$ for some $\lambda \in \mathbb{C}^*$. Coordinates in $\mathbb{P}^n$ are of the form $(x_0 : \ldots : x_n)$.

Our goal is to define projective varieties, but in order to do this we first need to define ideals of polynomials in projective space. This requires homogeneous polynomials.

Definition 1.2.2. A polynomial $F \in \mathbb{C}[x_0, \ldots, x_n]$ is *homogeneous* if and only if it satisfies the following equation.

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n), \quad \forall \lambda \in \mathbb{C}^*,$$

where $d = \deg F$.

Thus a homogeneous polynomial has a well-defined vanishing set. We can capture these using the homogeneous ideal.

Definition 1.2.3. An ideal $I \trianglelefteq \mathbb{C}[x_0 \ldots, x_n]$ is called *homogeneous* if it is generated by homogeneous polynomials $F_1, \ldots, F_r$.

We define the vanishing set such that it only considers the vanishing of homogeneous elements in $\langle F_1, \ldots, F_r \rangle$.

Definition 1.2.4. The *vanishing set* of a homogeneous ideal $I \trianglelefteq \mathbb{C}[x_0, \ldots, x_n]$ is

$$V(I) := \{x \in \mathbb{P}^n : F(x) = 0 \text{ for all homogeneous } F \in I\}.$$

A subset $X \subseteq \mathbb{P}^n$ is a *projective algebraic set* if $X = V(I)$ for some homogeneous ideal $I \trianglelefteq \mathbb{C}[x_0, \ldots, x_n]$. If X is irreducible then it is called a *projective variety.*

Just as in the affine case the union and intersection of projective algebraic sets are themselves projective algebraic sets. Setting algebraic subsets of $\mathbb{P}^n$ to be closed gives the same Zariski topology on $\mathbb{P}^n$. Similarly, an open subset of a projective algebraic set is called *quasi-projective.*

Note the variety $V(\langle x_0, \ldots, x_n \rangle) \subseteq \mathbb{P}^n$ gives the empty set, rather than being equal to $\{0\}$ in the affine case.
The set of polynomials vanishing on a set $X \subseteq \mathbb{P}^n$ is given by

$$I(X) := \{F \in \mathbb{C}[x_0, \ldots, x_n] : F(x) = 0 \text{ for all } x \in X\}.$$

The set $I(X)$ is a homogeneous ideal of $\mathbb{C}[x_0, \ldots, x_n]$.

Recall an algebra A is $\mathbb{N}$-*graded* if it decomposes into a direct sum

$$A = \bigoplus_{n=0}^{\infty} A_i,$$

where A_i are additive groups such that $A_n A_m \subseteq A_{n+m}$. We now wish to define the coordinate ring in projective space.
Definition 1.2.5. Given $X \subseteq \mathbb{P}^n$, the *homogeneous coordinate ring* of X is

$$R(X) := \frac{\mathbb{C}[x_0, \ldots, x_n]}{I(X)}.$$

It is a graded $\mathbb{C}$-algebra with grading

$$R(X) = \bigoplus_{d \geq 0} R(X)_d,$$

where $R(X)_d$ are the homogeneous polynomials of degree d modulo $I(X)$.

A construction that will be used later is the gluing of affine varieties to get projective varieties. Consider the patch U_i of $\mathbb{P}^n$ given by

$$U_i := \{(x_0 : \ldots : x_n) \in \mathbb{P}^n : x_i \neq 0\},$$

for $i \in \{0, \ldots, n\}$. Thus we can define maps $\phi_i : U_i \to \mathbb{C}^n$ by

$$(x_0 : \ldots : x_n) \longmapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

The inverse map $\phi_i^{-1} : \mathbb{C}^n \to U_i$ is given by

$$(x_1, \ldots, x_n) \longmapsto (x_1 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n).$$

An example that occurs frequently in geometric invariant theory is the projective line $\mathbb{P}^1$. The two affine patches are given by

$$U_0 = \{(1 : x_1) \in \mathbb{P}^1\}, \qquad U_1 = \{(x_0 : 1) \in \mathbb{P}^1\}.$$

Their coordinate rings are then

$$\mathbb{C}[U_0] = \mathbb{C}[x_1/x_0], \qquad \mathbb{C}[U_1] = \mathbb{C}[x_0/x_1].$$

Another characterisation of $\mathbb{C}[U_0]$ is $\mathbb{C}[x_0, x_1][x_0^{-1}]_0$, where the 0 subscript denotes taking the degree 0 homogeneous elements. This is identical to taking the degree 0 element in the grading of $\mathbb{C}[x_0, x_1, x_0^{-1}]$. This method does not work in general, necessarily when the generators of the algebra are not of degree 1. This will be studied in the next chapter.

Chapter 2

Geometric Invariant Theory

Geometric Invariant Theory (GIT) studies the method of constructing quotients of geometric spaces by group actions. These geometric spaces are identical to affine varieties, although for the main results that follow in Chapter 3 we take the case that our affine variety is the whole space. For an affine variety X over field $\mathbb{C}$ and group G, we denote the group action as $g \cdot x$ for $g \in G$, $x \in X$. Furthermore if $f \in \mathbb{C}[X]$ we have $g \cdot f(x) = f(g \cdot x)$. One may think the orbit space $X/G := \{G \cdot x : x \in X\}$ will suffice, but this involves taking the quotient by an equivalence relation, which do not act well with the morphisms in algebraic geometry. Thus instead we study the G-invariant functions on X, denoted $\mathbb{C}[X]^G$ which is a subalgebra of $\mathbb{C}[X]$ given that X is an affine variety. Furthermore we take its spectrum $\operatorname{Spec} \mathbb{C}[X]^G$ and denote this as $X/\!\!/G$, the *GIT quotient*. This quotient morphism $X \to X/\!\!/G$ is constructed to identify orbits whose closures intersect.

Victoria Hoskins' *Moduli Problems and Geometric Invariant Theory* [Hos16] studies moduli spaces using geometric invariant theory. This chapter aims to extract the results on the level of algebraic geometry in the previous chapter.

2.1 Affine Geometric Invariant Theory

In this section we only consider affine varieties over $\mathbb{C}$. However we require more structure from the groups that we will be quotienting by.

Definition 2.1.1. An *affine algebraic group* G over $\mathbb{C}$ is an affine variety over $\mathbb{C}$ such that multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are affine morphisms.

An example is the group $\mathrm{GL}_n(\mathbb{C})$, which is an affine variety in $\mathbb{C}^{n^2}$ cut out by the points where the determinant is non-zero. A consequence of this definition is that affine algebraic groups act *rationally* on the $\mathbb{C}$-algebra $\mathbb{C}[X]$. That is, every element $f \in \mathbb{C}[X]$ is contained in a finite dimensional G-invariant linear subspace of $\mathbb{C}[X]$.

Another requirement is that our groups are reductive. Alternative definitions include linearly reductive and geometrically reductive. These definitions are cumbersome, but luckily in characteristic 0 the three notions coincide. For those interested, Theorem 4.16 in Hoskins' text [Hos16] proves this. Thus for the purposes of this thesis, we adhere to the following definition.

Definition 2.1.2. An affine algebraic group G is *reductive* if every finite dimensional linear representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is completely reducible, that is, the representation splits into a direct sum of irreducible representations.

Remark 2.1.3. Some examples of reductive groups are $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$ and finite groups. The proof that $\mathrm{GL}_n(\mathbb{C})$ is reductive is no trivial task and will thus be omitted.

The following theorem by Hilbert and Mumford gives a necessary characteristic to the quotients we will study. Note a theorem by Nagata gives the same result. Both theorems are referenced below.

Theorem 2.1.4. Let G be a reductive group acting on a finitely generated $\mathbb{C}$-algebra A. Then A^G is finitely generated.
([Hos16] Theorem 4.20, Theorem 4.25)

Definition 2.1.5. An action of an affine algebraic set G on an affine variety X is *closed* if all G-orbits of X are closed.

Example 2.1.6. Consider the group $\mathbb{C}^*$ acting on $\mathbb{C}^2_{x,y}$ by $t \cdot (x, y) = (tx, t^{-1}y)$. The action has the following distinct orbits.

- The origin $\{(0, 0)\}$,
- the punctured x-axis $\{(x, 0) : x \neq 0\}$,
- the punctured y-axis $\{(0, y) : y \neq 0\}$,
- the hyperbolas $\{(x, y) : xy = \alpha\}$ for all $\alpha \in \mathbb{C}^*$.

Note only the hyperbolas and origin are closed, whereas the closure of the punctured axes both differ by the origin.

Let G be an affine algebraic group acting on affine variety X, and $x \in X$. Recall the following notations.

i) The *orbit* $G \cdot x = \{g \cdot x : g \in G\} \subseteq X$.
ii) The *stabiliser* $G_x = \{g \in G : g \cdot x = x\} \leq G$.

For the remainder of this chapter, suppose that G is an affine algebraic group acting on an affine variety X. We wish to study different quotients for the action of G on X. The first quotient we study is the categorical quotient.

Definition 2.1.7. A *categorical quotient* of Y by G is given by the pair (Y, φ), where Y is an affine variety and $\varphi : X \to Y$ is a G-invariant morphism such that the universal property holds: For any G-invariant morphism $\varphi' : X \to Z$, there exists a unique morphism $\gamma : Z \to Y$ such that the diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle \varphi'}\downarrow & \nearrow_{\gamma} & \\ Z & & \end{array}$$

The categorical quotient is unique up to isomorphism.

For further examples of the categorical quotient, seek Le Potier's *Lectures on Vector Bundles* [LePo04] Chapter 6.

As φ is G-invariant and thus constant on orbit spaces, it is also constant on orbit closures. Thus a categorical quotient is an orbit space if the action of G on X is closed.

We now have the tools necessary to construct the affine GIT quotient.

Definition 2.1.8. The *GIT quotient* is the morphism $\varphi : X \to X/\!\!/G := \operatorname{Spec} \mathbb{C}[X]^G$ of affine varieties associated to the inclusion $\varphi^* : \mathbb{C}[X]^G \to \mathbb{C}[X]$ by the algebra-geometry duality.

In the general case we denote the quotient as $X/\!\!/G$. However in some nice cases, the quotient is in fact geometric and thus we can just use X/G for the orbit space. Consider the following example which highlights the downfalls of the GIT quotient.

Example 2.1.9. Let $\mathbb{C}^*$ act on $\mathbb{C}^n$ by $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$. Then the two types of orbits are as follows.

- The origin $\{0\}$,
- all punctured lines through the origin.

Again, every orbit contains zero in its closure. Note that $\mathbb{C}[X]^G = \mathbb{C}$ and thus our GIT quotient is just $\varphi : \mathbb{C}^n \to \operatorname{Spec} \mathbb{C} = \{0\}$. But φ is a G-invariant morphism and thus it has to be constant.

This issue arises from 0 being contained in the closure of every orbit. The geometric intuition is to remove that point and then take the quotient. For this example it is precisely the same process as constructing the complex projective space $\mathbb{P}^{n-1}$, which actually serves as an orbit space for the action.

We have discovered that the GIT quotient is not always the best option as a quotient. There is a notion of a good quotient, unremarkably named a *good quotient*. Hoskins' text [Hos16] covers this definition extensively, although for the purpose of this thesis only the following theorem is required.

Theorem 2.1.10. Let G be a reductive group acting on an affine variety X. Then the GIT quotient $\varphi : X \to X/\!\!/G$ exhibits the following properties.

i) $X/\!\!/G$ is an affine variety.
ii) φ is G-invariant and surjective.
iv) $(X/\!\!/G, \varphi)$ is a categorical quotient.

([Hos16] Definition 3.27, Theorem 4.30)

This theorem is pivotal in proving the results that follow in Chapter 3 involving GIT quotients of quiver representations. This proves GIT quotients give information on the G-orbits. However, they do not always parameterise the orbits precisely.

Definition 2.1.11. Let $\varphi : X \to Y$ be a quotient and $y \in Y$. Then if $\varphi^{-1}(y)$ only contains a single orbit, we say φ is a *geometric quotient*.

Example 2.1.12. Consider the setting from Example 2.1.6. So $G = \mathbb{C}^*$ acts on $X = \mathbb{C}^2_{x,y}$ by $t \cdot (x, y) = (tx, t^{-1}y)$. We wish to calculate $\mathbb{C}[x, y]^G$, the G-invariant subalgebra of $\mathbb{C}[x, y]$. Consider the decomposition of $\mathbb{C}[x, y]$:

$$\mathbb{C}[x, y] = \bigoplus_{i,j \geq 0} \mathbb{C}x^i y^j.$$

Thus we only have to consider the action on each monomial $x^i y^j$.

$$t \cdot (x^i y^j) = (tx)^i (t^{-1}y)^j = t^{i-j} x^i y^j.$$

Equating this to the original monomial gives the condition that $i = j$, that is

$$\mathbb{C}[x, y]^G = \bigoplus_{i \geq 0} \mathbb{C}(xy)^i = \mathbb{C}[xy] \cong \mathbb{C}[z].$$

Thus $X /\!\!/ G = \operatorname{Spec} \mathbb{C}[xy] = \mathbb{C}_{xy}$. So the GIT quotient is the affine morphism

$$\begin{aligned} \mathbb{C}^2_{x,y} &\longrightarrow \mathbb{C}_{xy}, \\ (x, y) &\longmapsto xy. \end{aligned}$$

This quotient is clearly G-invariant by construction and furthermore it is surjective. Also $X /\!\!/ G \cong \mathbb{C}_z$ is an affine variety. It is also a categorical quotient. Furthermore, note that $\varphi^{-1}(z) = H_z$ for $z \in \mathbb{C}^*$ and $\varphi^{-1}(0) = \{(0, 0)\}$. So the preimage of φ is always a single orbit and thus φ is a geometric quotient.

The following result is a corollary of the GIT quotient being a *good quotient*. For further reading on good quotients, seek Hoskins' text [Hos16].

Corollary 2.1.13. Let G be a reductive group acting on an affine variety X and let $\varphi : X \to X /\!\!/ G$ be the GIT quotient. Then for $x, x' \in X$,

$$\varphi(x) = \varphi(x') \iff \overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset.$$

Furthermore, for all $y \in X /\!\!/ G$, the preimage $\varphi^{-1}(y)$ contains a unique closed orbit. Additionally, if the action of G on X is closed, then φ is a geometric quotient. ([Hos16] Corollary 4.31)

Thus if φ is the GIT quotient, it must separate closed orbits.

As seen in earlier examples, our quotients are not necessarily geometric. We have concluded that a set of points needs to be removed before quotienting by the group. Specifically, the points which not need be removed are those which are contained in already closed orbits. These points are called *stable points*, which form an open subset in the space and furthermore induce a geometric quotient. First we need some facts of the G-orbits.

Proposition 2.1.14. Let G be an affine algebraic group and X an affine variety. Then

a) The boundary $\overline{G \cdot x} \backslash G \cdot x$ is the union of orbits with strictly smaller dimension.
b) Each orbit closure contains a closed orbit.
c) For all $x \in X$,

$$\dim G = \dim(G_x) + \dim(G \cdot x).$$

([Hos16] Proposition 3.20)

We may now present the definition for stability.

Definition 2.1.15. A point $x \in X$ is *stable* if its orbit is closed in X and $\dim G_x = 0$. Denote $X^s \subseteq X$ to be the set of stable points.

Note the condition $\dim G_x = 0$ is equivalent to $\dim G \cdot x = \dim G$.

Proposition 2.1.16. Let G be a reductive group acting on affine variety X and let $\varphi : X \to Y := X /\!\!/ G$ be the GIT quotient. Then

a) $X^s \subseteq X$ is an open G-invariant subset,
b) $Y^s := \varphi(X^s) \subseteq Y$ is open,
c) and $X^s = \varphi^{-1}(Y^s)$.

Thus $\varphi : X^s \to Y^s$ is a geometric quotient.
([Hos16] Proposition 4.36)

Observing the GIT quotient restricted to this open set instantly gives the results we are after.

Example 2.1.17. Consider the setting from example 2.1.9. So $G = \mathbb{C}^*$ acts on $X = \mathbb{C}^n$ by scalar multiplication. We wish to find X^s. Recall the orbits were the punctured lines and the origin, but the origin is the only closed orbit. However the stabiliser of G at zero is the entire group G, which has positive dimension. Thus $X^s = \emptyset$.

We defined stable points to determine the points to remove when quotienting, but this has not aided in the example. The solution actually requires semistable points, which we will introduce in the following section.

2.2 Projective Geometric Invariant Theory

So far we have only considered the quotient of a group action on an affine variety. We can also define the notion for the projective case. However before we delve into more invariant theory, we need to define the construction of Proj.

Proj is analogous to the Spec construction presented earlier, whereas it produces projective spaces instead of affine. Let R be a finitely generated $\mathbb{C}$-algebra. Suppose it is also graded as follows:

$$R = \bigoplus_{n \geq 0} R_n.$$

If R has generators $\{x_1, \ldots, x_n\}$, then $\operatorname{Proj} R$ has a finite open cover $\{U_i\}_{i=1}^{n}$ defined as

$$U_i := \operatorname{Spec}(R[x_i^{-1}]_0), \qquad U_i \cap U_j = \operatorname{Spec}(R[x_i^{-1}, x_j^{-1}]_0),$$

where the 0 subscript denotes the degree 0 subspace.
For example, consider $R = \mathbb{C}[x, y]$ which is graded by the additive group of homogeneous polynomials of equal degree. Then $\operatorname{Proj} \mathbb{C}[x, y]$ is covered by U_1, U_2 as follows.

$$\begin{aligned} U_1 &= \operatorname{Spec}(\mathbb{C}[x, y][x^{-1}]_0) \\ &= \operatorname{Spec}(\mathbb{C}[x, y, x^{-1}]_0) \\ &= \operatorname{Spec} \mathbb{C}[y/x] \\ &= \mathbb{C}_{y/x}. \end{aligned}$$

Similarly $U_2 = \mathbb{C}_{x/y}$. Taking the disjoint union we get

$$\operatorname{Proj} \mathbb{C}[x, y] = \mathbb{C}_{y/x} \sqcup \mathbb{C}_{x/y} = \mathbb{P}^1_{x:y}.$$

The final equality is a result of 'gluing' the two complex planes in order to form the Riemann sphere, which can be interpreted with projective coordinates to get the projective line $\mathbb{P}^1$.

This gives the following inclusions, where $\mathbb{C}_{y/x} \cap \mathbb{C}_{x/y} = \mathbb{C} \setminus \{0\}$.

$$\begin{array}{ccccc} \mathbb{C}_{y/x} & & & & \mathbb{C}_{x/y} \\ & \nwarrow & & \nearrow & \\ & & \mathbb{C} \setminus \{0\} & & \end{array} \qquad \begin{array}{ccccc} \mathbb{C}[y/x] & & & & \mathbb{C}[x/y] \\ & \searrow & & \swarrow & \\ & & \mathbb{C}[\frac{y}{x}, \frac{x}{y}] & & \end{array}$$

In the example, the coordinates of $\mathbb{P}^1$ were of the form $(x : y)$. The following example gives a different outcome.

Example 2.2.1. Consider the group $G = \mathbb{C}^*$ acting on $X = \mathbb{C}^2_{x,y}$ by $t \cdot (x, y) = (tx, t^2y)$ for $t \in \mathbb{C}^*$. As usual, the G-action has the following orbits.

- The origin $\{(0, 0)\}$,
- the punctured x and y axes,
- the punctured parabolas $S_\alpha = \{(x, y) \in \mathbb{C}^2 : y = \alpha x^2, (x, y) \neq (0, 0)\}$, for all $\alpha \in \mathbb{C}^*$.

Again the closure of all orbits contains the origin. Instead of calculating the GIT quotient, we will proceed with a new method.

We may *lift* any point $(x, y) \in X$ to the point $(x, y, z) \in X \times \mathbb{C}$, where we decide G acts on $\mathbb{C}$ by $t \cdot z = t^{-1}z$. Thus the G-invariant functions in the coordinate ring of $X \times \mathbb{C}$ is

$$\mathbb{C}[x, y, z]^G = \mathbb{C}[xz, yz^2].$$

The calculation is left as an exercise to the reader. Denoting $R := \mathbb{C}[xz, yz^2]$ which has generators $\{xz, yz^2\}$, the open cover $\{U_1, U_2\}$ of $\operatorname{Proj} R$ is

$$U_1 = \mathbb{C}_{y/x^2}, \qquad U_2 = \mathbb{C}_{x^2/y}.$$

Thus in fact we have

$$\operatorname{Proj} R = \mathbb{P}^1_{x^2 : y}.$$

Note this is independent of z, so taking the Proj has eliminated the extra dimension which we lifted x into. The choice of the action on the extra dimension $\mathbb{C}$ will be studied in the stability section of Chapter 3.

This gives a morphism $X \to \mathbb{P}^1$; $(x, y) \mapsto (x^2 : y)$. This map has parameterised the orbits of all stable points $X^s = X \setminus \{(0, 0)\}$ and in fact it is a geometric quotient. This is an example of the *projective GIT quotient* which we will introduce shortly.

We wish to extend the notion of the GIT quotient into projective space. As we have just seen, we can do this by 'gluing' affine GIT quotients together. We do this by mapping open subsets of our affine variety X to other open invariant affine subsets before gluing. This poses a challenge which can be resolved using *ample line bundles*, however for the purpose of the thesis we do not need to define this.

Definition 2.2.2. Let $X \subseteq \mathbb{P}^n$ be a projective variety and G an affine algebraic group. A *linear action* of G on X is an action induced by a linear representation $\rho : G \to \mathrm{GL}_{n+1}(\mathbb{C})$.

Suppose a reductive group G acts linearly on a projective variety $X \subseteq \mathbb{P}^n$. We can associate this action to the affine cone $\mathbb{C}^{n+1}$ over $\mathbb{P}^n$. Furthermore note the coordinate ring of $\mathbb{C}^{n+1}$ is graded. Furthermore, its G-invariant subalgebra is of the form

$$\mathbb{C}[x_0, \ldots, x_n]^G = \bigoplus_{r \geq 0} \mathbb{C}[x_0, \ldots, x_n]_r^G.$$

Similarly we have $R(X)^G = \oplus_{r\geq 0} R(X)_r^G$ for the coordinate ring of X. This is finitely generated. Thus the inclusion $R(X)^G \hookrightarrow R(X)$ induces the morphism between projective varieties

$$X \dashrightarrow \operatorname{Proj} R(X)^G.$$

This construction is only a quotient when equipped with the right domain. Hoskins' covers the notions of stability in her text [Hos16]. The following definition is simplified from a lemma.

Definition 2.2.3. Suppose a reductive group G acts linearly on a projective variety $X \subset \mathbb{P}^n$. Then the point $x \in X$ is

(i) *semistable* if there exists a G-invariant homogeneous polynomial $f \in R(X)_r^G$ for $r > 0$ such that $f(x) \neq 0$,

(ii) *stable* if x is semistable, $G \cdot x$ is closed and $\dim G_x = 0$.

We denote the set of stable and semistable points of X as X^s and X^{ss}, respectively. Thus $X^s \subseteq X^{ss} \subseteq X$.

([Hos16] Definition 5.2, Lemma 5.9)

This definition suits for our purpose in projective GIT and will be refined later in Chapter 3 to suit the context of quiver representations.

With this in mind, we can now define the projective GIT quotient.

Definition 2.2.4. Suppose a reductive group G acts linearly on a projective variety $X \subset \mathbb{P}^n$. Then the *projective GIT quotient* is

$$\varphi : X^{ss} \to X/\!\!/G := \operatorname{Proj} R(X)^G$$

It exhibits the following properties.

i) $X/\!\!/G$ is a projective variety.

ii) φ is G-invariant and surjective.

iii) $(X/\!\!/G, \varphi)$ is a categorical quotient.

The properties follow from the notion of a good quotient just as in the affine case.

Example 2.2.5. Consider $G = \mathbb{C}^*$ acting on $X = \mathbb{P}^2$ by $t \cdot (x : y : z) = (t^{-1}x : ty : tz)$. The coordinate ring of X is just $R(X) = \mathbb{C}[x, y, z]$, so calculating the G-invariant functions gives

$$R(X)^G = \mathbb{C}[x, y, z]^G = \mathbb{C}[xy, xz].$$

Thus $X/\!\!/G = \operatorname{Proj} \mathbb{C}[xy, xz] = \mathbb{P}^1_{xy:xz}$ with projective GIT quotient map

$$\varphi : X^{ss} \to \mathbb{P}^1$$
$$(x : y : z) \mapsto (xy : xz).$$

Where $X^{ss} = \{(x : y : z) \in \mathbb{P}^1 : x \neq 0, (y, z) \neq (0, 0)\}$. Note $X^{ss} \cong \mathbb{C}^2$ and furthermore it is just the affine chart of $\mathbb{P}^2$ where $x \neq 0$. The preimage of φ is a single orbit and thus the quotient is geometric. Furthermore, each point in X^{ss} has a 0 dimensional stabiliser and thus $X^{ss} = X^s$.

The past examples have all involved calculating the G-invariant functions explicitly, or at least just its generators, in order to determine the stability of points. In some cases, it can be difficult to calculate these generators and hence find the quotient map. This issue arises later in the thesis, when the group $G = \mathrm{GL}_2(\mathbb{C}) \times (\mathbb{C}^*)^4$ acts on an affine space isomorphic to $\mathbb{C}^8$. For such cases we look to the *Hilbert-Mumford criterion*, a numerical criterion which can be used to determine the stability of points.

For context, let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$. Instead of requiring calculations of explicit points $x \in X$, we look to one-parameter subgroups of G. That is, a one-parameter subgroup (1-PS in literature) is a group homomorphism

$$\lambda : \mathbb{C}^* \longrightarrow G.$$

We will define these more precisely in the following chapter. This lends us the following theorem.

Theorem 2.2.6 (Hilbert-Mumford Criterion). Let G be a reductive group acting linearly on a projective variety $X \subseteq \mathbb{P}^n$. Then for $x \in X$.

- $x \in X^{ss} \iff \mu(x, \lambda) \geq 0$ for all 1-PSs of λ of G,
- $x \in X^s \iff \mu(x, \lambda) > 0$ for all 1-PSs of λ of G.

Where $\mu(x, \lambda)$ is the *Hilbert-Mumford weight* of x at λ.
([Hos16] Theorem 6.11, [Mum92] Theorem 2.1)

This is found in both Hoskins' [Hos16] and Mumford's texts [Mum92]. We will not cover the Hilbert-Mumford weight, but rather introduce it as its own object in the next chapter alongside quiver representations. This definition for stability will prove to be a more accessible method for the classification of points.

Chapter 3
Representations of Quivers

The following chapter will evolve the concepts of GIT quotients to be used with the representations of quivers. The first two sections on stability and quivers are presented in the style of Alastair D. King's *Moduli of Representations of Finite Dimensional Algebras* [King94]. This will be followed by a brief overview of the path algebras induced by quivers and then finally we will cover two key examples, chosen by Daniel Chan, in order to grasp the power of GIT when analysing quiver representations. We invoke GIT on representations of quivers in order to solve the problem of finding the isomorphism classes of their representations. This is simple in some cases, as the first key example will prove. However, it can be a gruelling task in more complex cases.

3.1 Stability

Given a $\mathbb{C}$-vector space $\mathcal{R}$ and group G which is a subgroup of $\mathrm{GL}(\mathcal{R})$, we may consider the group action as $G \times \mathcal{R} \to \mathcal{R}; (g, x) \mapsto g \cdot x$. We introduce a character $\chi : G \to \mathbb{C}^\times$ which aids to lift the G-action to the trivial line bundle $L^{-1} = \mathcal{R} \times \mathbb{C}$, where elements of L^{-1} are of the form (x, z), $x \in \mathcal{R}$, $z \in \mathbb{C}$. G now acts on this as follows;

$$g \cdot (x, z) = (g \cdot x, \chi^{-1}(g)z).$$

Note that for the purpose of this thesis only this mention of the trivial line bundle appears and no further knowledge is required.

Definition 3.1.1. A function $f \in \mathbb{C}[\mathcal{R}]$ is a *relative invariant of weight* χ^n if $f(g \cdot x) = \chi(g)^n f(x)$ for all $g \in G$. We denote this as $f \in \mathbb{C}[\mathcal{R}]^{G,\chi^n}$.

An *invariant section* of L^n is the function $f(x)z^n \in \mathbb{C}[\mathcal{R}\times\mathbb{C}]$, where $f \in \mathbb{C}[\mathcal{R}]^{G,\chi^n}$ and z is the variable of $\mathbb{C}[z]$.

We may now give the definition of stability. It shares resemblance to the definitions provided in the context of projective GIT, although these should act more broadly. First define Δ to be the subgroup of scalar matrices in $\mathrm{GL}(\mathcal{R})$.

Definition 3.1.2. A point $x \in \mathcal{R}$ is

(i) χ-*semistable* if there exists $f \in \mathbb{C}[\mathcal{R}]^{G,\chi^n}$, $n \geq 1$ such that $f(x) \neq 0$.

(ii) χ-*stable* if it is χ-semistable, $\dim(G \cdot x) = \dim(G/\Delta)$ and the G-action on $\{y \in \mathcal{R} \mid f(y) \neq 0\}$ is closed.

We will denote the set of χ-semistable and χ-stable points as $\mathcal{R}^{ss}$ and $\mathcal{R}^s$, respectively. Thus we have $\mathcal{R}^s \subseteq \mathcal{R}^{ss} \subseteq \mathcal{R}$.

Note the dimension condition is analogous to finding the stabiliser $H = G_x \leq G$, where $\Delta \leq H$ as Δ acts trivially. Thus we have the isomorphism

$$G \cdot x \cong G/H \cong \frac{G/\Delta}{H/\Delta}.$$

We can now give a definition for the geometric quotient.

Definition 3.1.3. The *geometric quotient* of $\mathcal{R}$ by G with respect to character χ is

$$\mathcal{R}/\!\!/(G,\chi) := \operatorname{Proj} \mathbb{C}[\mathcal{R} \times \mathbb{C}]^G.$$

An alternative definition involving the natural grading is

$$\mathcal{R}/\!\!/(G,\chi) = \operatorname{Proj}\left(\bigoplus_{n \geq 0} (\mathbb{C}[\mathcal{R}]z^n)^G\right)$$

This quotient is not to be confused with the GIT quotient $\mathcal{R}/\!\!/G = \operatorname{Spec} \mathbb{C}[\mathcal{R}]^G$.

Remark 3.1.4. A. D. King [King94] prefers the notation

$$\mathcal{R}/\!\!/(G,\chi) = \operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[\mathcal{R}]^{G,\chi^n}\right).$$

We depart from the notion of relative invariants in favour of the GIT notation.

The geometric quotient has a geometric description, which is its equivalence to $\mathcal{R}^{ss}/\sim$, where $x \sim y$ if and only if $\overline{G \cdot x} \cap \overline{G \cdot y}$ is non-empty in $\mathcal{R}^{ss}$. So the quotient classifies distinct closed orbits of χ-semistable points. If two points or orbits satisfy this relation, then they are *GIT equivalent*. A more substantial description can be found in King's text [King94].

The notion of stability can be further explored with the use of one-parameter subgroups.

Definition 3.1.5. A *one-parameter subgroup* of G is a function $\lambda : \mathbb{C}^\times \to G$ which is a continuous group homomorphism.

Note that given a character χ, then $\chi \circ \lambda : \mathbb{C}^\times \to \mathbb{C}^\times$ is also a character. We may now revise our definition of stability.

Theorem 3.1.6. Lift $x \in \mathcal{R}$ to $\hat{x} = (x, z) \in \mathcal{R} \times \mathbb{C}$, $z \neq 0$. Then $x \in \mathcal{R}$ is

(i) χ*-semistable* if and only if, for all one-parameter subgroups λ of G, we have $\lim_{t\to 0} \lambda(t) \cdot \hat{x} \notin \mathcal{R} \times \{0\}$,

(ii) χ*-stable* if and only if all one-parameter subgroups λ of G, for which the limit $\lim_{t\to 0} \lambda(t) \cdot \hat{x}$ exists, are precisely in Δ.

([King94] Lemma 2.4)

This theorem is not directly present in Hoskins' [Hos16], as she opts to use different machinery to arrive at the Hilbert-Mumford criterion. This is covered in Chapter 6 of [Hos16].

We can again revise this definition. Denote the *integral pairing* between the one-parameter subgroup λ and character χ to be $\langle \chi, \lambda \rangle = m$, where $\chi(\lambda(t)) = t^m$. This gives us a version of "Mumford's Numerical Criterion".

Proposition 3.1.7. A point $x \in \mathcal{R}$ is

(i) χ-*semistable* if and only if $\chi(\Delta) = \{1\}$ and for all one-parameter subgroups λ of G, where $\lim_{t\to 0} \lambda(t) \cdot x$ exists, then $\langle \chi, \lambda \rangle \geq 0$.

(ii) χ-*stable* if and only if it is χ-semistable and all one-parameter subgroups λ of G, where $\lim_{t\to 0} \lambda(t) \cdot x$ exists and $\langle \chi, \lambda \rangle = 0$, are in Δ.

([King94] Proposition 2.5)

King omits the proof, so we do it ourselves. The aim is to prove the respective (semi)stability conditions are identical. We write out the condition for each definition and they are equivalent for each case.

Proof. The proposition follows from Theorem 3.1.6. Denote $\mathcal{R}_T$, $\mathcal{R}_P$ to be the vector spaces in the theorem and proposition respectively. We wish to show that $\mathcal{R}_T^{ss} = \mathcal{R}_P^{ss}$ and $\mathcal{R}_T^{s} = \mathcal{R}_P^{s}$, where $\mathcal{R}_T^{ss}$, $\mathcal{R}_T^{s}$ and $\mathcal{R}_P^{ss}$, $\mathcal{R}_P^{s}$ are the set of semistable and stable points of the theorem and proposition, respectively. For this proof we will interchange $m = \langle \chi, \lambda \rangle$. This renders the expression $\lambda(t) \cdot \hat{x} = (\lambda(t) \cdot x, t^{-m} z)$.

i) To prove $\mathcal{R}_T^{ss} = \mathcal{R}_P^{ss}$, we wish to show that

$$\lim_{t\to 0} \lambda(t) \cdot \hat{x} \notin \mathcal{R} \times \{0\} \iff \left(\lim_{t\to 0} \lambda(t) \cdot x \text{ exists } \Rightarrow \langle \chi, \lambda \rangle \geq 0 \right).$$

Assume that $\lim_{t\to 0} \lambda(t) \cdot \hat{x} \notin \mathcal{R} \times \{0\}$.
Then if $\lim_{t\to 0} \lambda(t) \cdot x$ exists, we must have $\lim_{t\to 0} \lambda(t) \cdot \hat{x} = \lim_{t\to 0} t^{-m} z \neq 0$. We define z to be non-zero, so this admits $m \geq 0$.

Conversely, assume that if $\lim_{t\to 0} \lambda(t) \cdot x$ exists, then $\langle \chi, \lambda \rangle \geq 0$.
Consider the case where $\lim_{t\to 0} \lambda(t) \cdot x$ does not exist. Then clearly $\lim_{t\to 0} \lambda(t) \cdot \hat{x} \notin \mathcal{R} \times \{0\}$. Then if $\lim_{t\to 0} \lambda(t) \cdot x$ exists, then $m \geq 0$. So $\lim_{t\to 0} \lambda(t) \cdot z = \lim_{t\to 0} t^{-m} z$ does not exist or is equal to $z \neq 0$. Thus $\lim_{t\to 0} \lambda(t) \cdot \hat{x} \notin \mathcal{R} \times \{0\}$.

ii) To prove $\mathcal{R}_T^{s} = \mathcal{R}_P^{s}$ it suffices to show that

$$\left(\lim_{t\to 0} \lambda(t) \cdot \hat{x} \text{ exists } \Rightarrow \lambda(t) \in \Delta \right)$$
$$\Updownarrow$$
$$\left(\lim_{t\to 0} \lambda(t) \cdot x \text{ exists } \Rightarrow \langle \chi, \lambda \rangle \geq 0, [\langle \chi, \lambda \rangle = 0 \Rightarrow \lambda(t) \in \Delta] \right)$$

Assume that if $\lim_{t\to 0} \lambda(t) \cdot \hat{x}$ exists, then $\lambda(t) \in \Delta$.
Suppose that $\lim_{t\to 0} \lambda(t) \cdot x$ exists, but $\lim_{t\to 0} \lambda(t) \cdot z =$ does not. Thus $m > 0$. Alternatively, suppose $\lim_{t\to 0} \lambda(t) \cdot \hat{x}$ exists, so $\lambda(t) \in \Delta$. Thus by definition, $\langle \chi, \lambda \rangle = 0 \geq 0$.

Conversely, assume that if $\lim_{t\to 0}\lambda(t)\cdot x$ exists, then $\langle\chi,\lambda\rangle \geq 0$, and furthermore if $\langle\chi,\lambda\rangle = 0$, then $\lambda(t)\in\Delta$.
Suppose that $\lim_{t\to 0}\lambda(t)\cdot\hat{x}$ exists. So by assumption $\langle\chi,\lambda\rangle \geq 0$. And $\lim_{t\to 0}\lambda(t)\cdot z$ exists, so $m \leq 0$. Thus $m = 0$ and so $\lambda(t)\in\Delta$. □

Thus we have a numerical method of determining the stability of a point. Notice we have also dissolved the need to lift the point to $\mathcal{R}\times\mathbb{C}$. We now have all the machinery required to being studying the representations of quivers.

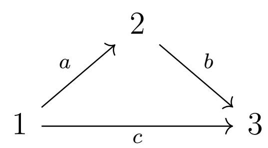
3.2 Quivers

We begin by defining the main object of study.

Definition 3.2.1. A *quiver* $Q = (Q_0, Q_1)$ is a directed graph, where Q_0 is a set of vertices and Q_1 a set of directed edges, also called arrows. We denote the head and tail of an arrow $a \in Q_1$ to be ha and ta, respectively.

The following example of a quiver will be studied in-depth in a future section.

Example 3.2.2. Consider $Q = (\{1,2,3\},\{a,b,c\})$ as follows:



The example above is just a quiver, and does not contain a representation.

Definition 3.2.3. A *representation* of $Q = (Q_0, Q_1)$ is $M = (W_v, \phi_a)$, where W_v is a $\mathbb{C}$-vector space for each $v \in Q_0$ and $\phi_a : W_{ta} \to W_{ha}$ a $\mathbb{C}$-linear map for each $a \in Q_1$.

There is a notion of isomorphism given two representations.

Definition 3.2.4. Let $M = (W_v, \phi_a)$ and $M' = (U_v, \psi_a)$ be two representations of a quiver $Q = (Q_0, Q_1)$. A *morphism* between the two representations is a collection of linear maps $f_v : W_v \to U_v$, $\forall v \in Q_0$ with the condition that $f_{ha}\phi_a = \psi_a f_{ta}$, $\forall a \in Q_1$. It is an isomorphism if and only if each f_v is.

That is, the following diagram commutes for any $a \in Q_1$.

$$\begin{array}{ccc} W_{ta} & \xrightarrow{\phi_a} & W_{ha} \\ {\scriptstyle f_{ta}}\downarrow & & \downarrow{\scriptstyle f_{ha}} \\ U_{ta} & \xrightarrow{\psi_a} & U_{ha} \end{array}$$

Example 3.2.5. Consider the quiver Q and representation M given by the following diagram.

$$\mathbb{C} \xrightarrow{1} \mathbb{C}$$

Observe the following two representations N_1, N_2 of Q displayed in respective order below.

$$0 \longrightarrow \mathbb{C}, \qquad \mathbb{C} \longrightarrow 0$$

Thus we have the two diagrams below.

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{1} & \mathbb{C} \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{1} & \mathbb{C} \end{array}$$

Note while the first commutes for a trivial choice of morphisms, but the second never commutes.

The example motivates the following.

Definition 3.2.6. Given a representation $M = (W_v, \phi_a)$ of a quiver Q, a *subrepresentation* of M is a representation $M' = (W'_v, \phi'_a)$ of Q such that W'_v is a subspace of W_v for all $v \in Q_0$ and there exists morphism from M' to M.

Definition 3.2.7. The *dimension vector* of a quiver representation $M = (W_v, \varphi_a)$ is $\alpha \in \mathbb{N}^{Q_0}$ with components $\alpha_v = \dim_{\mathbb{C}} W_v$.

Given a dimension vector α, we can now introduce the representation space

$$\mathcal{R}(Q, \alpha) = \bigoplus_{a \in Q_1} \mathrm{Hom}(W_{ta}, W_{ha}).$$

Each point $x \in \mathcal{R}(Q, \alpha)$ is a choice of homomorphism for each arrow. It is a representation of the given quiver. The space is isomorphic to $\oplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha_{ta}}, \mathbb{C}^{\alpha_{ha}})$, which is affine and thus too is the representation space.

This space is acted on by the group

$$\mathrm{GL}(\alpha) = \prod_{v \in Q_0} GL(W_v).$$

An element $g \in GL(\alpha)$ acts on a point $x \in \mathcal{R}(Q, \alpha)$ by

$$(g \cdot x)_a = g_{ha} \phi_a g_{ta}^{-1},$$

where $x = \bigoplus_{a \in Q_1} \phi_a$ and $(g \cdot x)_a$ is the resulting homomorphism of the arrow $a \in Q_1$. Notice that $\Delta = \{(t1, \ldots, t1) \mid t \in \mathbb{C}^\times\}$ acts trivially.

Consider a character $\chi : \mathrm{GL}(\alpha) \to \mathbb{C}^\times$, which is determined by the choice of a fixed $\theta \in \mathbb{Z}^{Q_0}$. It is given by

$$\chi(g) = \prod_{v \in Q_0} \det(g_v)^{\theta_v}.$$

Consider a representation $M = (U_v, \psi_a)$. The condition that $\chi(\Delta) = \{1\}$ is equivalent to $\theta(M) := \sum_{v \in Q_0} \theta_v \alpha_v = 0$, where $\alpha_v = \dim U_v$. This can be seen as

$$\chi(\Delta) = \left\{ \prod_{v \in Q_0} (t^{\dim W_v})^{\theta_v} \right\} = \left\{ \prod_{v \in Q_0} t^{\alpha_v \theta_v} \right\} = \left\{ t^{\sum_{v \in Q_0} \alpha_v \theta_v} \right\} = \left\{ t^{\theta(M)} \right\}.$$

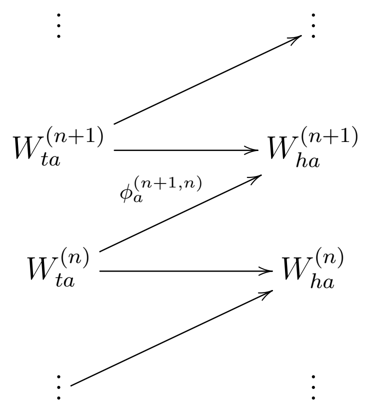
A filtration of modules is a descending sequence of submodules.

Lemma 3.2.8. Each one-parameter subgroup of $\mathrm{GL}(\alpha)$ corresponds to a filtration $(M_n)_{n \in \mathbb{Z}}$ of a representation M. Additionally, with the condition that $\sum_v \theta_v \alpha_v = 0$, it renders the expression $\langle \chi_\theta, \lambda \rangle = \sum_{n \in \mathbb{Z}} \theta(M_n)$.
([King94] pp.520-521)

This is not expressed as a result in King's text but rather just some exploration he uses to motivate a proposition.

Proof. Consider a representation $M = (W_v, \phi_a)$ and one-parameter subgroup $\lambda : \mathbb{C}^\times \to \mathrm{GL}(\alpha)$. For each $v \in \mathbb{Q}_0$ define the $\mathbb{Z}$-grading $W_v = \oplus_{n \in \mathbb{Z}} W_v^{(n)}$, where $\lambda(t)$ acts on each $W_v^{(n)}$ by multiplying by t^n. Next define $W_v^{(\geq n)} = \oplus_{m \geq n} W_v^{(m)}$ and $\phi_a^{(m,n)} : W_{ta}^{(n)} \to W_{ha}^{(m)}$, which $\lambda(t)$ acts on by multiplying by t^{m-n}. This map sends the component of weight n of a vertex to the component of weight m of another vertex, given that an arrow between them exists.

Notice the limit $\lim_{t \to 0} t^{m-n}$ does not exist if $m - n < 0$. Thus $\lim_{t \to 0} \lambda(t)\phi_a$ only exists if $\phi_a^{(m,n)} = 0$ in the cases where $m < n$. That is, ϕ_a only increases the degree of the vector space, as shown in the picture below.



Thus all $W_v^{(\geq n)}$ give subrepresentations M_n of M, where $(M_n)_v = W_v^{(\geq n)}$. This gives the filtration

$$M \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots \supseteq 0.$$

Now by denoting the limit to be $\lim_{t\to 0} \lambda(t) \cdot \phi_a = \psi$, we get an infinite grid of homomorphisms $\psi = (\psi^{(m,n)})_{m,n}$, where if $m = n$, then $\psi^{(m,n)} = \phi_a$, and if $m \neq n$, then $\psi^{(m,n)} = 0$. The latter can be observed as $m < n$ implies that $\phi^{(m,n)} = 0$, while $m > n$ gives a t^{m-n} term in the limit, which also goes to 0. Thus this limit only sends each n-weighted subspace to itself. Each of these homomorphisms is none other than M_n/M_{n+1}, so the limit in the end is

$$\lim_{t\to 0} \lambda(t) \cdot \phi_a = \bigoplus_{n\in\mathbb{Z}} \left(W^{(n)}, \phi_a^{(n,n)}\right) = \bigoplus_{n\in\mathbb{Z}} M_n/M_{n+1}.$$

This follows from the fact that $W^{(n)} = W^{(\geq n)}/W^{(\geq n+1)}$ paired with the homomorphism $\phi_a^{(n,n)} : W_{ta}^{(n)} \to W_{ha}^{(n)}$.

Now assuming that $\sum_v \theta_v \alpha_v = 0$, the filtration $(M_n)_{n\in\mathbb{Z}}$ also gives us an expression of the pairing $\langle \chi_\theta, \lambda\rangle$. Note that

$$\chi_\theta(\lambda(t)) = \prod_{v\in Q_0} \left(\det(\lambda(t); W_v\right)^{\theta_v},$$

where $(\lambda(t), W_v)$ refers to $\lambda(t)$ restricted to acting on W_v.
Note that $\det(\lambda(t); W_v) = \prod_{n\in\mathbb{Z}} \det(\lambda(t); W_v^{(n)})$, as each term in the product is the determinant of a block in the block matrix $\lambda(t)$. We have just proved that each of these blocks is just a diagonal matrix with common elements t^n, for each $W_v^{(n)}$. Thus $\det(\lambda(t); W_v^{(n)}) = (t^n)^{\dim W_v^{(n)}}$.

This expression for the determinant gives us

$$\begin{aligned}\langle \chi_\theta, \lambda\rangle &= \sum_{v\in Q_0} \theta_v \sum_{n\in\mathbb{Z}} n \dim W_v^{(n)} \\ &= \sum_{n\in\mathbb{Z}} n\theta(M_n/M_{n+1}) \\ &= \sum_{n\in\mathbb{Z}} \theta(M_n).\end{aligned}$$

The second equality follows from the definition of $\theta(M_n/M_{n+1})$,
where $\dim(M_n/M_{n+1}) = \dim W_v^{(n)}$. The third equality takes more work. First we prove that $\theta(M_n/M_{n+1}) = \theta(M_n) - \theta(M_{n+1})$. It suffices to observe the following:

$$\begin{aligned}\dim(M_n) - \dim(M_{n+1}) &= \sum_{k\geq n} \dim(W_v^{(k)}) - \sum_{k\geq n+1} \dim(W_v^{(k)}) \\ &= \dim(W_v^{(n)}).\end{aligned}$$

Thus we have that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} n\theta(M_n/M_{n+1}) &= \sum_{n\in\mathbb{Z}} n\theta(M_n) - n\theta(M_{n+1}) \\ &= \sum_{n\in\mathbb{Z}} n\theta(M_n) - \sum_{n\in\mathbb{Z}}(n-1)\theta(M_n) \\ &= \sum_{n\in\mathbb{Z}} \theta(M_n).\end{aligned}$$

Note this requires that $\theta(M_k)$ is non-zero for only finitely many k.
This completes the proof. □

There is another notion of stability with respect to the character. Note that the character χ is purely dependant on the choice of θ.

Definition 3.2.9. Given fixed $\theta \in \mathbb{Z}^{Q_0}$, a representation M is

i) *θ-semistable* if $\theta(M) = 0$ and $\theta(M') \geq 0$ for all submodules $M' \leq M$.
ii) *θ-stable* if it is θ-semistable and for any submodule $M' \leq M$, for which $\theta(M') = 0$, then $M' = 0$ or $M' = M$.

We can now give a different method of computation of the stability of representations.

Proposition 3.2.10. A point in $\mathcal{R}(Q, \alpha)$ which corresponds to representation M is χ_θ-(semi)stable if and only if M is θ-(semi)stable.
([King94] Proposition 3.1)

This proposition is crucial in proving results for quivers with higher dimensional vertices. We began with a notion of stability in Chapter 2 where we needed to calculate the invariant subalgebra, and now the only information we need is the choice of θ and dimension vector α of the quiver. Although this only applies to stability in quiver representations, it is still a great progression. Some examples of θ-stability will follow in the last two sections of this chapter, where we encounter less trivial examples of quivers.

3.3 Representations as Algebras

This section is based on the Michel Brion's text *Representations of Quivers* [Br08]. We delve into such algebras to explore other methods to study representations of quivers. This method of study revolves around the algebraic approach, where we observe $\mathbb{C}Q$-algebras, where Q is a fixed quiver.

Consider a quiver $Q = (Q_0, Q_1)$ and a representation $M = (W_v, \phi_a)$. Denote the total vector space

$$W = \bigoplus_{v\in Q_0} W_v.$$

Also define the projection π_v and inclusion ι, given by the following diagram.

$$\begin{array}{ccccc} W & \xrightarrow{\pi_{ta}} & W_{ta} & \xhookrightarrow{\iota} & W \\ & & {\scriptstyle \phi_a}\big\downarrow & & \\ W & \xrightarrow{\pi_{ha}} & W_{ha} & \xhookrightarrow{\iota} & W \end{array}$$

We can define the two maps $f_v,\ f_a : W \to W$ for $v \in Q_0,\ a \in Q_1$ by the composition of maps

$$f_v := \iota\pi_v, \qquad f_a := \iota\phi_a\pi_{ta}.$$

Notice that $\pi_v\iota$ is the identity map on W_v. Then $f_{ha}f_a = \iota\pi_{ha}\iota\phi_a\pi_{ta} = \iota\phi_a\pi_{ta} = f_a$ and similarly $f_af_{ta} = f_a$. It is also straightforward to check that $f_v^2 = f_v$ and for $v, w \in Q_0$, $f_vf_w = 0$ if and only if $v \neq w$. This construction allows us to define the following.

Definition 3.3.1. A *path* is a sequence of arrows $a_1, \ldots, a_n$, for $a_i \in Q_1$, such that $ha_i = ta_{i+1}$, $\forall i \in \{1, \ldots, n-1\}$.

The length of such a path $a_1, \ldots, a_n$ is defined to be n. We may now map $f_v \mapsto e_v$ and $f_a \mapsto a$ as indeterminates to create an object with the same structure given by the relations constructed as above.

Definition 3.3.2. The *path algebra* of a quiver Q is the $\mathbb{C}$-algebra $\mathbb{C}Q$ with generators $\{e_v\ :\ v \in Q_0\}$. For any $a \in Q_1$, they exhibit the relations

$$e_v^2 = e_v, \qquad e_ve_w = 0 \text{ iff } v \neq w, \qquad e_{ha}a = a = ae_{ta}.$$

That is, if $a_1, a_2, \ldots, a_{n-1}, a_n$ is a path in Q, then $a_na_{n-1}\cdots a_2a_1$ is a generator of $\mathbb{C}Q$. Thus if P is the set of all such generators of $\mathbb{C}Q$, we have

$$\mathbb{C}Q = \bigoplus_{p \in P} \mathbb{C}p.$$

Where multiplication is defined by path concatenation. Note that each e_v is a '0 length path'.

The first two relations enforce that the generators e_v are idempotent and pairwise orthogonal. Furthermore, consider the sum

$$e_w \sum_{v \in Q_0} e_v = e_w^2 = e_w.$$

Thus we have $\sum_{v \in Q_0} e_v = 1$.

Example 3.3.3. Consider the quiver $Q = (\{1,2,3\},\{a,b\})$ given by the following picture.

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Then the generators of $\mathbb{C}Q$ are $\{e_1, e_2, e_3\}$ such that

$$e_2 a e_1 = a, \qquad e_3 b e_2 = b, \qquad e_i^2 = e_i \ \ \forall i, \qquad e_i e_j = 0 \ \ (i \neq j).$$

Thus the length 0 paths are e_1, e_2, e_3, the length 1 paths are a, b and the length 2 path is ba. The path algebra is then

$$\mathbb{C}Q = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}a \oplus \mathbb{C}b \oplus \mathbb{C}ba.$$

Note that $\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$ is a subalgebra of $\mathbb{C}Q$. Additionally $\mathbb{C}a \oplus \mathbb{C}b \oplus \mathbb{C}ba$ is a two-sided ideal. Furthermore, each $\mathbb{C}a$, $\mathbb{C}b$, $\mathbb{C}ba$ is nilpotent.

The next example poses an interesting construction, although quivers with loops will not be discussed again.

Example 3.3.4. Consider the quiver $Q = (\{1\},\{a\})$ with a single arrow being a loop.

$$1 \circlearrowright a$$

Listing all paths, we have $\{e_1, a, a^2, \dots\}$. Thus the path algebra is just

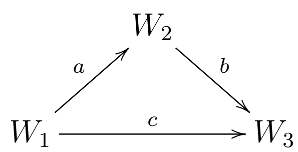
$$\mathbb{C}Q = \bigoplus_{n \geq 0} \mathbb{C}a^n,$$

where we may define $a^0 := e_1$. This is clearly isomorphic to $\mathbb{C}[x]$ by the map $a \mapsto x$, $e_1 \mapsto 1$. So $\mathbb{C}Q \cong \mathbb{C}[x]$.

We end our exploration of quivers as algebras here, although readers still interested should refer to Brion's text [Br08]. We return back to King's perspective of quiver representations by considering an example.

3.4 Key Example

The following example was originally posed by my supervisor, Daniel Chan, who established it in order to teach GIT and representations of quivers in unison.
Consider the following quiver Q with dimension vector $\alpha = (1,1,1)$.



Thus each $W_i \cong \mathbb{C}^{\alpha_i} = \mathbb{C}$. The representation space is $\mathcal{R}(Q,\alpha) = \mathbb{C}^3$, where the arrow labels correspond to $(a,b,c) \in \mathbb{C}^3$. We act on this space with group $G = \mathrm{GL}(\alpha) = (\mathbb{C}^\times)^3$ by conjugation, giving

$$(t_1, t_2, t_3) \cdot (a,b,c) = (t_2 t_1^{-1} a, t_3 t_2^{-1} b, t_3 t_1^{-1} c).$$

We choose the vector $\theta = (-2, 1, 1)$ such that $\sum_{v \in Q_0} \theta_v \alpha_v = 0$. This will aid to lift this action to $\mathcal{R} \times \mathbb{C}$ by using the character $\chi_\theta : G \to \mathbb{C}; (t_1, t_2, t_3) \mapsto t_1^{-2} t_2 t_3$. Now observe the geometric quotient

$$\mathcal{R}/\!\!/(G, \chi_\theta) = \text{Proj}\left(\mathbb{C}[\mathcal{R} \times \mathbb{C}]^{\mathbb{C}^{\times 3}}\right) = \text{Proj}\left(\mathbb{C}[a, b, c, z]^{\mathbb{C}^{\times 3}}\right).$$

First we wish to compute $\mathbb{C}[a, b, c, z]^{\mathbb{C}^{\times 3}}$ by observing the invariant terms. We have that

$$(t_1, t_2, t_3) \cdot (a^i b^j c^k z^l) = t_1^{-i-k+2l} t_2^{i-j-l} t_3^{j+k-l} a^i b^j c^k z^l.$$

We require that the t_i terms vanish and thus have exponent 0. This gives a system of equations with solutions $i = 2j + k$, $l = j + k$ which give

$$\begin{aligned} a^i b^j c^k z^l &= a^{2j+k} b^j c^k z^{j+k} \\ &= (a^2 bz)^j (acz)^k. \end{aligned}$$

Thus we have that $\mathbb{C}[a, b, c, z]^{\mathbb{C}^{\times 3}} = \mathbb{C}[a^2 bz, acz]$ and so the geometric quotient is $\text{Proj}\,\mathbb{C}[a^2 bz, acz] = \mathbb{P}^1_{a^2bz\,:\,acz} = \mathbb{P}^1_{a^2b\,:\,ac}$. This completes our calculation.

Points in the quotient are of the form $(ab : c)$. Thus the case where $ab = 0 = c$ give unstable points.
That is, $\{(0, 0, 0)\}$, $\{(a, 0, 0) : a \in \mathbb{C}^\times\}$, $\{(0, b, 0) : b \in \mathbb{C}^\times\} \subseteq \mathcal{R}(\mathbb{Q}, \alpha)$ are the unstable orbits.
Thus the complement of these two sets give $\mathcal{R}^{ss}$, the set of semistable representations with dimension vector α and character dependent on θ.

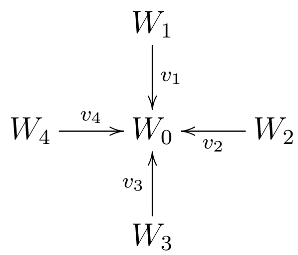
We can verify that the ratio $(ab : c)$ is invariant to the group action. Consider the following.

$$\begin{aligned} (t_1, t_2, t_3) \cdot (ab : c) &= (t_2 t_1^{-1} a t_3 t_2^{-1} b : t_3 t_1^{-1} c) \\ &= (t_3 t_1^{-1} ab : t_3 t_1^{-1} c) \\ &= (ab : c). \end{aligned}$$

Using Proposition 3.2.10, we can show $\mathcal{R}^{ss} = \mathcal{R}^s$. We know $\mathcal{R}^s \subseteq \mathcal{R}^{ss}$. It remains to prove the reverse inclusion. Consider a representation $x \in \mathcal{R}^{ss}$ with associated representation M. Suppose $M' \subseteq M$ is a submodule with associated dimension vector α', so $0 \leq \alpha'_v \leq \alpha_v$ for all $v \in Q_0$. Consider if $\theta(M') = 0$. Then $-2\alpha'_1 + \alpha'_2 + \alpha'_3 = 0$. But over $\{0, 1\}$ the only solutions are $\alpha' = (0, 0, 0)$ or $\alpha' = (1, 1, 1)$. So $M' = 0$ or $M' = M$ and thus M is θ-stable so $x \in \mathcal{R}^s$.
Thus $\mathcal{R}^s = \mathcal{R}^{ss}$.

3.5 Larger Example

Consider the quiver $Q = (Q_0, Q_1)$ pictured below. Note in texts studying quiver algebras it is known as the extended Dynkin quiver, $\widetilde{D}_4$.



We attribute it dimension vector $\alpha = (2, 1, 1, 1, 1)$ indexed from 0 so $\dim W_0 = 2$ and $\dim W_i = 1$, $\forall\, i \neq 0$. We work over $\mathbb{C}$ so each $W_i \cong \mathbb{C}^{\alpha_i}$. Thus the representation space is $\mathcal{R}(Q, \alpha) = (\mathbb{C}^2)^4 \cong \mathbb{C}^8$, so each $v_i \in \mathbb{C}^2$. It is acted on by

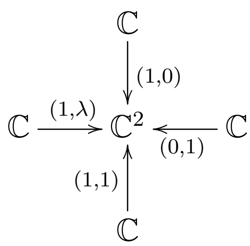
$$G = \mathrm{GL}(\alpha) = \mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times.$$

We choose $\theta = (2, -1, -1, -1, -1)$ such that $\chi(\Delta) = \{1\}$. The choice of θ can be anything in $\mathbb{Z}^{Q_0}$, but ours will give conclusive results as we will see.

Our main goal is to prove the following theorem.

Theorem 3.5.1. Given the setting above, $\mathcal{R}^{ss} /\!\!/ G \cong \mathbb{P}^1$.

We will construct $\psi : \mathbb{P}^1 \to \mathcal{R}^{ss} /\!\!/ G$ and show it is injective. To achieve this, we first define the quiver representation $M(\lambda)$ by the diagram below.



It corresponds to the point $((1, 0),\ (0, 1),\ (1, 1),\ (1, \lambda)) \in \mathcal{R}$.

Lemma 3.5.2. We have $M(\lambda) \in \mathcal{R}^{ss}$ for all $\lambda \in \mathbb{C}$.

Proof. Denote M as the module representation associated to $M(\lambda)$. Thus $\theta(M) = 0$. It suffices to show all submodules $M' \leq M$ satisfy $\theta(M') \geq 0$. Suppose the vertices of M' are labelled W_i' for $i \in Q_0$. We wish to minimise the value of $\theta(M')$. Consider each case of α_0', the dimension of W_0' where it's possible values lie in $\{0, 1, 2\}$.

If $\alpha_0' = 0$, then $W_0' = 0$. But as M' is a subrepresentation of M, the morphisms between vertices and arrows commute which forces $W_i' = 0$ for all i. Thus $\theta(M') = 0$.

On the other hand if $\alpha_0' = 2$ then $\theta(M') \geq 0$ for all subrepresentations.

Finally we check for $\alpha_0' = 1$, so $W_0' \cong \mathbb{C}$ and thus $W_0' = \mathbb{C}w$ for some $w \in \mathbb{C}^2$. Assume that $\lambda \notin \{0, 1\}$ and thus all maps v_i are pairwise non-parallel. But the image of any two non-parallel maps is the entire $\mathbb{C}^2$, which forces $W_i' = 0$ for all but one $i \in \{1, 2, 3, 4\}$. Thus in this case we have $\theta(M') = 2 \cdot 1 - 1 \cdot 1 = 1$.
Thus consider the final case where $\lambda \in \{0, 1\}$. If $\lambda = 0$ then $v_4 = (0, 1) = v_1$ so the following subrepresentation M' exists.

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & & \downarrow{\scriptstyle v_4} & & \\ \mathbb{C} & \xrightarrow{v_4} & \mathbb{C}v_4 & \longleftarrow & 0 \\ & & \uparrow & & \\ & & 0 & & \end{array}$$

Clearly $\theta(M') = 0$. Adding non-zero W_2 or W_3 will force $\dim W_0' = 2$ so we cannot achieve a negative value of $\theta(M')$. Thus $\theta(M') \geq 0$ for all submodules $M' \leq M$ and so M is θ-semistable. So by proposition 3.2.10, the associated point $M(\lambda)$ is χ-semistable. □

The proof also indicates that $M(\lambda) \in \mathcal{R}^s$ if $\lambda \in \mathbb{C} \setminus \{0, 1\}$. The semistability condition in the quiver is essentially enforcing that each map is pairwise non-parallel.

An alternative definition is $N(\lambda)$, which corresponds to the point $((1, 0), (0, 1), (1, 1), (\lambda, 1))$. One can show that $M(\lambda)$ and $N(\lambda^{-1})$ are isomorphic quiver representations for all $\lambda \in \mathbb{C}$. It suffices to show the morphisms between $(1, \lambda)$, $(\lambda^{-1}, 1) : \mathbb{C} \to \mathbb{C}^2$ form a commutative diagram. A similar proof to the one before shows that $N(\lambda) \in \mathcal{R}^{ss}$, with inclusion in $\mathcal{R}^s$ when $\lambda \neq \{0, 1\}$. We know $M(\lambda) \cong N(\lambda^{-1})$ and both are not stable for $\lambda \in \{0, 1\}$. We can extend the range of λ to $\mathbb{C} \cup \{\infty\}$, so now both $M(\lambda), N(\lambda)$ are not stable for $\lambda \in \{0, 1, \infty\}$. These three points which correspond to the only semistable representations will be discussed later.

Consider the map $\mathbb{P}^1 \to \mathcal{R}^{ss}$ defined on each patch U_0, U_1 by

$$U_0 \ni (1 : \lambda) \longmapsto M(\lambda), \qquad U_1 \ni (\lambda : 1) \longmapsto N(\lambda).$$

However taking the union of the two images in $\mathcal{R}^{ss}$ will not give an injective map. Note that the three exceptional points $\{0, 1, \infty\}$ associate as follows.

$$\begin{aligned} (1 : 0) &\longmapsto M(0), \\ (1 : 1) &\longmapsto M(1), \\ (0 : 1) &\longmapsto N(0) \cong M(\infty). \end{aligned}$$

Thus we can take $\mathbb{P}^1 \to \mathcal{R}^{ss}$ to only take values in the patch U_0 and consider the exceptional cases above otherwise. This gives an injective map with image $\{M(\lambda) \ : \ \lambda \in \mathbb{C} \cup \{\infty\}\}$.

To prove φ is injective, it suffices to show that $\{M(\lambda) : \lambda \in \mathbb{C} \cup \{\infty\}\} \to \mathbb{R}^{ss} /\!\!/ G$ is injective. However this requires that each $M(\lambda)$ is non-isomorphic.

Lemma 3.5.3. If $\lambda, \mu \in \mathbb{C}$ are distinct, then the associated module representations of $M(\lambda)$ and $M(\mu)$ are not isomorphic.

Proof. The module representations of $M(\lambda)$, $M(\mu)$ share the same vector spaces W_i, so they only differ by $v_4 : \mathbb{C} \to \mathbb{C}^2$. Denote v_i, w_i to be the corresponding representations of $M(\lambda)$, $M(\mu)$, respectively. It suffices to show that no family of linear maps $\{f_i : W_i \to W_i\}_{i \in Q_0}$ exist such that $f_0 v_i = w_i f_i$ for $i \in \{1, 2, 3, 4\}$ (Note this is a specific case of definition 3.2.4 where $ha = \mathbb{C}^2$ and $ta = \mathbb{C}$ for all arrows $a \in Q_1$). That is, the following diagram does not commute for all i.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{v_i} & \mathbb{C}^2 \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_0} \\ \mathbb{C} & \xrightarrow{w_i} & \mathbb{C}^2 \end{array}$$

As all maps in the diagram are linear, it suffices to observe the image of the unit under each composition. Denote $f_0 = A \in M_{2\times 2}(\mathbb{C})$. For $i \in \{1, 2\}$ we have $Av_1 = f_1 v_1$, $Av_2 = f_2 v_2$ as $w_i = v_i$ and thus f_1, f_2 are the two eigenvalues of A. Furthermore v_1, v_2 are the standard basis elements of $\mathbb{C}^2$ and thus we can take A to be a diagonal matrix with f_1, f_2 as its diagonal entries.
Observe for $i = 4$, we have $Av_4 = f_4 w_4$ which equates to

$$\begin{pmatrix} f_1 \\ \lambda f_2 \end{pmatrix} = f_4 \begin{pmatrix} 1 \\ \mu \end{pmatrix}.$$

We may scale all f_i by the same factor and retain the equality, thus we may choose $f_1 = \lambda = f_4$, $f_2 = \mu$.
Finally, observe for $i = 3$ we have $Av_3 = f_3 v_3$ which evaluates to

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = f_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

But no solution exists for $f_3 \in \mathbb{C}$ as $\lambda \neq \mu$ and thus there is no linear morphism between $M(\lambda)$ and $M(\mu)$. □

We may now prove the following lemma.

Lemma 3.5.4. If $\lambda, \mu \in \mathbb{C}$ are distinct, then the associated module representations of $M(\lambda)$ and $M(\mu)$ are not S-equivalent.

Proof. Denote M, N to be the associated module representations of $M(\lambda)$, $M(\mu)$, respectively.
Suppose $\lambda, \mu \notin \{0, 1, \infty\}$. Then $M(\lambda)$ is θ-stable and thus M has composition series $0 \leq M$ with one composition factor $M/0 = M$. A similar process grants the series $0 \leq N$. The previous lemma states that these modules are non-isomorphic, and thus M and N have distinct composition factors. So M, N are not S-equivalent.

Conversely, consider $\lambda, \mu \in \{0, 1, \infty\}$. Suppose for example that $\lambda = 0$. Then $v_4 = (1, 0) = v_1$ and so M has composition series $0 \leq M_1 \leq M$, with M_1 pictured below.

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & & \Big\downarrow{\scriptstyle v_1} & & \\ \mathbb{C} & \xrightarrow{v_1} & \mathbb{C}v_1 & \longleftarrow & 0 \\ & & \Big\uparrow & & \\ & & 0 & & \end{array}$$

The other two choice for λ give similar composition series involving two other subrepresentations denoted M_2, M_3. To prove $M(\lambda)$, $M(\mu)$ are not S-equivalent, it suffices to show that the composition factors are non-isomorphic, that is, $M_i \ncong M_j$. Consider two such M_i, M_j where $i, j \in \{1, 2, 3\}$ correspond to the choice of $\lambda \in \{0, 1, \infty\}$, respectively. Observe the two morphisms f_0, f_4 between each W_0 and W_4 of each M_i, M_j, which should commute by the diagram below.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{v_i} & \mathbb{C}v_i \\ {\scriptstyle f_4}\Big\downarrow & & \Big\downarrow{\scriptstyle f_0} \\ \mathbb{C} & \xrightarrow{v_j} & \mathbb{C}v_j \end{array}$$

So the condition $M_i \cong M_j$ imposes that $f_4 v_j = f_0 v_i$. But v_i, v_j are non-parallel for all $i, j \in \{1, 2, 3\}$ distinct, and $f_0, f_4 \in \mathbb{C}$ so the equation only holds when $i = j$. This only occurs when $\lambda = \mu$. Thus the composition factors of the module representation of two points $M(\lambda)$, $M(\mu)$ are non-isomorphic and hence they are not S-equivalent. □

Thus each representation is not GIT equivalent and so the quotient $\mathcal{R}^{ss} /\!\!/ G$ separates their orbits. Hence the map $\{M(\lambda) : \lambda \in \mathbb{C}\} \to \mathcal{R}^{ss} /\!\!/ G$ is injective.
Thus $\phi : \mathbb{P}^1 \to \mathcal{R}^{ss} /\!\!/ G$ is injective.

We now wish to prove φ is surjective.
To prove this, we first need a morphism called the cross ratio, which is G-invariant.
Definition 3.5.5. The *cross-ratio* is the map $\psi : \mathcal{R}^{ss} \to \mathbb{P}^1$ defined as follows.

$$(v_1, v_2, v_3, v_4) \longmapsto (\det(v_1, v_3)\det(v_2, v_4) : \det(v_1, v_4)\det(v_2, v_3)) .$$

Where $\det(v_i, v_j)$ is the determinant of the 2×2 matrix with columns v_i, v_j.

This map is well-defined, as $(0 : 0)$ is not in the image of ψ. Suppose that $\psi(v_1, v_2, v_3, v_4) = (0 : 0)$. Then $\det(v_i, v_j) = 0$ and $\det(v_j, v_k) = 0$ for some $i, j, k \in \{1, 2, 3, 4\}$ and so v_i, v_j, v_k are all parallel. Without loss of generality, assume that v_1, v_2, v_3 are parallel. Then consider a subrepresentation in which $W_0 = \mathbb{C}v_1$ and $W_4 = 0$, where v_4 is non-parallel to the others. Suppose it has corresponding module representation M, so $\theta(M) = -1$ and thus M is not θ-semistable. Thus $(v_1, v_2, v_3, v_4) \notin \mathcal{R}^{ss}$. So ψ is well-defined.

The cross-ratio aids in defining an inverse and proving surjectivity for ϕ. We first wish to prove the following lemma.

Lemma 3.5.6. The cross ratio $\psi : \mathcal{R}^{ss} \to \mathbb{P}^1$ is G-invariant and surjective.

Proof. The group G is composed of $\mathrm{GL}_2(\mathbb{C})$ and $(\mathbb{C}^*)^4$, where $\mathrm{GL}_2(\mathbb{C})$ acts by multiplication from the left and $\mathbb{C}^*$ acts by multiplication from the right. The map ψ is clearly invariant under $\mathrm{GL}_2(\mathbb{C})$ as $\det(Av_i, Av_j) = \det(A)\det(v_i, v_j)$ for $A \in \mathrm{GL}_2(\mathbb{C})$.
The group $(\mathbb{C}^*)^4$ acts by scalar multiplication on each v_i distinctly. As det is linear in each input, we have $\det(t_i v_i, t_j v_j) = t_i t_j \det(v_i, v_j)$. So clearly ψ is also invariant under $(\mathbb{C}^*)^4$. Thus ψ is G-invariant.

To show surjectivity, observe the image of $M(\lambda)$ under ψ for $\lambda \notin \{0, 1, \infty\}$.

$$\begin{aligned}\psi(M(\lambda)) &= \psi\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\\lambda\end{pmatrix}\right)\\ &= (1 : \lambda).\end{aligned}$$

Similarly $M(\lambda^{-1})$ maps to $(\lambda : 1)$ and the remaining values $\{0, 1, \infty\}$ give the following.

$$M(1) \longmapsto (1:1), \qquad M(0) \longmapsto (1:0), \qquad M(\infty) \longmapsto (0:1).$$

These cover the entire projective line $\mathbb{P}^1$, and thus ψ is surjective. $\square$

Finally we may now show $\varphi : \mathbb{P}^1 \to \mathcal{R}^{ss}/\!\!/G$ is surjective.
First note G is reductive and denote $\pi : \mathcal{R}^{ss} \to \mathcal{R}^{ss}/\!\!/G$. Note $\mathcal{R}^{ss}$ is an affine variety. Thus by theorem 2.1.10, $(\mathcal{R}^{ss}/\!\!/G, \varphi)$ is a categorical quotient. So the diagram below commutes, where ψ is G-invariant.

$$\begin{array}{ccc} \mathcal{R}^{ss} & \xrightarrow{\pi} & \mathcal{R}^{ss}/\!\!/G \\ {\scriptstyle\psi}\downarrow & \nearrow_{\varphi} & \\ \mathbb{P}^1 & & \end{array}$$

But both π and ψ are surjective and hence φ is too.
Thus $\mathcal{R}^{ss}/\!\!/G \cong \mathbb{P}^1$.

We now wish to find the inverse of φ.
We can restrict the cross-ratio $\psi|_{\mathcal{R}^{ss}/\!\!/G} = \psi' : \mathcal{R}^{ss}/\!\!/G \to \mathbb{P}^1$. This gives a candidate for the inverse of φ.

Lemma 3.5.7. The map ψ' is the inverse of φ.

Proof. It suffices to show that ψ' satisfies $\psi' \circ \varphi = \mathrm{id}_{\mathbb{P}^1}$ and $\varphi \circ \psi' = \mathrm{id}_{\mathcal{R}^{ss}/\!\!/G}$. Consider $(1 : \lambda) \in \mathbb{P}^1$ for any $\lambda \in \mathbb{C} \setminus \{0, 1, \infty\}$. Then

$$(\psi' \circ \varphi)(1 : \lambda) = \psi'(M(\lambda)) = (1 : \lambda).$$

Similarly $(\psi' \circ \varphi)(\lambda : 1) = (\lambda : 1)$. We have explicitly defined the image of the points $(1 : 0), (0 : 1), (1 : 1)$ under both maps earlier and they are all fixed by $\psi' \circ \varphi$. Thus $\psi' \circ \varphi = \mathrm{id}_{\mathbb{P}^1}$.

Next we wish to show $\varphi \circ \psi' = \mathrm{id}_{\mathcal{R}^{ss}/\!\!/G}$. Consider $(v_1, v_2, v_3, v_4) \in \mathcal{R}^{ss}/\!\!/G$ such that all v_i, v_j are non-parallel, so

$$\begin{aligned}\psi'(v_1, v_2, v_3, v_4) &= (\det(v_1, v_3)\det(v_2, v_4) : \det(v_1, v_4)\det(v_2, v_3)) \\ &= \left(1 : \frac{\det(v_1, v_4)\det(v_2, v_3)}{\det(v_1, v_3)\det(v_2, v_4)}\right).\end{aligned}$$

For clarity denote $\beta_{ij} = \det(v_i, v_j)$. Finally we can map this by φ to get

$$\varphi\left(1 : \frac{\beta_{14}\beta_{23}}{\beta_{13}\beta_{24}}\right) = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\ \frac{\beta_{14}\beta_{23}}{\beta_{13}\beta_{24}}\end{pmatrix}\right).$$

Note the resultant point is in $\mathcal{R}^{ss}/\!\!/G$ and thus we can act on it by G, which includes individually scaling each vector or multiplying all of them by the same matrix in $\mathrm{GL}_2(\mathbb{C})$. We wish to scale the four vectors (v_1, v_2, v_3, v_4) in this way to get back the vectors given above. Scaling the last vector gives $(\beta_{13}\beta_{24}, \beta_{14}\beta_{23})$.

Denote the components of each vector as $v_i = (x_i, y_i)^T$ and consider matrix $A \in \mathrm{GL}_2(\mathbb{C})$ given by

$$A = \begin{pmatrix} y_3\beta_{24} & -x_3\beta_{24} \\ y_4\beta_{23} & -x_4\beta_{23}\end{pmatrix}.$$

Note $\det(A) = \beta_{24}\beta_{23}\beta_{34} \neq 0$. One can check that $Av_i = (\beta_{i3}\beta_{24},\, \beta_{i4}\beta_{23})^T$. This gives the following maps of each v_i after multiplication by A and scaling. Note that $\beta_{ii} = 0$.

$$\begin{aligned} Av_1 &= \begin{pmatrix}\beta_{13}\beta_{24}\\ \beta_{14}\beta_{23}\end{pmatrix}, \\ Av_2 &= \begin{pmatrix}\beta_{23}\beta_{24}\\ \beta_{24}\beta_{23}\end{pmatrix} \longmapsto \begin{pmatrix}1\\1\end{pmatrix}, \\ Av_3 &= \begin{pmatrix}\beta_{33}\beta_{24}\\ \beta_{34}\beta_{23}\end{pmatrix} \longmapsto \begin{pmatrix}0\\1\end{pmatrix}, \\ Av_4 &= \begin{pmatrix}\beta_{34}\beta_{24}\\ \beta_{44}\beta_{23}\end{pmatrix} \longmapsto \begin{pmatrix}1\\0\end{pmatrix}. \end{aligned}$$

Thus we may apply the inverse of the scaling and multiply by A^{-1} to get back to (v_1, v_2, v_3, v_4) in $\mathcal{R}^{ss}/\!\!/G$.
We must also consider the case that two entries v_i, v_j are parallel. Note that if three vectors are parallel then the point is no longer in $\mathcal{R}^{ss}$. If we choose any pair of vectors which appear in a determinant in the cross-ratio, then we end up with $\varphi(0 : 1)$ or $\varphi(1 : 0)$. If the pair is not present in the cross-ratio, like v_1, v_2 for example, then we have $\varphi(1 : 1)$. Each of these give four vectors which can be scaled back to their original points in $\mathcal{R}^{ss}$. Finding the matrix which does this for each case is an exercise.

Thus $\varphi \circ \psi' = \mathrm{id}_{\mathcal{R}^{ss}/\!\!/G}$.
So ψ' is the inverse of φ. □

Thus we have proven that $\mathcal{R}^{ss}/\!\!/G \cong \mathbb{P}^1$. So we have the quotient

$$\mathcal{R}^{ss} \longrightarrow \mathbb{P}^1.$$

Conclusion

Algebraic geometry gives zero sets of polynomials an algebraic interpretation. This allows us to study them intrinsically and even consider group actions on them. Geometric Invariant Theory offers a conclusive method to calculating and assessing quotients of group actions on such spaces. With this power we are able to give a geometric structure to the space of orbits of group actions. Although, this was not always exact as we saw in some cases where all orbits coalesced to a single point. Thus by restricting the domain of the space, we can extract more valuable information from this quotient.

Representations of quivers are also affine spaces, whose isomorphism classes under a change of basis can be studied through constructing appropriate quotients. If the quiver's vertices only consist of one-dimensional complex space, then the quotients can be calculated directly through the invariant subalgebra. However, when dealing with any spaces of more dimension, this instantly becomes a challenge.

Thus the knowledge of GIT aids in finding the quotient without the need for such calculation. Through this, the cross-ratio is used, a purely geometric construct.

The notions in algebraic geometry presented just scratch the surface, as the first chapter only covers the content required to be introduced to Geometric Invariant Theory. If explored more deeply, one can uncover regular functions and then sheaves, which unify some of the notions in algebraic geometry that we have explored in the thesis.

We hope the reader steps away from this thesis with an understanding of algebraic geometry and a knowledge in the construction of isomorphism classes of quiver representations. For further exploration in each topic, the reader is encouraged to approach each topic through the texts referenced at the beginning of the respective chapters.

References

[Hos16] Hoskins, V., (2016). *Moduli Problems and Geometric Invariant Theory.* Lecture Notes, http://userpage.fu-berlin.de/hoskins/M15_Lecture_notes.pdf.

[King94] King A.D. (1994). *Moduli of Representations of Finite Dimensional Algebras.* The Quarterly Journal of Mathematics, 45(4), pp.515–530. doi:10.1093/qmath/45.4.515.

[Br08] Brion, Michel. (2008). *Representations of quivers.* 10.1007/978-1-4020-5141-8_2.

[Hart10] Hartshorne, R. (2010). *Algebraic geometry.* New York: Springer.

[Shaf94] Shafarevich, I.R. (1994). *Basic Algebraic Geometry.* Berlin ; New York: Springer-Verlag.

[Shaf13] Shafarevich, I.R. (2013). *Basic Algebraic Geometry 2.* Springer.

[Mum92] David Mumford and Fogarty, J. (1982). *Geometric Invariant Theory.* Berlin: Springer.

[LePo04] Joseph Le Potier (2004). *Lectures on Vector Bundles.* Cambridge: Cambridge Univ. Press.