CLASSIFICATION OF SIMPLE PLANE CURVE SINGULARITIES AND THEIR AUSLANDER-REITEN QUIVER

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Abstract

This thesis is largely a consolidation of parts of lectures given by Yoshino, Y. at Tokyo Metropolitan University in 1987 (*cf.* [17]). We aim to investigate the Cohen-Macaulay modules of simple plane curve singularities. We consider such simple plane curve singularities algebraically as quotient rings $R = k[\![x, y]\!]/(f)$ of the formal power series ring in two variables over an algebraically closed field k. In fact, the study of Cohen-Macaulay modules over such rings reduces to the study of torsion-free R-modules. We prove a classification theorem of one-dimensional rings R by the power series f, as well as an easy extension of this theorem to higher dimensions. We label the resultant classes of R by the classical Dynkin types A_n, D_n, E_6, E_7, E_8 (*cf.* [6]). For the case of A_n in dimension one, we construct the quiver of morphisms between indecomposable torsion-free modules. We will make reference to an extra structure element of this quiver, which indicates the existence of Auslander-Reiten sequences of the torsion-free modules. A complete list of all such Auslander-Reiten quivers will be provided in appendix A.

Contents

List of Figures

Chapter 1

Introduction

An algebraic plane curve can be defined as the locus of points which satisfy the equation $f(x, y) = 0$, where $f(x, y)$ can be thought of as a Taylor series centred at $(0, 0)$ over an algebraically closed field k. Then the germ of the curve at $(0, 0)$ is classed as *smooth* if both of the partial derivatives of $f(x, y)$ vanish at $(0, 0)$. Otherwise, we class it as *singular*. To study this algebraically, we introduce the ring $S = k[\![x, y]\!]$ of formal power series over k, consisting of all expressions in powers of x and y. While we do not ask that a power series be convergent analytically like the Taylor series, any sequence of polynomials in $k[x, y]$ converges to a power series in S and we can build an appropriate $f(x, y)$ thusly. In this sense, it is perfectly sound reasoning to admit the function f to be among the set S. This allows an algebraic plane curve to be encoded in the quotient ring given by $R = S/(f)$, in which $f \equiv 0$ under the natural quotient structure.

The question arises, for which power series f is the algebraic plane curve singular; in fact, does the meaning of the term singularity even correlate with the analytic version above? The answer is yes, and in fact, the algebraic distinction between smooth and singular is even more profound. We will see that the plane curve described by $f = 0$ is smooth if and only if its ring $R = S/(f)$ is a principal ideal domain (PID) isomorphic to $k[\![x]\!]$ (which has even more special properties). We will also classify plane curves by this f for some mild cases of singularities known as simple singularities (theorem 3.2.4).

The most basic example of these simple singularities is the case when $f = x^2 - y^3$, shown in figure 1.1. The figure illustrates nicely the fact that we obtain the same

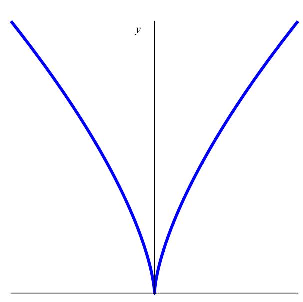


Figure 1.1: The curve described by $f = x^2 - y^3$ is singular at $(0, 0)$

locus of points which satisfy the equation $f = 0$ via both methods to produce the same graph. From the ring perspective, there is an algebraic variety of points (*cf.* [14]) corresponding to the zeros of f in $k[x,y]/(f)$; to extend this to $S/(f)$ provides no further structure for the curve because sequences of polynomials in $k[x,y]$ converge to limits in S (see section 2.2). Therefore, we obtain the same graph below from either approach. The check that $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = -3y^2$ which both vanish at $(0,0)$ is thus enough to call the curve singular.

In this thesis, we provide an algebraic criterion for smoothness, singularity, and simple singularity. It is easy to verify by the method used here that the simple singularities in theorem 3.2.4 are in fact singular, and we will make clear the distinction between singularity and simple singularity in the algebra of the ring R.

A natural extension of the notion of plane curves is plane surfaces for the case of three dimensions, and in general we refer to the ring $k[\![x_0,\dots,x_d]\!]/(f)$ as a *hypersurface* defined by f of dimension d. In fact, we prove an easy extension of the classification theorem 3.2.4 to hypersurfaces of finite dimension.

Once we understand well the classification of our plane curve R, we naturally move on to classifying its algebraic structure. To do this, we must study various structure maps known as modules over R, and investigate how they interact with one another. This is where the use of quivers becomes important. A quiver is simply a directed graph which we will use to compile information about the modules we are interested in (the vertices of the quiver) and the morphisms between them (the arrows of the quiver). But which are the modules that we should include? To answer this question, we first need some more background.

The reader may have already come across the *structure theorem for finitely generated modules over a principal ideal domain* (*cf.* [8]). This states that any such module decomposes into a direct sum of a free submodule and a torsion submodule. This makes study of such principal ideal domains (PID's) nice and already well understood if we consider the torsion-free submodules (which will simply be free modules in the PID case). This is because free modules have a basis, so are the closest we can get to a vector space whilst maintaining a module structure map over the ring in question rather than scalar multiplication. Thus, one can investigate torsion-free modules over PID's assured that these are free and easily computable using ideas from linear algebra. In addition, we immediately obtain all the morphisms between submodules of free modules over PID's as nice matrices by simply observing the images of basis elements.

For the study of torsion-free modules over our plane curve singularities R, it is easy to work with finitely generated modules. However, R is not a PID in the singular case. The question then becomes, is it still of any worth to study torsion-free modules over singularities? The answer is yes. In general, torsion is a measure of the failure for a module M to be free. Of course, if $T(M)$ is a maximal torsion submodule of M, we have the *exact sequence* (see section 2.1)

$$0 \to T(M) \hookrightarrow M \twoheadrightarrow M/T(M) \to 0,$$

where $M/T(M)$ is torsion-free. Thus if we start with a module of the form $M/T(M)$, we can learn about an arbitrary module via this sequence. So we study torsion-free modules as the closest ones to free modules in hope of alleviating the complexity of the algebra involved as much as possible.

In fact, torsion-free modules deliver very useful properties when studying plane curve singularities. There is a theorem by Buchweitz, Greuel and Schreyer (*cf.* [2]) which states that if a plane curve has finitely-many non-isomorphic indecomposable torsion-free modules, then it is a simple singularity. We say that plane curves with this property have *finite representation type* (FRT). We study simple plane curve singularities R because once we know about those then we can build any plane curve singularity with FRT as an intermediate ring extension of R in the integral closure of R. This is a re-stating of the birational dominance theorem of Greuel and Knörrer (*cf.* [5]), which provides the converse of the Buchweitz-Greuel-Schreyer theorem for plane curves and illustrates the significance of FRT.

In general, to study a surface or hypersurface, one would need to replace torsion-free modules with *Cohen-Macaulay* (CM) modules. These require some knowledge of the depth of a module M in terms of regular sequences, and perhaps even some homological algebra. This is not the intent of the thesis, which is more a pragmatic approach to the plane curve case for the naive reader. It is important to note, however, that in dimension one (the case of plane curves) the category of CM modules coincides with the category of torsion-free modules. Thus our approach inserts logically into the work of Auslander and Reiten (*cf.* [1, 17]). Many aspects of this thesis will still hold for CM modules, so we will use CM and torsion-free synonymously for the most part; we will take special care to discern their differences when considering purely one-dimensional arguments.

With the pragmatic approach to this topic as it is, we will have to bypass much of the profound Auslander-Reiten (AR) theory, such as almost split sequences of CM modules; this make for a thesis in its own right. As such, we will bypass the discussion of AR translations of CM modules in our construction of the *AR quiver* of a simple plane curve singularity. The AR quiver is simply the quiver of morphisms between isomorphism classes of indecomposable torsion-free modules that we seek, together with an extra structure element: a dotted line between two vertices M and N of the AR quiver indicates that there exists an AR sequence ending N with AR translation of N given by M. A brief overview of these concepts can be found in appendix A, but for thorough understanding, see [1, 17].

1.1 Outline

The main thesis will begin with some background on ring theory. In that chapter, we will introduce some useful results and ideas that will aid many proofs later on. In addition, we will introduce the ring of formal power series. There will be some very useful results involving power series, including the Weierstrass Preparation theorem 2.2.9 and Hensel's lemma 2.2.10, which both serve to deliver our main classification theorem 3.2.4.

From there, we will then move on to formally introduce the algebra of simple singularities and prove the main classification theorem 3.2.4.

We will then be interested in discussing structure on our simple singularities. We will provide a brief discussion on Cohen-Macaulay modules, and introduce our notion of torsion-free modules for the plane curve. We will make clear the distinction between writing torsion-free and Cohen-Macaulay throughout the thesis. It will become apparent that it is still worth talking of Cohen-Macaulay modules in this thesis, as we then provide a brief interlude of category theory in which we define the category of Cohen-Macaulay modules in general. Furthermore, we will move on to a description of matrix factorisations of Cohen-Macaulay modules from the general setting. The matrix factorisations will be shown to provide a means for computing our torsion-free modules in later examples, via Eisenbud's equivalence theorem 4.3.4 of the categories of Cohen-Macaulay modules and matrix factorisations.

We then will introduce the Auslander-Reiten quiver. We will observe the rules of construction and make brief mention of the Auslander-Reiten translation information. Then we will provide an in-depth construction of the quiver for the simplest simple singularity type, less the Auslander-Reiten aspect.

All other quiver types will be constructed somewhat and drawn for the purpose of completeness; their construction is an easy exercise in the concepts we discuss for the simplest simple singularity type. In addition, for completeness, the full Auslander-Reiten quivers will be drawn in appendix A.

Finally, we will make some closing remarks, with the implications of what we have achieved as well as suggestions for further study.

1.2 Notation

Where possible, I have tried to use traditional notation to denote the fundamental objects used. For full disambiguation, below is a list of some of the notation that I will be using, as well as my convention for symbols that sometimes have different interpretations in different mathematical circles.

- $\mathbb{Z}$ will denote all integers.
- $\mathbb{Z}_+$ will denote only the positive integers.
- $\mathbb{N}$ will denote exactly the non-negative integers (the natural numbers); note $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$.
- $\operatorname{Spec} R$ will denote the spectrum of the ring R consisting of all prime ideals of R.
- $X := Y$ will be interpreted as meaning X is defined to be Y, often used for applying meaning to new symbols.
- R^* will denote the group of units of the ring R.
- $k^* = k - 0$ if k is a field where $k - 0 := k \backslash \{0\}$
- $(R, \mathfrak{m}, k)$ will signify a local ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k \simeq R/\mathfrak{m}$. Sometimes simply $(R, \mathfrak{m})$ will suffice where the residue field is implied from the context.
- $\bar{f}$ will usually mean the image of $f \in R$ modulo some ideal $I \subseteq R$; the definition will be clear from the context.
- $\varphi\psi := \varphi \circ \psi$ for composition of morphisms φ and ψ where the context will be evident. Also, for full disambiguation, $\varphi \circ \psi(x) = \varphi(\psi(x))$ for any x upon which ψ acts.
- $\mathbf{1}_R := \mathrm{id}_R$ will be the identity mapping on the ring (or structure) R.

Chapter 2

Some Background on Ring Theory

In this chapter, we present some basic facts and general results about ring theory which will fuel the "engine" used to churn out most of the main results in subsequent chapters of this thesis. The nature of this layout will also provide a solid foundation upon which the reader can be confident of following ring theoretic arguments; very little will be assumed that the undergraduate mathematics student is not familiar with. We assume that the reader is familiar with basic concepts such as the axioms of rings, modules and algebras, and has some experience working with these objects. Of particular importance for matters concerning algebraic geometry, is an understanding of the ideal structure of a ring R encoded in its prime spectrum (consisting of all prime ideals of R), denoted $\operatorname{Spec} R$ when referred to below. A good introductory reference for this material is Reid (*cf.* [14]), and wherever unsure throughout this chapter, the reader need only refer to this text for consolidation of ideas.

2.1 Rings and Modules

Henceforth in this thesis, R shall denote a commutative ring with identity and M an R-module, unless otherwise stated.

We begin with some brief dimension theory. Whenever referring to the dimension of a ring R, we mean the Krull dimension of R which is the supremum of the number of strict inclusions in a chain of prime ideals in R. Formally, we have the following definition.

Definition 2.1.1. For a prime ideal $P \in R$, we define the *height* of P to be the supremum of the lengths of all strictly decreasing chains of prime ideals beginning with P; i.e.

$$\operatorname{height}(P) := \sup\{n \in \mathbb{N} : P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n,\ P_i \text{ prime}\}$$

Then the (Krull) dimension of R is given by

$$\dim R := \sup_{P \in \operatorname{Spec} R} \operatorname{height}(P).$$

Theorem 2.1.2. *If R is an Artinian commutative ring then* $\dim R = 0$.

Proof. Suppose R is Artinian and consider a prime $P \in \operatorname{Spec} R$. Then the integral domain R/P is Artinian, for if $I_1 \supseteq I_2 \supseteq \cdots$ is a descending chain of ideals in R/P, then it can be lifted to one in R via the correspondence of primes under the quotient morphism (*cf.* [14]). Thus, for any $0 \neq x \in R/P$, there exists a positive integer n such that $(x^n) = (x^{n+1})$. Consequently, we may write $x^n = yx^{n+1}$ for some $y \in R/P$, and this implies that $1 = yx$ after integral domain cancellation. Thus R/P is in fact a field so P is maximal in R. Hence we have shown that P has height 0 and so $\dim R = 0$. ❑

We now move on to some results with modules. Of particular importance will be the notion of exact sequences explored below, but first, the following will be a useful form of Nakayama's lemma for local rings in later proofs; see Reid (*cf.* [14]) for alternate formulations if necessary, but we will mainly be concerned with this version.

Lemma 2.1.3. (Nakayama)
Suppose R is a local ring with maximal ideal $\mathfrak{m}$. Let M be a finitely generated R-module, N a submodule of M, and I an ideal of R contained in $\mathfrak{m}$. Then $M = N + IM$ if and only if $M = N$.

Proof. The statement $M = N + IM$ is equivalent to $M + N = IM + N$ since N is a submodule of M, so in fact we have $M/N = I(M/N)$. Let $g_1, \ldots, g_m$ be generators of M. Then M/N is generated by $g_1 + N, \ldots, g_m + N$ because if $(\sum_{i=1}^m h_i g_i) + N \in M/N$, then this is the same as $\sum_{i=1}^m h_i(g_i + N)$; the reverse inclusion is clear. Let $G := \{g_i + N\}_{i=1}^{\ell}$ be a minimal generating set for M/N where $\ell \leq m$. We suppose that $\ell > 0$ and obtain a contradiction to conclude that M/N needs no generators so must be the zero module. If $\ell > 0$, since $M/N = I(M/N)$, we may express the element $g_1 + N \in M/N$ as a linear combination $g_1 + N = \sum_{i=1}^{\ell} a_i g_i + N$ where $a_i \in I$. Rearrangement yields $(1 - a_1)g_1 + N = \sum_{i=2}^{\ell} a_i g_i + N$. But $a_1 \in I \subseteq \mathfrak{m}$ implies that $1 - a_1$ is a unit in R. Thus $g_1 + N = \sum_{i=2}^{\ell}(1 - a_1)^{-1} a_i g_i + N$, so we may omit $g_1 + N$ from our generating set G, contrary to the minimality of G. Thus we have shown that $M/N = 0$, and this is true if and only if $M = N$. ❑

Later on we will see that Nakayama's lemma proves useful when determining results on a module by simply studying the structure modulo $\mathfrak{m}$. More on lifting elements will be discussed with Hensel's lemma 2.2.10.

We now have a look at the property of exactness for sequences of maps between modules.

Definition 2.1.4. For the ring R as discussed in this chapter, an *exact* sequence of R-modules

$$\cdots \to M_{i-1} \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \to \cdots$$

is a sequence in which, for each i, we have $\ker \varphi_i = \operatorname{im} \varphi_{i-1}$. In particular, a *short exact* sequence of R-modules L, M and N is an exact sequence given by

$$0 \to L \to M \to N \to 0.$$

Short exact sequences are crucial in many of the arguments to come in this thesis, so it is worth exploring some of their properties.

Observe that in the definition above, the strategic use of zeros means that the two outermost maps are the zero map; zero cannot map to anything but zero in L, and everything in N must map to zero. As such, we see that $L \to M$ is an injection since its kernel is given by the image of the zero map on the left, and $M \to N$ is a surjection since the kernel of the zero map on the right is all of N. In fact, this combined with the first isomorphism theorem (*cf.* [14]) enables us to deduce that $N \simeq M/L$. More precisely, if α is the map $L \to M$ and β the map $M \to N$, then we have

$$N = \operatorname{im} \beta \simeq M/\ker \beta = M/\operatorname{im} \alpha.$$

Actually, because L, M and N are all submodules of R with the same R-action we may say that $L = \alpha(L)$ as the canonical inclusion map, thus $M/\operatorname{im} \alpha = M/L$ and all previous isomorphisms become equalities, so $N = M/L$.

Often, when we have a submodule $L \subseteq M$ we will equivalently denote this with the *left* short exact sequence $0 \to L \to M$. Similarly we may use *right* exact sequences for quotients of modules.

The following definition will become necessary for the AR theory found in appendix A.

Definition 2.1.5. A short exact sequence of R-modules is called a *split* exact sequence if it is isomorphic to one of the form

$$0 \to L \to L \oplus N \to N \to 0.$$

Now, equipped with an understanding of some of the techniques that we will be using throughout this thesis, we are ready to move on to studying the specific ring which is the basis upon which we will be constructing simple plane curve singularities; the ring of formal power series.

In many senses, the ring of formal power series is a natural way of studying curves in a plane: we will be using two indeterminates, so these are easily associated with the coordinate axes of a plane for comfortable visualisation of the geometry involved. More often than not, we will be working like this, although it should be stated now that these indeterminates may be interchanged with other elements when they are suitably independent of one another in the usual basis sense in terms of k-linearity. A common theme during the main classification theorem 3.2.4 will be to perform these changes of variables in order to obtain a more tractable form of our power series that describes the curve. More on this to come, and now the properties of the formal power series ring.

2.2 Properties of the Formal Power Series Ring

In this section, we will investigate some useful properties of the formal power series ring that will be critical in subsequent chapters. Some basic concepts of commutative algebra will be assumed; these include Noetherian rings and local rings, all of which can be found in Reid (*cf.* [14]). These are vital concepts that we assume the reader understands the definition and properties of fairly well.

The ring of polynomials $k[x]$ over an algebraically closed field k can be defined as an infinite sequence of coefficients in k of which only finitely-many are non-zero; i.e. an element of $k[x]$ is given by

$$a_0 + a_1 X + \cdots + a_n X^n \simeq (a_0, a_1, \ldots, a_n, 0, 0, \ldots).$$

A natural extension of this is to consider arbitrarily long sequences $(a_n)_{n\in\mathbb{N}}$. We associate all such sequences with expressions in the indeterminate X and denote this the ring of formal power series $k[\![X]\!]$. We cannot obtain such sequences from the well-known ring operations of the polynomials, but we formally define their analog to be

$$\left(\sum_{i\in\mathbb{N}} a_i X^i\right) + \left(\sum_{i\in\mathbb{N}} b_i X^i\right) = \sum_{i\in\mathbb{N}} (a_i + b_i) X^i,$$

$$\left(\sum_{i\in\mathbb{N}} a_i X^i\right) \times \left(\sum_{i\in\mathbb{N}} b_i X^i\right) = \sum_{n\in\mathbb{N}} \left(\sum_{j=0}^{n} a_j b_{n-j}\right) X^n.$$

With these operations, $k[\![x]\!]$ does indeed form a commutative ring, with identity represented by the sequence $(1, 0, 0, \ldots)$, and zero represented by $(0, 0, \ldots)$. In fact, as we will see in subsection 2.2.1, this ring forms a principal ideal domain.

In several indeterminates $X_1, \ldots, X_d$, we let $X^\alpha = {X_1}^{\alpha_1} \cdots {X_d}^{\alpha_d}$ for $\alpha \in \mathbb{N}^d$. Then a power series in $k[\![X_1, \ldots, X_d]\!]$ is expressed as $\sum_{\alpha\in\mathbb{N}^d} a_\alpha X^\alpha$ and the ring operations are a clear extension of the above. Although now the ideals are not principal — e.g. $(X_1, \ldots, X_d)$ is an ideal — we do indeed still have unique factorisation in this ring. This comes from the Weierstrass Preparation theorem 2.2.9 below, which is used to write a power series as an associate to a polynomial in a single variable. Then the fact that the polynomial ring is a unique factorisation domain yields the result we are after.

From the formulation given, it is clear that the polynomials embed as a subring of the formal power series ring. In fact, any sequence of polynomials converges to a unique power series via the following proposition. We use a pragmatic approach to the notion of the completion of $k[z_1, z_2, \ldots, z_d]$. We let $\mathfrak{m} = (z_1, z_2, \ldots, z_d)$ and consider the $\mathfrak{m}$-adic topology consisting of open sets given by the powers of $\mathfrak{m}$ (*cf.* [4]). We can thus obtain the completion of $k[z_1, z_2, \ldots, z_d]$ by providing a prescription for constructing a power series from an infinite sequence of cosets of $\mathfrak{m}^i$, $i \geq 1$. For this, we have the following proposition.

Proposition 2.2.1. *The ring $S = k[\![z_1, z_2, \ldots, z_d]\!]$ is the completion of the ring $P = k[z_1, z_2, \ldots, z_d]$ with respect to the maximal ideal $\mathfrak{m} = (z_1, z_2, \ldots, z_d)$ of P.*

Proof. Consider the sequence of polynomials given by $(f_1 + \mathfrak{m}, f_2 + \mathfrak{m}^2, \ldots)$. Then we may form a well-defined power series given by

$$g := f_1 + (f_2 - f_1) + (f_3 - f_2) + \ldots$$

Note that we have $f_{i+1} - f_i \in \mathfrak{m}^{i+1}$, so the coefficient of ${z_1}^{j_1}{z_2}^{j_2}\cdots{z_d}^{j_d}$ in g where $j_1 + j_2 + \cdots + j_d < i$ is obtained solely from the polynomial f_i. ❑

The following is a crucial definition for studying simple curve singularities.

Definition 2.2.2. A *regular local ring* is a Noetherian local ring with the property that the minimal number of generators of its maximal ideal is equal to its (Krull) dimension. We call such generators *regular parameters*

Of note is that a field is a regular local ring, since it has dimension zero and unique maximal ideal (0). This is clear, as any non-zero element generates the entire field.

We build up the fact that $k[\![x, z_2, \ldots, z_d]\!]$ is a regular local ring by first showing its local structure.

Proposition 2.2.3. *The ring $k[\![x, z_2, \ldots, z_d]\!]$ is a local ring with maximal ideal $\mathfrak{m}_d = (x, z_2, \ldots, z_d)$.*

Proof. We prove this by induction on d. To see that $k[\![x]\!]$ is local, we show that it has a unique maximal ideal given by (x). We do this by showing that the ideal $1 + (x)$ contains only units of $k[\![x]\!]$ (see [14] if not convinced). Let $r \in (x)$; i.e. $r = a_1x + a_2x^2 + \ldots$ for some $a_i \in k$. Observe that we may obtain the inverse of $1 - r$ via the sum of the geometric series $(1 - r)^{-1} = \sum_{i=0}^{\infty} r^i$. To see that this is a well-defined element of $k[\![x]\!]$, we use an argument similar to that in proposition 2.2.1 above. Consider the partial sums $S_n := \sum_{i=0}^{n} r^i$. We seek the coefficient of x^j in $(1-r)^{-1}$ for each j. Note that $r^j \in (x^j)$, so for $n > j$, we have

$$S_n - S_j = \sum_{i=j+1}^{n} r^i = \sum_{i=0}^{n-j-1} r^{i+j+1} = r^{j+1} S_{n-j-1} \in (x^{j+1}).$$

Thus the coefficient of x^j in $(1-r)^{-1}$ remains the same after j partial sums and we determine it to be this coefficient obtained from the polynomial S_j. Thus $k[\![x]\!]$ is local with maximal ideal $\mathfrak{m}_1 = (x)$. In fact, it is not too difficult to see that the same argument holds if we replace k with $k[\![x, z_2, \dots, z_{d-1}]\!]$; if $a_i \in k[\![x, z_2, \dots, z_{d-1}]\!]$ in the above reasoning, then these remain unchanged elements of the power series ring after j partial sums. Thus the inductive step yields that $k[\![x, z_2, \dots, z_{d-1}]\!][\![z_d]\!]$ is a local ring with maximal ideal generated by z_d over $k[\![x, z_2, \dots, z_{d-1}]\!]$; i.e. the maximal ideal is $\mathfrak{m}_d$. ❑

Proposition 2.2.4. *The ring $k[\![x, z_2, \dots, z_d]\!]$ is a regular local ring of dimension d and regular parameters $x, z_2, \dots, z_d$.*

Proof. With proposition 2.2.3 above, we need only show that $k[\![x, z_2, \dots, z_d]\!]$ is Noetherian of dimension d. We do this by induction on d. See that $k[\![x]\!]$ is generated by the regular parameter x over k, so the base case of the induction involves showing that $k[\![x]\!]$ is Noetherian of dimension 1. Now, the proper ideals of $k[\![x]\!]$ are linearly ordered by inclusion since they must have the form (x^i) for $i \geq 1$, and these are all contained in the maximal ideal $\mathfrak{m}_1 = (x)$. Thus any ascending chain of ideals must terminate, hence $k[\![x]\!]$ is Noetherian. Also, the linear ordering yields the maximum height over all prime ideals of $k[\![x]\!]$ (indeed the only prime ideals are (0) and $\mathfrak{m}_1$) to be height $\mathfrak{m}_1 = 1$. Thus $k[\![x]\!]$ is Noetherian of dimension 1; i.e. $k[\![x]\!]$ is a regular local ring of dimension 1. Now suppose that $R := k[\![x, z_2, \dots, z_{d-1}]\!]$ is Noetherian of dimension $d-1$. Then $R[\![x_d]\!]$ is Noetherian as a consequence of Hilbert's basis theorem (*cf.* [14]) and proposition 2.2.1. In addition, we see that $R[\![z_d]\!]$ has dimension 1 over R, so must have dimension d over k. But we have seen, via proposition 2.2.3, that at least d parameters are needed to generate $\mathfrak{m}_d$, so $R[\![z_d]\!]$ is a regular local ring. ❑

2.2.1 The Special Case With One Indeterminate

We begin this subsection with some theory on *discrete valuation rings.*

Definition 2.2.5. For a field K, a *discrete valuation* is a surjection $v : K - 0 \to \mathbb{Z}$ with the properties that for all $x, y \in K - 0$

(a) $v(xy) = v(x) + v(y)$,

(b) $v(x \pm y) \geq \min\{v(x), v(y)\}$.

Then a *discrete valuation ring* (DVR) V is formed by the elements $x \in K$ for which $v(x) \geq 0$, with the convention that $v(0) = \infty$ to resolve conditions (a) and (b).

Verification that V forms a subring of K is clear. For $x, y \in V$, say with $v(x) \geq v(y) \geq 0$, we have $v(xy) = v(x) + v(y) \geq 0$ and $v(x - y) \geq v(y) \geq 0$. Thus $xy, x - y \in V$, and $v(1) + v(x) = v(x) \Rightarrow v(1) = 0$ so $1 \in V$.

Now, any DVR V is actually also a local PID because of the following. Firstly, to see that V is local, let $x \in V$ such that $v(x) > 0$. Then for any $y \in V$, observe that $v(xy) = v(x) + v(y) \geq v(x) > 0$. In addition, suppose that $v(y) > 0$. Then we are assured that $v(x + y) > 0$, observing (b). Thus we form an ideal of V consisting of all elements $x \in V$ such that $v(x) > 0$. Now for $u \in K$, since $0 = v(1) = v(u^{-1}u) = v(u^{-1}) + v(u)$, then $v(u^{-1}) = -v(u)$, and this reveals that u is a unit in V if and only if $v(u) = 0$. Thus V is partitioned into units and the ideal of elements with positive valuation, hence V is local with maximal ideal $\mathfrak{m} = \{x \in V : v(x) > 0\}$. The PID structure of V is a consequence of the following proposition.

Proposition 2.2.6. *Let V be a DVR over the field K, v its valuation, and $\mathfrak{m}$ its maximal ideal. Then $\mathfrak{m} = (t)$ for some t such that $v(t) = 1$, and any ideal $I \in V$ is either the zero ideal or has the form $I = (t^n) = \mathfrak{m}^n$ for some $n \in \mathbb{N}$.*

Proof. Take $t \in K$ with minimal valuation in $\mathfrak{m}$ as in the statement of the proposition. We show that any non-zero ideal $I \in V$ has the form $I = (t^n)$ for some $n \in \mathbb{N}$. Clearly the case that $n = 0$ has that $I = (t^0) = (1) = V$. Let $I \subsetneq V$ be a non-zero ideal and observe the following. For $a \in I$ with minimal valuation, say $v(a) = n$, in K we note that $v(at^{-n}) = v(a) + v(t^{-n}) = 0$. So $u := at^{-n}$ is a unit in V and $a = ut^n$. Thus we see that $t^n = u^{-1}a \in I$. This means that we have $(t^n) \subseteq I$. On the other hand, for any other $b \in I$ we have, say $m = v(b) \geq n$ by the minimality of n. Similarly as above we obtain $b = u't^m$ for some unit $u' \in V$. But $m \geq n$ has that $b = u't^{n+k} = u't^k t^n$ for some positive integer K, thus $b \in (t^n)$. Clearly (t) is maximal amongst these proper ideals so $\mathfrak{m} = (t)$. ❑

Remark 2.2.7. The power series ring $k[\![x]\!]$ is a DVR with valuation $v(f)$ equal to the minimal power of x in the expression of f and maximal ideal $\mathfrak{m} = (x)$. In particular, we can define a notion of degree on this ring by taking $\deg f = v(f)$ for any $f \in k[\![x]\!]$. Intuitively, we see that this means, where there is no maximal power of x in the expression of an arbitrary $f \in k[\![x]\!]$, it suffices to observe the *minimal* power of x for the degree of a power series. In what follows, we shall implicitly understand that $\deg f$ means exactly this when we are in the context of a DVR.

2.2.2 Weierstrass Preparation

Now that we have an understanding of the ring $k[\![x]\!]$, we move on to proving the Weierstrass Preparation theorem, which will be critical later on when classifying simple singularities in dimension one (and even extending this to higher dimensions). First, we need a lemma. The proof of the following is a consolidation of a result found in Lang (*cf.* [12]).

Lemma 2.2.8. (Weierstrass Division)
Let S be a complete regular local ring with maximal ideal $\mathfrak{m}$ and consider

$$f(X) = \sum_{i=0}^{\infty} a_i X^i \in S[\![X]\!]$$

such that not all $a_i \in \mathfrak{m}$, say $a_n \in S^*$, $a_0, \ldots, a_{n-1} \in \mathfrak{m}$. Then for any $g \in S[\![X]\!]$ we can solve uniquely the equation

$$g = qf + r, \quad q \in S[\![X]\!],\ r \in S[X],\ \deg_X r \leq n-1. \tag{2.1}$$

Proof. (*cf.* Manin, [12])
Let α and τ be *head* and *tail* projections given by

$$\alpha : \sum_{i=0}^{\infty} b_i X^i \mapsto \sum_{i=0}^{n-1} b_i X^i,$$

$$\tau : \sum_{i=0}^{\infty} b_i X^i \mapsto \sum_{i=n}^{\infty} b_i X^{i-n}.$$

Note that these are k-linear maps which is clear from their definitions. The problem of finding q and r for which (2.1) holds is equivalent to the problem of finding q such that

$$\tau(g) = \tau(qf). \tag{2.2}$$

This is because if (2.1) holds, then $\tau(g - qf) = \tau(r) = 0$ and we obtain (2.2) by linearity of τ. Conversely, if (2.2) holds for some power series $q \in S[\![x]\!]$, then $\tau(g - qf) = 0$ implies that $g - qf = \alpha(g - qf)$. Thus $g = qf + r$ where $r =$

$\alpha(g-qf)$ is an appropriate polynomial with degree less than n. Now, quite clearly, $qf = q\alpha(f) + q\tau(f)X^n$ by splitting f into its head and tail components. Then applying τ, we obtain $\tau(qf) = \tau(q\alpha(f)) + q\tau(f)$, so (2.2) may be re-written

$$\tau(g) = \tau(q\alpha(f)) + \tau(q\tau(f)X^n) = \tau(q\alpha(f)) + q\tau(f). \tag{2.3}$$

The hypothesis $a_n \in S^*$ ensures that $\tau(f)$ is invertible, because a_n is separated from any power of X under the map τ. Thus we obtain q by finding $Z = q\tau(f)$ and then computing $q = Z(\tau(f))^{-1}$. In terms of Z, (2.3) becomes

$$\tau(g) = \tau\left(Z\frac{\alpha(f)}{\tau(f)}\right) + Z = \left(\text{id} + \tau \circ \frac{\alpha(f)}{\tau(f)}\right) Z =: \varphi Z.$$

The map $\tau \circ \frac{\alpha(f)}{\tau(f)}$ sends a power series into $\mathfrak{m}S[\![X]\!]$. This is because each of $a_0, \ldots, a_{n-1} \in \mathfrak{m}$, so $\alpha(f) \in \mathfrak{m}S[\![X]\!]$, and τ does not change this (nor multiplication by the unit $\tau(f)^{-1}$). Thus we may invert φ to obtain $Z = \varphi^{-1}\tau(g)$. ❑

We are now ready to prove the Weierstrass Preparation theorem. The following is an original statement of the theorem as it pertains to this thesis.

Theorem 2.2.9. (Weierstrass Preparation)
Let $f \in k[\![X_1, \ldots, X_d]\!]$ and assume that when each X_i, $i = 1, \ldots, d-1$, is evaluated at zero, we have $f(0, \ldots, 0, X_d) = \alpha {X_d}^n + g(X_d)$, $\deg g \geq n+1$ where $\alpha \in k - 0$. Then there are unique functions $u \in k[\![X_1, \ldots, X_d]\!]$ with $u(0) \neq 0$, and $a_i \in k[\![X_1, \ldots, X_{d-1}]\!]$, $1 \leq i \leq n$ with $a_i(0) = 0$, such that

$$f = u({X_d}^n + a_1{X_d}^{n-1} + \cdots + a_{n+1});$$

i.e. f differs by a unit from a polynomial in $k[\![X_1, \ldots, X_{d-1}]\!][X_d]$.

Proof. We use the Weierstrass division with $S = k[\![X_1, \ldots, X_{d-1}]\!]$ and $f \in S[\![X_d]\!]$ to write uniquely ${X_d}^n = qf + r$. Then since $\deg r \leq n-1$ we had better have ${X_d}^n$ a summand in the expression of qf. But this can only occur if q has constant term $q_0 \in k - 0$ (specifically α^{-1}), hence q is invertible and we obtain

$$f = q^{-1}({X_d}^n - r), \quad r \in S[X_d].$$

such an expression is unique via Weierstrass division. ❑

We now move on to a discussion of Hensel's lemma.

2.2.3 Hensel's Lemma

Hensel's lemma is a fundamentally important lemma when studying hypersurface singularities, with many applications for lifting approximate results modulo the maximal ideal $\mathfrak{m}$ of a local ring to exact results in the ring itself. Here we will present a carefully selected version of Hensel's lemma which will have a high degree of relevance to the reader in the context of this thesis. The aim is to provide a pragmatic approach to the lemma and prove some results which require its use, in order to illustrate its importance when studying rings as in this thesis.

The following statement of Hensel's lemma is a version that appears in Eisenbud (*cf.* [4]), the proof of which is original work based on exercises found in the source. We include it here with proof and simple example in attempt to bestow upon the reader some useful experience in dealing with power series.

Lemma 2.2.10. *(*Hensel*)*
Let R be a Noetherian local ring, complete with respect to its maximal ideal $\mathfrak{m}$. Let $f \in R[x]$ be a polynomial in one variable with coefficients in R, and let $\bar{f}$ be the polynomial over $R/\mathfrak{m}$ obtained by reducing the coefficients of f modulo $\mathfrak{m}$. If $\bar{f}$ has factorisation

$$\bar{f} = \bar{g}_1\bar{g}_2 \in (R/\mathfrak{m})[x]$$

where $\bar{g}_1$ is monic and coprime to $\bar{g}_2$, then there is a unique factorisation

$$f = g_1g_2 \in R[x]$$

where g_1 (monic) and g_2 reduce to $\bar{g}_1$ and $\bar{g}_2$ respectively modulo $\mathfrak{m}$.

Proof. We first approximate g_1 and g_2 by any two polynomials g_{11} and g_{21} respectively for which g_{11} is monic and $\bar{g}_{11} = \bar{g}_1$, $\bar{g}_{21} = \bar{g}_2$ after reducing the coefficients modulo $\mathfrak{m}$. Then g_{11} is coprime to g_{21}. Since $\bar{g}_1$ is coprime to $\bar{g}_2$, this is equivalent to $\mathfrak{m}R[x]/(g_1, g_2) = R[x]/(\bar{g}_1, \bar{g}_2) = 0$. Now, $R[x]/(g_{11}, g_{21})$ is a finitely generated module over R. Since g_{11} is monic of, say, degree d, then $R[x]/(g_{11}, g_{21}) = R + Rx + \cdots + Rx^{d-1}$. Thus the proof of Nakayama's lemma 2.1.3 shows that $\mathfrak{m}R[x]/(g_{11}, g_{21}) = R[x]/(\bar{g}_1, \bar{g}_2) = 0$ implies that $R[x]/(g_{11}, g_{21}) = 0$. Consequently, we may write

$$f - g_{11}g_{21} = g_{11}h_1 + g_{21}h_2, \tag{2.4}$$

for some $h_1, h_2 \in (R/\mathfrak{m})[x]$. We then take a second lot of approximations $g_{12} = g_{11} + h_2$ and $g_{22} = g_{21} + h_1$. For these, we have

$$g_{12}g_{22} = g_{11}g_{21} + (g_{11}h_1 + g_{21}h_2) + h_1h_2 = f + h_1h_2. \tag{2.5}$$

Note that h_1 and h_2 both have coefficients in $\mathfrak{m}$ because g_{11} and g_{21} both have coefficients in R, so (2.4) can be formulated in this manner, with $g_{i2} \equiv g_{i1} \bmod \mathfrak{m}$. In addition, the product $h_1 h_2$ has coefficients in $\mathfrak{m}^2$, so (2.5) yields $f \equiv g_{12} g_{22} \bmod \mathfrak{m}^2$; i.e. g_{12}, g_{22} produce a factorisation of f modulo $\mathfrak{m}^2$. Since R is complete with respect to $\mathfrak{m}$, we may obtain successive approximations with the same reasoning, replacing $\mathfrak{m}$ with $\mathfrak{m}^i$ and $\mathfrak{m}^2$ with $\mathfrak{m}^{i+1}$; i.e. the leading terms remain unchanged as we observe a sequence of approximations up to increasing powers of $\mathfrak{m}$. We continue in this fashion to obtain the sequence of approximations g_{1n}, g_{2n} modulo $\mathfrak{m}^n$, and since R is complete, this sequence converges to a unique limit g_1, g_2 for which $f = g_1 g_2 \in R[x]$. ❑

Naturally, we have the following definition.

Definition 2.2.11. A local ring in which Hensel's lemma holds is called a *Henselian* ring.

Now we are able to discuss the following corollary to Hensel's lemma.

Corollary 2.2.12. *The ring of formal power series $k[\![x_0, x_1, \ldots, x_d]\!]$ is Henselian.*

Proof. Apply Hensel's lemma with the local ring $R = k[\![x_0, x_1, \ldots, x_d]\!]$ that is complete with respect to its maximal ideal $\mathfrak{m} = (x_0, x_1, \ldots, x_d)$. ❑

By far the most crucial application of Hensel's lemma is lifting solutions to algebraic equations modulo the maximal ideal of a ring to actual solutions; this will be fundamental later on and we will see an application of this at the end of this section. The notion of lifting solutions in general is a consequence of the following, another corollary of Hensel's lemma which we prove as original work.

Corollary 2.2.13. *Let R be local and complete with respect to its maximal ideal $\mathfrak{m}$ as in Hensel's lemma. Let $f \in R[x]$ and suppose that its image $\bar{f}$ in $(R/\mathfrak{m})[x]$ has a simple root $b \in (R/\mathfrak{m})[x]$. Then f has a root $a \in R[x]$ such that $\bar{a} = b$.*

Proof. We apply Hensel's lemma to the factorisation of $\bar{f}$ given by

$$\bar{f}(x) = (x - b)g(x)$$

for some $g \in (R/\mathfrak{m})[x]$ of which $(x - b)$ is not a factor (the latter owing to the fact that b is a simple root of $\bar{f}$). Notice that $x - b$ is monic and coprime to $g(x)$ for precisely the reasons just stated, thus we are ready to apply Hensel's lemma and obtain the factorisation of f given by

$$f(x) = (x - a)g'(x)$$

so that $\bar{a} = b$ (and $\bar{g'} = g$). Hence it is clear that a is a root of f and remains a root after reducing all the coefficients modulo $\mathfrak{m}$. ❑

It is worth mentioning explicitly that any quotient $(S/I, \mathfrak{n})$ of a Henselian ring $(S, \mathfrak{m})$ is Henselian via the canonical quotient map $S \to S/I$ which corresponds one-to-one the ideals of S and S/I (*cf.* [14]). Under this we correspond $\mathfrak{n} \mapsto \mathfrak{m}$, as well as approximations to solutions of $f + I \in S/I[x]$ modulo $\mathfrak{n}$ to those of $f \in S[x]$ modulo $\mathfrak{m}$. These are then lifted to solutions in S and projected onto S/I. This idea will ensure us that the hypersurfaces $k[\![x,y]\!]/(f)$ for some $f \in k[\![x,y]\!]$ in section 3.1 are Henselian because we have seen that $k[\![x,y]\!]$ is Henselian.

Below we have a proposition that implements the Henselian property, so this is a good exercise for the reader to realise the significance of the lemma and understand how it is applied. The proof is a clarified version of one found in Yoshino (*cf.* [17]), which will be crucial for the main classification theorem 3.2.4

Proposition 2.2.14. *Let $R = k[\![x_0, x_1, \ldots, x_d]\!]$ be a power series ring with maximal ideal $\mathfrak{m} = (x_0, x_1, \ldots, x_d)$ where k is an algebraically closed field of characteristic zero. Then for any unit $u \in R$ there is an automorphism of R which sends ${x_0}^n \mapsto u{x_0}^n$ while fixing each $x_i \mapsto x_i$, $1 \leq i \leq d$.*

Proof. Let $u \in R$ be a unit with image $u_0 \in k = R/\mathfrak{m}$. The algebraic equation $X^n - u_0 \equiv 0 \bmod \mathfrak{m}$ must have n solutions in k so since R is Henselian it follows that $v^n = u$ for some unit $v \in R$. Thus the map $\varphi : x_0 \mapsto vx_0$ and $x_i \mapsto x_i$, $1 \leq i \leq d$ is clearly an automorphism which satisfies $\varphi(x_0) = vx_0 = u^n x_0$. ❑

Chapter 3

Simple Singularities

We now move towards the notion of simple singularities. In this chapter, we establish the precise meaning of simple singularities and investigate some properties of these. The main result will provide a classification of such simple singularities in section 3.2, but first we investigate the basic concepts we will need.

3.1 Simple Singularities

For what follows, in this chapter, we will work with the regular local ring S and consider the hypersurface given by $R = S/(f)$ for some $f \in \mathfrak{n}$, where $\mathfrak{n}$ is the maximal ideal of S. We also denote $\mathfrak{m} = \mathfrak{n}/(f)$ the maximal ideal of R.

Firstly, we have the following crucial definition.

Definition 3.1.1. Let S be a regular local ring and $R = S/(f)$ a hypersurface. The *multiplicity* $e(R)$ of R is given by the maximal integer e for which $f \in \mathfrak{n}^e - \mathfrak{n}^{e+1}$.

Now this enables us to clarify what is meant by smooth and singular.

Definition 3.1.2. Let $(S, \mathfrak{n}, k)$ be a regular local ring and $R = S/(f)$ for some $f \in \mathfrak{n}$, with $\mathfrak{m}$ the maximal ideal of R. We say that R is *smooth* if $e(R) = 1$. Alternatively, R is called a *singularity* if $e(R) \geq 2$.

An heuristic approach to the above definition sheds some light on the ideas presented. We will be concerned with the example $S = k[\![x, y]\!]$ where it is easy to visualise the curve described by R as the locus of points associated with the power series $f = 0$. As such, in this case $\mathfrak{n} = (x, y)$ so that the germ of the curve R at the origin is considered smooth precisely when $f \in \mathfrak{n} - \mathfrak{n}^2$; i.e. either x or y appears in the expression of f, so f has an affine component. On the other hand, R is considered a singularity precisely when $f \in \mathfrak{n}^2$ so no such x or y appear. This is consistent with intuition derived from analytic methods for determining whether a curve is smooth or singular.

In fact, in such a case that $f \in \mathfrak{n} - \mathfrak{n}^2$, we may assume that $f = x + g$, up to a change of the variable x, for some $g \in S$. Then Weierstrass Preparation allows us to write $f = u(x^n - h_1 x^{n-1} - \cdots - h_0)$ for some unit $u \in S$ and power series $h_i \in (y)$. Then, since the only x^i term that is independent of y is x^n, we must have $n = 1$ since $f = x + g$. Thus we have $f(x, y) = u(x, y)(x - b_0(y))$ and so we obtain a relation in R that allows us to replace each x with $b_0(y)$; i.e. R is naturally isomorphic to the DVR $k[\![y]\!]$. Consequently, we see a consolidation of the reason that smooth curves have such nice properties in both calculus and algebra (recall section 2.2.1 for the properties of $k[\![y]\!]$).

The next question naturally arises – what types of singularities are tractable and worth studying in this context? To answer this, we classify such singularities as *simple* when the size of the set defined below is tractable.

Definition 3.1.3. Let $R = S/(f)$ be a singularity and consider the set of ideals in S given by

$$c(f) := \{I \subsetneq S : I \text{ is an ideal of } S \text{ with } f \in I^2\}.$$

We call R a *local ring of simple singularity* if the set $c(f)$ is finite.

Roughly speaking, $c(f)$ is a measure of the "size" of the singularity in the sense that the smaller the cardinality of $c(f)$, the more tractable the singularity will be in practice. In addition, with our example $S = k[\![x, y]\!]$, it is actually easy to characterise any such R by this set $c(f)$ as smooth, simply singular, or singular.

In general this definition determines the form of f up to the dimension of R, so that if $\dim(R) = 1$ then $e(R) \leq 3$, otherwise $e(R) \leq 2$ if $\dim(R) \geq 2$. We prove this in the following lemma.

Lemma 3.1.4. *Let $R = S/(f)$ be a local ring of simple singularity and assume that $k = S/\mathfrak{n}$ is an algebraically closed field.*

1. *If* $\dim(R) = 1$, *then* $e(R) \leq 3$,
2. *if* $\dim(R) \geq 2$, *then* $e(R) \leq 2$.

Proof. Let $d = \dim(R)$, so in particular, $\dim(S) = d + 1$. Then we may choose $\{z_0, z_1, \ldots, z_d\}$ to be a set of regular parameters of S. Thus, the natural projection $\pi : \mathfrak{n} \to \mathfrak{n}/\mathfrak{n}^2$ yields a k-basis $\{\pi(z_0), \pi(z_1), \ldots, \pi(z_d)\}$ of the vector space $\mathfrak{n}/\mathfrak{n}^2$. Suppose that $e(R) \geq 4$. For any subspace λ of $\mathfrak{n}/\mathfrak{n}^2$, let $J_\lambda = \pi^{-1}(\lambda)$. Then quite clearly $J_\lambda \neq J_{\lambda'}$ if $\lambda \neq \lambda'$, because if $J_\lambda = J_{\lambda'}$ then the images of the generators are the same under π. Now, $f \in {J_\lambda}^2$ for any λ, because $f \in \mathfrak{n}^4$ by assumption. So if $d \geq 1$, there are infinitely-many distinct subspaces in $\mathfrak{n}/\mathfrak{n}^2$ and R is not a simple singularity. Thus we have established that $e(R) \leq 3$ for any $d \geq 1$ by contradiction. If $e(R) = 3$ when $d \geq 2$ we again find that I_λ form an infinite subset of $c(f)$. The proof of this is left to the reader (*cf.* [17]). ❑

Pertaining to our case it is quite clear and intuitive; we have $S = k[\![x, y]\!]$, $\mathfrak{n} = (x, y)$ and $c(f)$ is finite only if $e \leq 3$. So we now have that $c(f)$ distinguishes between simple singularities or otherwise, but also we can even consider this set on any hypersurface $R = S/(f)$ and obtain the following criterion for R to be smooth. The following is a completely original observation, and although quite simple, it is useful to make explicit the power of the set $c(f)$.

Proposition 3.1.5. *Let $S = k[\![x, y]\!]$ and $R = S/(f)$ for some non-unit $f \in (x, y)$. Then $c(f)$ is empty if and only if the multiplicity of R is 1.*

Proof. If $c(f)$ is empty then f lies in no square ideal, so certainly $f \notin (x, y)^2$. Conversely, if there exists $I \in c(f)$, then $f \in I^2 \subseteq (x, y)^2$, so $e(R) \geq 2$. ❑

Thus we can simply investigate the set $c(f)$ to determine exactly what is necessary about the nature of the hypersurface.

Before we can discuss our main classification result for simple singularities, we first need some lemmas. The following lemmas 3.1.6 and 3.1.7 are elaborations and consolidations of proofs from Yoshino (*cf.* [17]).

Lemma 3.1.6. *If $R = S/(f)$ is a simple singularity of positive dimension, then R is reduced.*

Proof. Suppose that R is not reduced. Then f may be written as $f = g^2h$ for some $g, h \in S$. Thus for any ideal I of S containing g, it is clear that $hf \in I^2$. Now $S/(g)$ has the maximal ideal $\mathfrak{n}/(g)$ which contains a prime ideal, say $(x)/(g)$, so $\mathfrak{n}/(g)$ has height at least one. Thus $S/(g)$ has dimension at least one, hence it is not Artinian by the contrapositive of theorem 2.1.2. Thus there is a chain of ideals containing g with no minimal element, so the set $c(f)$ has infinitely-many elements. This contradicts the fact that R is a simple singularity. ❑

Lemma 3.1.7. *Let $R = S/(f)$ be a simple singularity of dimension 1. Assume that S contains an infinite field F. Then for any two elements $x, y \in \mathfrak{n}$ we have $f \notin (x^3, x^2y^2, xy^4, y^6) = (x, y^2)^3$.*

Proof. Suppose that $f \in (x, y^2)^3$ for some $x, y \in \mathfrak{n}$ and consider the ideals $I_\lambda = (x + \lambda y^2, y^3)$ indexed by all $\lambda \in F$. We will contradict the fact that $c(f)$ is finite by showing that $\{I_\lambda\}_{\lambda \in F} \subseteq c(f)$ and each I_λ is distinct. Now $f \in {I_\lambda}^2$ because $(x, y^2)^3 \subseteq {I_\lambda}^2$, so we need only show that $I_\lambda \neq I_{\lambda'}$ if $\lambda \neq \lambda'$. If $I = I_\lambda = I_{\lambda'}$ for some $\lambda \neq \lambda'$, then $(\lambda - \lambda')y^2 \in I$, so $y^2 \in I$ and hence $x \in I$. Thus we must have $I = (x, y^2)$. Now in $S/(x + \lambda y^2)$, $(x, y^2) = (y^2)$ because we may write $x = (x + \lambda y^2) - \lambda y^2 = -\lambda y^2$, and $(x + \lambda y^2, y^3) = (y^3)$ because $x + \lambda y^2$ is the zero element. Thus, for some $a \in S$, we have that $y^2 - ay^3 = y^2(1-ay) \in (x+\lambda y^2)$, so $y^2 \in (x+\lambda y^2)$ because $(1-ay)$ is a unit since $y \in \mathfrak{n}$. Thus we see that $x \in (x+\lambda y^2)$. Now we note that $I^3 = (x^3, x^2y^2, xy^4, y^6)$ by taking all possible triples of generators, and so $f \in I^3 \subseteq (x+\lambda y^2)^3$. This means that R contains the subring $S/(x+\lambda y^2)^3$, and this is not reduced because $x + \lambda y^2$ is a non-zero nilpotent of power 3. Thus we have contradicted the previous lemma 3.1.6 so there are infinitely many $I_\lambda \in c(f)$ and we are done. ❑

3.2 Main Classification Result

For this section, we consider $S = k[\![x, y]\!]$ with k an algebraically closed field of characteristic zero, and denote the maximal ideal $\mathfrak{n} = (x, y)$, whose image in the simple singularity $R = S/(f)$ is given by $\mathfrak{m}$.

We wish to obtain a result which regulates the form of f for a simple singularity R. This will then prove useful in subsequent chapters to analyse the torsion-free modules of R. We already have a method of visualising some of these via matrix factorisations, so after this section we will be able to apply this to many such rings R at once by the classification theorem proven.

Before we can begin the classification theorem, we first need the notion of *minimal reductions*. The following treatment of minimal reductions for a *generic* element of $\mathfrak{m}$ is entirely original work (see the conditions on z in proposition 3.2.1 for a characterisation of the term generic in our context).

Let $n \geq e(R)$ and consider the finitely generated vector space of dimension $n+1$ given by

$$\mathfrak{n}^n/\mathfrak{n}^{n+1} = kx^n \oplus kx^{n-1}y \oplus \cdots \oplus ky^n.$$

Its image in R requires a few extra zeros via the cosets of (f), so we add to the quotient. In this manner we obtain

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} = \mathfrak{n}^n/(\mathfrak{n}^{n+1} + ((f) \cap \mathfrak{n}^n)).$$

We may write

$$f(x, y) = \sum_{i \geq e} f_i(x, y)$$

where each f_i is homogeneous of degree i. This makes it clear that any element of $f\mathfrak{n}^{n-e}$ is also in $\mathfrak{n}^n$ and (f). On the other hand, $fg \in \mathfrak{n}^n$ for some $g \in S$ implies that $n = \deg f + \deg g = e + \deg g$, so $g \in \mathfrak{n}^{n-e}$. Thus, we have shown that $(f) \cap \mathfrak{n}^n = f\mathfrak{n}^{n-e}$. So

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} = \mathfrak{n}^n/(\mathfrak{n}^{n+1} + f\mathfrak{n}^{n-e}) = \mathfrak{n}^n/(\mathfrak{n}^{n+1} + f_e\mathfrak{n}^{n-e}) \tag{3.1}$$

noting that $f_i \in \mathfrak{n}^{n+1}$ for all $i > e$.

Since $\dim \mathfrak{n}^n/\mathfrak{n}^{n+1} = n+1$, after multiplication by f_e we see that

$$\dim \mathfrak{m}^n/\mathfrak{m}^{n+1} = (n+1) - \dim \mathfrak{n}^{n-e}/\mathfrak{n}^{n-e+1} = (n+1) - (n-e+1) = e.$$

Now with these facts we shall prove the following original proposition.

Proposition 3.2.1. *Using the notation above, for $z \in \mathfrak{m} - \mathfrak{m}^2$ whose image $\bar{z} \in \mathfrak{m}/\mathfrak{m}^2$ is such that $\bar{z} \nmid f_e$, we have that $z\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for all $n \geq e$.*

Proof. It will suffice to prove the result modulo $\mathfrak{m}^{n+1}$ due to Nakayama's lemma 2.1.3, so if we obtain $z\mathfrak{m}^n/\mathfrak{m}^{n+1} = \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$ then we are done. We have just seen that $\dim \mathfrak{m}^n/\mathfrak{m}^{n+1} = \dim \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} = e$ for all $n \geq e$, so we need only show that the map given by multiplication by z is injective. Let $g(x,y) \in S$ of homogeneous degree n, corresponding to $\bar{g} \in \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Then if $\bar{z}\bar{g} = 0$ in $\mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$ we must have $f_e \equiv zg \,|\mathfrak{n}|^{n+2}$ via (3.1), but by assumption on z this means that $f_e \mid g$ so $\bar{g} = 0$. ❑

Definition 3.2.2. With the above notation we call z a *minimal reduction* of $\mathfrak{m}$.

We now use the theory developed above to prove the following proposition from Yoshino (*cf.* [17]) which is central to the main classification result. The proof of this is omitted in Yoshino, but it now comes easily with the theory that we have established.

Proposition 3.2.3. *Let $R = k[\![x,y]\!]/(f)$ be a simple singularity with minimal reduction z of its maximal ideal $\mathfrak{m}$. Then R is a finitely generated module over a subalgebra $T = k[\![z]\!]$ and there is an isomorphism of T-algebras $R \simeq T[X]/(g(X))$ with $\deg(g(X)) = e(R)$.*

Proof. Consider z as an element of the finite-dimensional k-vector space $\mathfrak{m}^n/\mathfrak{m}^{n+1}$. We claim that $R = T + Tw + Tw^2 + \cdots + Tw^n$ for some $w \in R$ that is k-linearly independent from T. Since zT is the Jacobson radical of T, it suffices to prove the claim modulo zR, for then an appeal to Nakayama's lemma 2.1.3 yields the result. See that $zR \supseteq z\mathfrak{m}^n = \mathfrak{m}^{n+1}$, so

$$R/zR = \sum_{j=0}^{n} \sum_{i=0}^{j} kw^i z^{j-i} + zR. \tag{3.2}$$

Thus we have proven the claim. The isomorphism in the statement of the theorem follows from the fact that T embeds into R via the formula (3.2). Any $g \in R$ for which $g \mapsto 0$ under this embedding has $\deg(g(X)) \leq n$, and we may consequently have $n = e(R)$. ❑

We now have enough to move towards our main classification theorem. Below we compile a necessary condition that a ring be a simple singularity — modelled on the proof found in Yoshino (*cf.* [17]) — together with an original proof that the converse is also true.

Theorem 3.2.4. *Let $S = k[\![x, y]\!]$ where k is an algebraically closed field of characteristic zero. Then $R = S/(f)$ is a simple singularity if and only if f is equal to one of the following polynomials after a suitable change of variables:*

$$\begin{array}{lll} (A_n) & x^2 + y^{n+1} & (n \geq 1) \\ (D_n) & x^2y + y^{n-1} & (n \geq 4) \\ (E_6) & x^3 + y^4 & \\ (E_7) & x^3 + xy^3 & \\ (E_8) & x^3 + y^5 & \end{array}$$

Proof. Firstly, if f is one of the polynomials given, then $f \in (x, y)^2$; i.e. R is not smooth since $c(f)$ has (x, y). In fact, we can list all the elements of $c(f)$. For instance, consider the case A_n where $f = x^2 + y^{n+1}$. Observe that in $k[\![x]\!]$, we have $f - y^{n+1} \in (x)^2 - (x)^3$ and the linear ordering of ideals of $k[\![x]\!]$ ensures us that there are no more. Similarly, in $k[\![y]\!]$, we have $f - x^2 \in (y^i)^2$ for $1 \leq i \leq \lfloor (n+1)/2 \rfloor$. Here, there are no more proper square ideals containing $f - x^2$, for if $i \geq \lfloor (n+1)/2 \rfloor + 1$ then $y^{n+1} \notin (y^i)^2$ since $2i > n+1$. Thus in S, there is only a finite number of ways to combine these ideals, namely $f \in (x)^2 \cup (y^i)^2$, $f \in (x)^2(y^i)^2$ or $f \in (x)^2 + (y^i)^2$, so any ideal I for which $f \in I^2$ must have $I^2 \supseteq (x, y^i)^2$. Note that we used the fact that $(x, y^i)^2$ contains each of the sum, product and union of $(x)^2$ with $(y^i)^2$. Thus since S is Noetherian, there must be finitely many such I and we see that $c(f)$ is finite, so R is a simple singularity.

For the case D_n where $f = x^2y + y^{n-1}$, observe that this time $f = y(x^2 + y^{n-2})$. Using the method above on the factor $x^2 + y^{n-2}$ we see that $f \in y(x, y^j)^2$ for $1 \leq j \leq \lfloor (n-2)/2 \rfloor$, but a little more work is required because $y(x, y^j)^2$ is obviously not square. Indeed $(x, y^j)^2 \supset y(x, y^j)^2$ for each j, and the Noetherian property again yields that there are only finitely-many possible square ideals containing f. Thus, again $c(f)$ is finite, so R is a simple singularity.

For the cases E_6, E_7 and E_8 corresponding to $f_6 = x^3 + y^4$, $f_7 = x^3 + xy^3$ and $f_8 = x^3 + y^5$ respectively, we have $c(f_6) = c(f_7) = \{(x, y)^2, (x, y^2)^2\}$ and $c(f_7) = \{(x, y)^2\}$ following similar methods as above.

We now move on to proving the converse statement. Suppose R is a simple singularity. We shall divide the proof into several cases by the values of $e(R)$:

$e(R) = 2$: By changing variables if necessary, we may take y as a minimal reduction of the maximal ideal $\mathfrak{m}$ of R. Then applying proposition 3.2.3 we may take $g(x) = x^2 + bx + c = (x + b/2)^2 + c - b^2/4$, or equivalently after a change of variables, write $R = T[x]/(x^2 + a)$ for some $a \in T$. So we may assume that $f = x^2 + a$ and write $a = uy^{n+1}$ for some unit $u \in T$. Then Proposition 2.2.14 yields the result $f = x^2 + y^{n+1}$; this is the case (A_n) (figure 3.1).

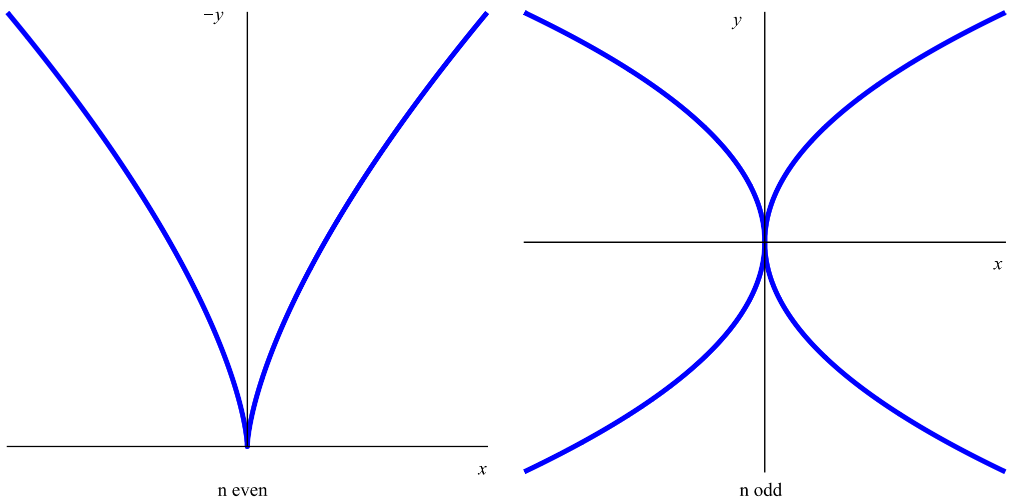


Figure 3.1: The curve $f = 0$ with $e = 2$ is the case A_n

$e(R) = 3$: The curve described by $f = 0$ may or may not have different tangent directions (compare figures 3.2 and 3.3), so we consider each case separately:

- If the curve described by $f = 0$ has two or three different tangent directions (see figure 3.2), then we may write $f = gh$ for some $g, h \in S$ where the curves described by $g = 0$ and $h = 0$ have distinct tangents. Since $e(R) = e(S/(g)) + e(S/(h))$, we may take $e(S/(g)) = 2$ and $e(S/(h)) = 1$; note if $e = 0$ for either of these we have a field and thus no tangents, so this is the only combination of choices. Take generic $y \in R$ whose image in $S/(g)$ and $S/(h)$ is a minimal reduction of the corresponding maximal ideal. Observe that $S/(h)$ is smooth, and $S/(g)$ is the case (A_{n-3}) above, so we have $S/(g) \simeq T[X]/(X^2 + y^{n-2})$ for $n \geq 4$. Now as $\deg f = e = 3$, we deduce that $R \simeq T[X]/((X - t)(X^2 + y^{n-2}))$ for some $t \in T$, so we may assume that $f = (x - t)(x^2 + y^{n-2})$. Consider the case $n \geq 5$. Since the curve described by $f = 0$ has different tangent directions, we had better ensure that $t = uy$ for some unit $u \in T$. We may take a change of variables, replacing y with $x - uy$. This yields

$f = y(x^2+(x-uy)^{n-2}) = y(ax^2+bxy^{n-3}+cy^{n-2})$, where $a-1$ is a unit in S with $(a-1)x^2$ equal to the sum of all but the last two terms in the expansion of $(x-uy)^{n-2}$; $b = (-u)^{n-3}$ and $c = (-u)^{n-2}$ are clearly also units in S. Then the substitution of regular parameters $\xi = a^{\frac{1}{2}}x + \frac{1}{2}a^{-\frac{1}{2}}by^{n-3}$ and $\eta = y(c - \frac{1}{4}a^{-1}b^2y^{n-4})^{\frac{1}{n-2}}$ yields $f = \eta(\xi^2 + \eta^{n-2}) = \xi^2\eta + \eta^{n-1}$ up to multiplication by a unit in S; this is the case (D_n) for $n \geq 5$. For the case that $n = 4$ we require a slightly different argument. This time we take h as a minimal reduction of $S/(g)$, so that $S/(g) \simeq T[h]/(h^2+y^2)$. Then we deduce that $R \simeq T[h]/(h(h^2+y^2))$, so we may assume that $f = x(x^2+y^2)$ up to a change of variables; this is the case (D_4).

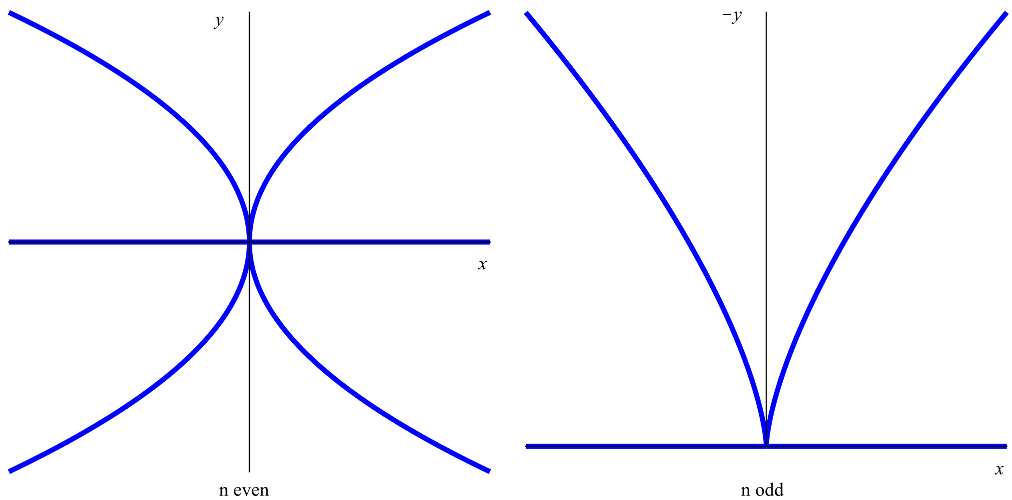


Figure 3.2: The curve $f = 0$ with different tangent directions is the case D_n

- If the curve described by $f = 0$ has a unique tangent direction, then f may or may not be irreducible (compare figures 3.3 and 3.4). If f is reducible, then for the minimal reduction y of $\mathfrak{m}$ we may write $f = x(x^2+ax+b)$, $a, b \in T$ by applying proposition 3.2.3. Note that this is the only degree 3 polynomial with unique tangent direction, for if $x - t$ were a factor for some $t \in T$ then we are in the case above. Now the coefficients a, b must be in y^2T, y^3T respectively so that x^3 is the initial form of f. Thus we may write $f = x(x^2+cxy^2+uy^3)$ where $c, u \in T$, noting that u is a unit in T so that we do not contradict lemma 3.1.7. However, then we are able to obtain f in the form $x(x^2+dxy^2+y^3)$, $d \in T$ via an automorphism using proposition 2.2.14. Changing the variable y to $y - \frac{1}{3}ex$ facilitates the following to obtain $f = x(x^2+y^3+sx^2y+tx^3)$, where $s = \frac{-1}{3}e^2$, $t = \frac{2}{27}e^3$. Thus we have $f = x(x^2(1+sy+tx)+y^3)$, and the change of variable $x(1+sy+tx)^{1/2}$ to x allows us to arrive at the form $f = x(x^2+y^3)$ for the case E_7.

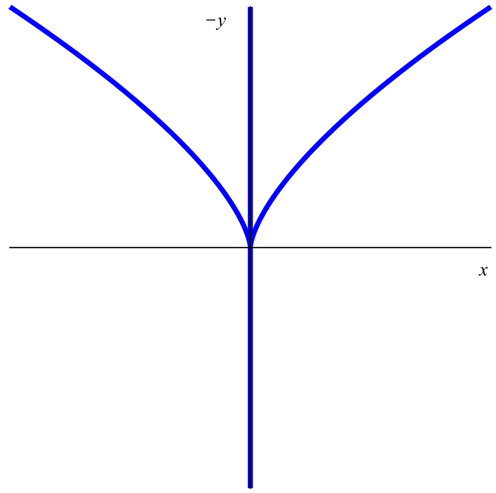


Figure 3.3: The reducible curve $f = 0$ with unique tangent direction is the case E_7

On the other hand, if f is already irreducible with unique tangent direction for the curve $f = 0$, we may immediately write (for the same minimal reduction) $R = T[x]/(x^3 + ax + b)$ for some $a, b \in T$, so it is assumed that $f = x^3 + ax + b$. Note also that $b \neq 0$ so that f is in fact irreducible. If $a = 0$, then the proof of proposition 2.2.14 reveals the fact that f can be made to have the form $x^3 + y^m$ for some integer m, and lemma 3.1.6 ensures that $m \leq 5$; we must also have that $m \geq 4$ so that f is irreducible and in $\mathfrak{m}^e - \mathfrak{m}^{e+1}$. Thus we have precisely the case E_6, E_8 for $m = 4, 5$ if $a = 0$ (see figure 3.4). We now show that the same is

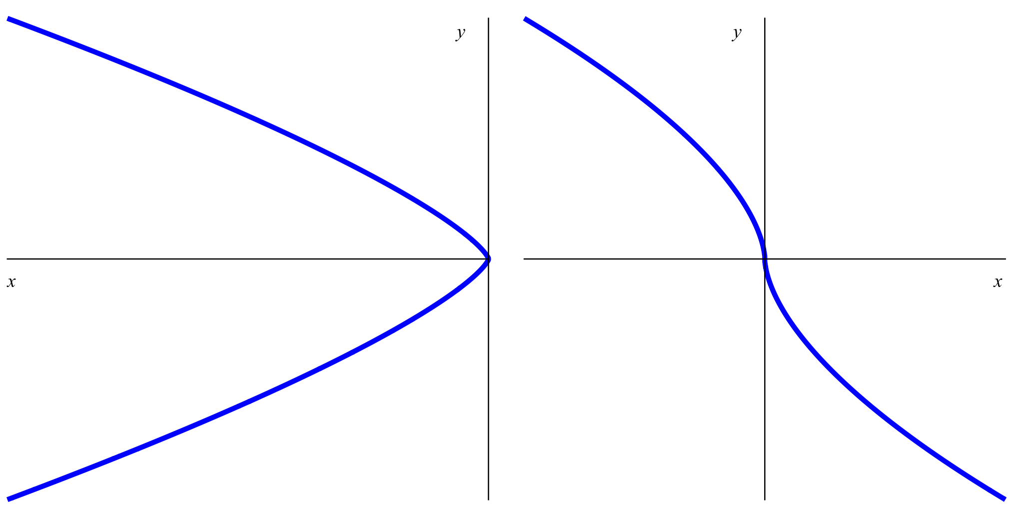


Figure 3.4: The irreducible curve $f = 0$ with unique tangent direction is either E_6 (left) or E_8 (right)

achieved if $a \neq 0$. Suppose that $a \neq 0$ and write f in the form

$$f = x^3 + uxy^n + y^m$$

for some unit $u \in T$. Since x^3 must be the initial form of f, we require $n \geq 3$ and $m \geq 4$. In fact we have either $n \geq 4$ or $m = 4$. To see this, suppose the contrary. We shall produce a factor of f which violates its irreducibility; we are now working with f in the form

$$f = x^3 + uxy^3 + y^m$$

where $m \geq 5$. Indeed, in this case the algebraic equation

$$X^3 + uX^2 + y^{2m-9} = 0 \tag{3.3}$$

over T has a solution. This is due to the fact that T is Henselian and, modulo y, there is the obvious simple root $-u$, so this lifts to a solution in T which we will denote $\xi \in T$. In fact, the proof of corollary 2.2.13 to Hensel's lemma provides a constructive means by which we can see that $\xi \in T$ is a unit since $u \in T$ is a unit. Consequently, we see that

$$f(\xi^{-1}y^{m-3}, y) = \xi^{-3}y^{3m-9} + u\xi^{-1}y^m + y^m = \xi^{-3}y^m(y^{2m-9} + u\xi^2 + \xi^3),$$

but this is zero by (3.3), hence $x - \xi^{-1}y^m$ is a factor of f.

So we have $n \geq 4$ or $m = 4$. If $m = 4$ then we are working with $f = x^3 + uxy^n + y^4$ for $n \geq 3$ and a simple replacement of y with $y - \frac{1}{4}uxy^{n-3}$ and applying proposition 2.2.14 yields f in the form $f = x^3 + gx^2y^2 + y^4$ for some $g \in S$. Finally, we change x to $x + \frac{1}{3}gy^2$ in order to obtain f in the form $f = x^3 + (1 - \frac{1}{3}g^2x - \frac{1}{27}g^3y^2)y^4$, and application of proposition 2.2.14 yields the case E_6 with f in the form $f = x^3 + y^4$.

If $m \neq 4$ then $m = 5$ so that we do not contradict lemma 3.1.7. So we are working with $f = x^3 + uxy^n + y^5$ for $n \geq 4$, and replacing y with $y + \frac{1}{5}ux$ yields f in the form $f = x^3 + hx^2y^3 + y^5$ for some $h \in S$. Finally, we change x to $x + \frac{1}{3}hy^3$ to obtain f in the form $f = x^3 + (1 - \frac{1}{3}h^2xy - \frac{1}{27}h^3y^4)y^5$ and application of proposition 2.2.14 yields the case E_8 with f in the form $f = x^3 + y^5$. ❑

This theorem quite easily extends to higher dimensions using the Weierstrass Preparation theorem.

Theorem 3.2.5. *Let $S = k[\![x, y, z_2, z_3, \ldots, z_d]\!]$ where k is an algebraically closed field of characteristic zero. Then $R = S/(f)$ is a simple singularity if and only if f is equal to one of the following polynomials after a suitable change of variables:*

$$\begin{array}{lll} (A_n) & x^2 + y^{n+1} + {z_2}^2 + {z_3}^2 + \cdots + {z_d}^2 & (n \geq 1) \\ (D_n) & x^2 y + y^{n-1} + {z_2}^2 + {z_3}^2 + \cdots + {z_d}^2 & (n \geq 4) \\ (E_6) & x^3 + y^4 + {z_2}^2 + {z_3}^2 + \cdots + {z_d}^2 & \\ (E_7) & x^3 + xy^3 + {z_2}^2 + {z_3}^2 + \cdots + {z_d}^2 & \\ (E_8) & x^3 + y^5 + {z_2}^2 + {z_3}^2 + \cdots + {z_d}^2 & \end{array}$$

Proof. Firstly, note that if f is one of the polynomials above, then $f \in I^2$ for some proper ideal I of S only if I is the union of a corresponding ideal J from 3.2.4 with $(z_2, \ldots, z_d)$; i.e. $I = (J, z_2, \ldots, z_d)$. Thus, it follows from theorem 3.2.4 that $c(f)$ is finite. Hence R is a simple singularity.

Now note that d is the dimension of the ring R since S has dimension $d + 1$ as it is a regular local ring. The case $d = 1$ is true as it is simply the preceding theorem 3.2.4. We assume $d \geq 2$ and apply induction on d. Firstly, we know that $e(R) = 2$ because R is assumed to be a singularity, so $f \in \mathfrak{n}^2 - \mathfrak{n}^3$ where $\mathfrak{n} = (x, y, z_2, z_3, \ldots, z_d)$ is the maximal ideal of S. Thus we may assume that f has nonzero (indeed monic) term ${z_d}^2$ and prepare the form of f for the inductive step as follows: for a unit $u \in S$ and $a_1, a_2 \in S' = k[\![x, y, z_2, z_3, \ldots, z_{d-1}]\!]$ we apply Weierstrass Preparation to obtain

$$f = u({z_d}^2 + a_1 z_d + a_2) = u\left((z_d + \tfrac{1}{2}a_1)^2 + a_2 - \tfrac{1}{4}{a_1}^2\right).$$

But then f may as well be taken to be $u^{-1}f$. Moreover, after a change of the variable z_d with $z_d + \frac{1}{2}a_1$, we may write, for suitable $g \in S'$,

$$f(x, y, z_2, z_3, \ldots, z_d) = {z_d}^2 + g(x, y, z_2, z_3, \ldots, z_{d-1}).$$

So it remains to see that $S'/(g)$ is a simple singularity. Indeed, the map

$$c(g) \to c(f);\ I \mapsto (I, z_d)$$

is quite clearly injective, for if $(I, z_d) = (J, z_d)$ we must obviously have $I = J$ because z_d is a generator of S independent from S' of which I and J are contained in. Thus, $c(g)$ is finite since $c(f)$ is finite and hence $S'/(g)$ is a simple singularity. Thus the theorem is proved by induction. ❑

Chapter 4

Cohen-Macaulay Modules

We are now ready to introduce CM modules for the one-dimensional case and see some results describing their structure and their interaction with one another.

In this chapter, we present the elements used for classifying simple plane curve singularities – the CM modules. In general, the maximal CM modules of a ring have a more profound meaning, but in the one-dimensional case, we simply consider them to be the torsion-free modules. We will see what this means and develop a means for computing them, which will become valuable later.

4.1 Torsion-free Modules

We begin with a definition of torsion and then proceed to define CM modules as torsion-free modules; this is true in the one-dimensional case that we will be working in for the most part, so it suffices for the treatment in this thesis. A more general definition requires the notion of *depth* and some homological algebra (*cf.* [13]).

Definition 4.1.1. Let R be a commutative ring. The *torsion* submodule T of an R-module M is defined to be

$$T(M) := \{m \in M : rm = 0 \text{ for some non-zerodivisor } r \in R\}.$$

We say that M is *torsion-free* if $T(M) = 0$.

The check to see that $T(M)$ is a submodule of M is clear. For $m, n \in T(M)$, let $r, s \in R$ be non-zerodivisors such that $r \cdot m = s \cdot n = 0$. Then rs is also a non-zerodivisor and annihilates $m + n$; i.e. $rs \cdot (m + n) = s(rm) + r(sn) = 0$. In addition, for any $a \in R$, r annihilates am; i.e. $r \cdot (am) = a(rm) = 0$.

We will be mainly concerned with the ring $R = k[\![x, y]\!]/(f)$, where $f \in (x, y)$. Below we introduce the formal definition of a CM R-module where R is one-dimensional, and this will be our working definition whenever we refer to a CM module throughout the rest of the thesis. The terms “CM” and “torsion-free” will be used interchangeably (CM will be particularly used when the ideas presented do in fact extend to higher dimensions, although we are not at all concerning ourselves with this concept).

Definition 4.1.2. Let $R = k[\![x,y]\!]/(f)$, $f \neq 0$. A finitely generated R-module M is a *torsion-free* module if the only element $m \in M$ for which $(\bar{x},\bar{y})m = 0$ is $m = 0$, where $\bar{x}$, $\bar{y}$ denote respectively the images of $x, y \in k[\![x,y]\!]$ modulo (f).

The difference between definitions 4.1.1 and 4.1.2 is subtle. Definition 4.1.2 is the specialisation of definition 4.1.1 in the case of the power series rings which we are interested in. Firstly, if a unit $u \in k[\![x,y]\!]$ is such that $um = 0$ for some $m \in R$, then $m = u^{-1}0 = 0$, so it is trivial to check torsion with unit elements. Thus we check torsion with the maximal ideal of R given by $(\bar{x},\bar{y})$. Then note that $(\bar{x},\bar{y})m = 0 \Rightarrow m = 0$ is equivalent to $T(M) = 0$.

Naturally, we have the following definition.

Definition 4.1.3. A ring R is said to be a CM ring if it is CM as an R-module.

With a working notion of CM modules, we move on with the theory of matrix factorisations of general CM modules, in which we obtain a method of construction of these modules. First, we need some of the language of category theory.

4.2 Interlude – A Brief Overview of Category Theory

For what follows, we will require the language of categories and functors (see [7] for more), so this section will provide a brief overview of the basic definitions involved. In subsequent chapters it will be necessary to define the categories of CM modules and Matrix Factorisations in order to apply a powerful theorem 4.3.4 due to Eisenbud which shows their equivalence. The reader should only proceed with this specific goal in mind rather than invest too many mental faculties to the implications of such a profound subject. One should pay particular attention to the examples given which pertain specifically to our endeavours.

Definition 4.2.1. A *category* $\mathfrak{C}$ consists of the following:

(*i*) A class $\operatorname{Ob}\mathfrak{C}$ of *objects*

(*ii*) A collection of pairwise disjoint sets $\operatorname{Hom}_{\mathfrak{C}}(X,Y)$, one for each ordered pair $(X,Y) \in \operatorname{Ob}\mathfrak{C}$, containing *morphisms* $\varphi : X \to Y$.

(*iii*) A collection of associative *composition* mappings, one for each $(X,Y,Z) \in \operatorname{Ob}\mathfrak{C}$, for which

$$\begin{aligned}\operatorname{Hom}_{\mathfrak{C}}(X,Y) \times \operatorname{Hom}_{\mathfrak{C}}(Y,Z) &\to \operatorname{Hom}_{\mathfrak{C}}(X,Z); \\ (\varphi,\psi) &\mapsto \psi\varphi := \psi \circ \varphi.\end{aligned}$$

A morphism $\varphi : X \to Y$ in $\mathfrak{C}$ is called an *isomorphism* if there exists another morphism $\psi : Y \to X$ for which $\psi\varphi = \mathrm{id}_X$ and $\varphi\psi = \mathrm{id}_Y$.

We will denote $\operatorname{Mor}\mathfrak{C}$ the union of all Hom-sets between objects in $\mathfrak{C}$. Also we stipulate that for any $X \in \operatorname{Ob}\mathfrak{C}$, there exists an identity morphism $\operatorname{id}_X : X \to X$ such that $\operatorname{id}_X\varphi = \varphi$ and $\psi\operatorname{id}_x = \psi$ whenever these compositions are defined.

Example 4.2.2. For a Henselian CM ring R, the R-modules of R form a category $\mathfrak{M}(R)$ together with morphisms given by all R-homomorphisms in a natural way.

Definition 4.2.3. A category $\mathfrak{C}$ is called a *subcategory* of a category $\mathfrak{D}$ if

- (i) $\operatorname{Ob}\mathfrak{C} \subseteq \operatorname{Ob}\mathfrak{D}$
- (ii) $\operatorname{Hom}_{\mathfrak{C}}(X, Y) \subseteq \operatorname{Hom}_{\mathfrak{D}}(X, Y)$ for all $X, Y \in \operatorname{Ob}\mathfrak{C}$
- (iii) The composition of morphisms in $\mathfrak{C}$ coincide with the composition of morphisms in $\mathfrak{D}$.
- (iv) id_X in $\mathfrak{C}$ coincides with id_X in $\mathfrak{D}$.

$\mathfrak{C}$ is called a *full* subcategory of $\mathfrak{D}$ if $\operatorname{Hom}_{\mathfrak{C}}(X, Y) = \operatorname{Hom}_{\mathfrak{D}}(X, Y)$ for all $X, Y \in \operatorname{Ob}\mathfrak{C}$.

Example 4.2.4. For a Henselian CM local ring R, the CM modules of R form a full subcategory $\mathfrak{C}(R)$ of $\mathfrak{M}(R)$.

Definition 4.2.5. Let $\mathfrak{C}$ be a category whose Hom-sets are abelian groups. For a set $D \subseteq \operatorname{Ob}\mathfrak{C}$ of objects of $\mathfrak{C}$, we define $\mathfrak{C}/D$ to be the category with $\operatorname{Ob}(\mathfrak{C}/D) = \operatorname{Ob}\mathfrak{C}$ and morphisms between two $X, Y \in \operatorname{Ob}(\mathfrak{C}/D)$ given by

$$\operatorname{Hom}_{\mathfrak{C}}(X, Y)/D(X, Y).$$

Here, $D(X, Y)$ denotes the subgroup generated by all morphisms $X \to Y$ which pass through direct sums of objects in D. In addition, the objects of D are the zero object in $\mathfrak{C}/D$.

Naturally, we ask how to compare two categories, which gives rise to the notion of a functor.

Definition 4.2.6. Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. A (covariant) *functor* $F : \mathfrak{C} \to \mathfrak{D}$ consists of the following:

- (i) A mapping $\operatorname{Ob}\mathfrak{C} \to \operatorname{Ob}\mathfrak{D}$; $X \mapsto FX$
- (ii) A mapping $\operatorname{Mor}\mathfrak{C} \to \operatorname{Mor}\mathfrak{D}$; $\varphi \mapsto F(\varphi)$ such that if $\varphi \in \operatorname{Hom}_{\mathfrak{C}}(X, Y)$ then $F(\varphi) \in \operatorname{Hom}_{\mathfrak{D}}(FX, FY)$

Also, we stipulate that $F(\psi\varphi) = F(\psi)F(\varphi)$ for any $\varphi, \psi \in \operatorname{Mor}\mathfrak{C}$ for which $\psi\varphi$ is well-defined, and that $F(\operatorname{id}_X) = \operatorname{id}_{FX}$ for any $X \in \operatorname{Ob}\mathfrak{C}$.

We wish to say when two categories are equivalent, but to do this we first need the following.

Definition 4.2.7. For a functor $F : \mathfrak{C} \to \mathfrak{D}$,

(i) F is said to be *faithful* (resp. *full*) if for any $X, Y \in \operatorname{Ob}\mathfrak{C}$, the map

$$F : \operatorname{Hom}_{\mathfrak{C}}(X, Y) \to \operatorname{Hom}_{\mathfrak{D}}(FX, FY)$$

is injective (*resp.* surjective).

(ii) F is called *dense* if any object $Y \in \operatorname{Ob}\mathfrak{D}$ is isomorphic to an object of the form FX for some $X \in \operatorname{Ob}\mathfrak{C}$.

Definition 4.2.8. Let F, G be functors mapping $\mathfrak{C}$ to $\mathfrak{D}$. A *functorial morphism* η from F to G is a family of morphisms in $\mathfrak{D}$ given by $\eta_X : FX \to GX$, $X \in \operatorname{Ob}\mathfrak{C}$, such that for any morphism $f : X \to Y$ in $\mathfrak{C}$, the following diagram commutes

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FY & \xrightarrow{\eta_Y} & GY \end{array}$$

Also, η is called a *functorial isomorphism* if every η_X is an isomorphism in $\mathfrak{D}$.

Definition 4.2.9. A functor $F : \mathfrak{C} \to \mathfrak{D}$ is called an *equivalence* of categories if there exists a functor $G : \mathfrak{D} \to \mathfrak{C}$ of F and functorial isomorphisms $\eta : GF \xrightarrow{\sim} \operatorname{id}_{\mathfrak{C}}$, $\zeta : FG \xrightarrow{\sim} \operatorname{id}_{\mathfrak{D}}$. Such a functor G is called a *quasi-inverse* of F. If in fact $GF = \operatorname{id}_{\mathfrak{C}}$ and $FG = \operatorname{id}_{\mathfrak{D}}$ then F is called an *isomorphism* of categories.

Theorem 4.2.10. *A functor $F : \mathfrak{C} \to \mathfrak{D}$ is an equivalence of categories if and only if F is a fully faithful and dense functor.*

Proof. If F is an equivalence then there is a quasi-inverse $G : \mathfrak{D} \to \mathfrak{C}$ of F. Thus we have functorial isomorphisms $\eta : GF \to \operatorname{id}_{\mathfrak{C}}$ and $\zeta : FG \to \operatorname{id}_{\mathfrak{D}}$ such that for $X \in \operatorname{Ob}\mathfrak{C}$ and $Y \in \operatorname{Ob}\mathfrak{D}$, $\eta_X : GFX \to X$ and $\zeta_Y : FGY \to Y$ are isomorphisms. Firstly, $Y \simeq FX$ for $X = GY \in \operatorname{Ob}\mathfrak{C}$ so F is dense. Secondly, for $f \in \operatorname{Hom}_{\mathfrak{C}}(X, X')$, the following diagram commutes

$$\begin{array}{ccc} GFX & \xrightarrow{\eta_X} & X \\ {\scriptstyle GF(f)}\downarrow & & \downarrow{\scriptstyle f} \\ GFX' & \xrightarrow{\eta'_X} & X' \end{array}$$

so $f = \eta_{X'} GF(f) {\eta_X}^{-1}$ thus F is faithful. Finally, for any $g \in \operatorname{Hom}_{\mathfrak{D}}(FX, FX')$, define $f = \eta_{X'} G(g) {\eta_X}^{-1} \in \operatorname{Hom}_{\mathfrak{C}}(X, X')$, but also $f = \eta_{X'} GF(f) {\eta_{X'}}^{-1}$ so $g = F(f)$ since G is faithful. Thus F is fully faithful.

If F is fully faithful and dense then for any $Y \in \operatorname{Ob}\mathfrak{D}$ we may fix $X_Y \in \operatorname{Ob}\mathfrak{C}$ and an isomorphism $\zeta_Y : FX_Y \to Y$ by using the dense property. Then we construct a quasi-inverse $G : \mathfrak{D} \to \mathfrak{C}$ of F as follows. Set $GY = X_Y$ for all $Y \in \operatorname{Ob}\mathfrak{D}$ and for any $g \in \operatorname{Hom}_{\mathfrak{D}}(Y, Y')$, set Gg to be the unique morphism in $\operatorname{Hom}_{\mathfrak{C}}(GY, GY')$ such that $FG(g) = {\zeta_{Y'}}^{-1} g \zeta_Y$; we do the latter by using the fully faithful property with the fact that $\operatorname{Hom}_{\mathfrak{D}}(FGY, FGY') = F\operatorname{Hom}_{\mathfrak{C}}(GY, GY')$. So G is a functor for which

$$FGY \xrightarrow{\zeta_Y} Y \xrightarrow{g} Y' \xrightarrow{{\zeta_{y'}}^{-1}} FGY'.$$

Also, for any $X \in \operatorname{Ob}\mathfrak{C}$, $\zeta_{FX} : FGFX \to FX$ is an isomorphism, so F fully faithful means that there is a unique isomorphism $\eta_X : GFX \to X$ such that $\zeta_{FX} = F\eta_X$. Thus we define $\eta := \{\eta_X : X \in \operatorname{Ob}\mathfrak{C}\}$ to see that this is a functorial isomorphism of $\mathfrak{C}$ an $\mathfrak{D}$ hence we are done. ❑

We are now interested in obtaining a method of construction for the torsion-free modules over our simple plane curve singularities $R = k[\![x, y]\!]/(f)$. We consider this from the general perspective of CM modules.

4.3 Matrix Factorisations

Let $(S, \mathfrak{n})$ be a regular local ring and $R = S/(f)$ a Henselian quotient of S for some $f \in \mathfrak{n}$. The following is a presentation of what is known as a matrix factorisation of a CM module M over R. The main result of this section will be the equivalence theorem 4.3.4 due to Eisenbud which will be instrumental later on for computing CM modules of singularities. We only provide a sketch of the proof; the full proof requires the Auslander-Buchsbaum formula (*cf.* [10]) for CM modules. For our purposes, this formula yields that the free resolution of any CM module is short as in (4.1). With this, we proceed with the following.

For any CM R-module M, viewing M as an S-module, it has the free resolution

$$0 \to S^n \xrightarrow{\varphi} S^n \to M \to 0. \tag{4.1}$$

Note that M has rank 0 as an S-module because f is a zerodivisor in M and a non-zerodivisor in S; thus the middle and left terms may be taken to have the same rank n. From exactness of (4.1), we see that $fS^n \subset \varphi(S^n)$, so for any $x \in S^n$ there is a unique element $y \in S^n$ for which $f \cdot x = \varphi(y)$. We can certainly put $y = \psi(x)$ where $\psi : S^n \to S^n$ and so we obtain

$$\varphi\psi = f \cdot \mathbf{1}_{S^n}.$$

Composing this relation with φ we have $\varphi\psi\varphi = f \cdot \varphi$ and since (4.1) is exact we see that φ is injective, and this means that also

$$\psi\varphi = f \cdot \mathbf{1}_{S^n}.$$

Definition 4.3.1. Let S and R be as above. An ordered pair of maps (φ, ψ), each mapping $S^n \to S^n$, is called a *matrix factorisation* of f if

$$\varphi\psi = f \cdot \mathbf{1}_{S^n} = \psi\varphi.$$

We may regard φ and ψ as square matrices on S in the natural way using the standard basis of S^n.

Before we proceed to the statement of Eisenbud's matrix factorisation theorem 4.3.4 we must first establish the category of matrix factorisations $\mathfrak{MF}_S(f)$ of f over S. To do this, we must establish what we mean by *morphisms* of matrix factorisations.

Definition 4.3.2. A *morphism* between two matrix factorisations (φ_1, ψ_1) and (φ_2, ψ_2) is a pair of matrices (α, β) with $\alpha\varphi_1 = \varphi_2\beta$ and $\beta\psi_1 = \psi_2\alpha$ such that the following diagram commutes:

$$\begin{array}{ccccc} S^m & \xrightarrow{\psi_1} & S^m & \xrightarrow{\varphi_1} & S^m \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ S^n & \xrightarrow{\psi_2} & S^n & \xrightarrow{\varphi_2} & S^n \end{array} \tag{4.2}$$

Remark 4.3.3. Actually, the condition $\alpha\varphi_1 = \varphi_2\beta$ suffices in practice for (α, β) to be a morphism; the commutativity on the right square in (4.2) implies the commutativity on the left square, since

$$f\psi_2\alpha = \psi_2\alpha\varphi_1\psi_1 = \psi_2\varphi_2\beta\psi_1 = f\beta\psi_1$$

(study diagram (4.2) to see this) and thus $\psi_2\alpha = \beta\psi_1$.

Now we are in a position to define the category $\mathfrak{MF}_S(f)$ with objects the matrix factorisations of f and morphisms as defined above. This is an additive category by defining the direct sum

$$(\varphi_1, \psi_1) \oplus (\varphi_2, \psi_2) := \left(\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix} \right),$$

so the reader should begin to realise that this may behave like a category of R-modules; this will be consolidated below.

Naturally, two matrix factorisations (φ_1, ψ_1) and (φ_2, ψ_2) are called *equivalent* if α and β are isomorphisms in (4.2).

Note that $(1, f)$ and $(f, 1)$ naturally are matrix factorisations of f, but we wish to distinguish our pursuits from such unenlightening cases. We will see in Eisenbud's theorem 4.3.4, that the important matrix factorisations are those that are *reduced*; i.e. (φ, ψ) is reduced if each of the entries of the matrices φ and ψ are non-units.

In what follows, we shall see the statement of Eisenbud's matrix factorisation theorem and an outline of as much of the proof as possible in this basic setting. Unfortunately, we cannot prove this in its entirety without discussion of *syzygies*; perhaps a thesis in its own right. The full proof, as well as treatment of the vast theory involved, can be found in Yoshino (*cf.* [17]). For now, the reader should rest assured that with the equivalence of categories that we show, we are able to study matrix factorisations in order to obtain torsion-free modules in dimension one.

Theorem 4.3.4. *(*Eisenbud*)*
If $R = S/(f)$ for a regular local ring S, then coker *induces an equivalence of the category of CM R-modules (excluding R itself) and the category of reduced matrix factorisations of f; i.e. we have*

$$\mathfrak{C}(R)/\{R\} \simeq \mathfrak{MF}_S(f)/\{(1, f), (f, 1)\}.$$

Proof. (Sketch)
Since $\operatorname{coker}(1) = 0$ for the matrix factorisation $(1, f)$, coker induces the functor $\mathfrak{MF}_S(f)/\{(1, f)\} \to \mathfrak{C}(R)$ which we will label as Coker . We then define the functor $\Upsilon : \mathfrak{C}(R) \to \mathfrak{MF}_S(f)/\{(1, f)\}$ as follows: For a non-trivial CM module $M \in \mathfrak{C}(R)$, the claimed existence of the free resolution (4.1) allows us to obtain a matrix factorisation (φ, ψ) of f for which we will set $\Upsilon M = (\varphi, \psi)$.

Firstly, this is a unique object in $\mathfrak{MF}_S(f)$ because of the following. Suppose that φ is chosen to produce the free resolution of with S^n having least rank, and that (φ', ψ') is any other matrix factorisation of f obtained from M. We may as well assume that $\varphi' : S^{n'} \to S^{n'}$, so there are invertible matrices α and β such that the following commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^{n'} & \xrightarrow{\left(\begin{smallmatrix}\varphi & 0\\ 0 & 1\end{smallmatrix}\right)} & S^{n'} & \longrightarrow & M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\| & & \\
0 & \longrightarrow & S^{n'} & \xrightarrow{\varphi'} & S^{n'} & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Thus we have the equivalence $(\alpha, \beta) : \left(\begin{pmatrix}\varphi & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\psi & 0\\ 0 & f\end{pmatrix}\right) \to (\varphi', \psi')$ and so $\Upsilon M \in \mathfrak{MF}_S(f)\{(1, f)\}$ is uniquely determined; i.e. Υ is dense.

Secondly, there is an equivalence of morphisms of ΥM as follows. Let $g : M_1 \to M_2$ be a morphism in $\mathfrak{C}(R)$ and observe the following commutative diagram for arbitrary free resolutions of M_1 and M_2:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^{m} & \xrightarrow{\varphi_1} & S^{m} & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & S^{n} & \xrightarrow{\varphi_2} & S^{n} & \longrightarrow & M_2 & \longrightarrow & 0
\end{array} \tag{4.3}$$

Then (α, β) yields a morphism $\Upsilon M_1 \to \Upsilon M_2$ which we shall denote Υg. Suppose now that for the same φ_1 and φ_2 (paired with ψ_1 and ψ_2 *resp.*), (α', β') is another pair of matrices such that (4.3) commutes. Then there exists $\gamma : S^m \to S^n$ such that $\alpha - \alpha' = \varphi_2\gamma$ and $\beta - \beta' = \gamma\varphi_1$, so the morphism $(\alpha, \beta) - (\alpha', \beta')$ is a composition of $(\gamma, \gamma\varphi_1) : (\varphi_1, \psi_1) \to (\mathbf{1}_n, f \cdot \mathbf{1}_n)$ with $(\varphi_2, 1) : (\mathbf{1}_n, f \cdot \mathbf{1}_n) \to (\varphi_2, \psi_2)$. Hence we have $(\alpha, \beta) = (\alpha', \beta')$ as morphisms in $\mathfrak{MF}_S(f)/\{(1, f)\}$ and so Υg is uniquely determined; i.e. Υ is fully faithful. Thus we have the functor Υ which is fully faithful and dense hence it is an equivalence of the categories $\mathfrak{C}(R)$ and $\mathfrak{MF}_S(f)/\{(1, f)\}$ via theorem 4.2.10, and the equivalence $\mathfrak{MF}_S(f)/\{(1, f), (f, 1)\} \simeq \mathfrak{C}(R)/\{R\}$ follows easily, noting that $\operatorname{coker}(f) = R$. ❑

Chapter 5

The Auslander-Reiten Quiver

The purpose of this chapter is to provide a constructive method for visualising the most useful parts of the AR quiver as it pertains to our theoretical development. To include the *AR translation* information (*cf.* [17]) is a thesis in its own right; the reader can find the AR quivers complete with AR translation indicators in appendix A. For completeness, these are included, but one must study in-depth the AR theory of *almost split sequences* (*cf.* [1]) in order to gain meaning on this subject. Nonetheless, our treatment in this text is highly enlightening. We will obtain the indecomposable CM modules and observe rules for drawing arrows between them in the AR quiver. We will then apply these methods to the construction of the quiver of type A_n for n even.

5.1 Rules for Construction

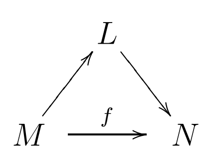
For this section, let R be a Henselian CM ring with maximal ideal $\mathfrak{m}$ and $k \simeq R/\mathfrak{m}$ an algebraically closed field. Let M, N be indecomposable CM modules over R.

Definition 5.1.1. The *radical* of M and N is the submodule of $\mathrm{Hom}_R(M, N)$ consisting of all non-invertible morphisms $f : M \to N$ and is denoted $\mathrm{rad}\,(M, N)$.

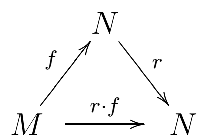
From this we have that $\mathrm{rad}^2(M, N) = \sum \mathrm{rad}\,(L, N)\mathrm{rad}\,(M, L)$ as L ranges over the subcategory of R-modules where these morphisms are defined. The elements of $\mathrm{rad}^2(M, N)$ are sums of morphisms $f : M \to L \to N$. Then, we naturally arrive at the definition of the submodule of *irreducible* morphisms

$$\mathrm{Irr}\,(M, N) = \frac{\mathrm{rad}\,(M, N)}{\mathrm{rad}^2(M, N)}.$$

If $f \in \mathrm{Irr}\,(M, N)$ then f cannot be decomposed as above because whenever there exists a commutative diagram



we factor this out. In fact, $\operatorname{Irr}(M,N)$ is a vector space over k because of the following. Fix $f \in \operatorname{rad}(M,N)$. Then for any $r \in \mathfrak{m}$, the action of r on f is a composition of the map f with the multiplication map $r : N \to N$; $n \mapsto rn$ as in the following commutative diagram:



Thus $r \cdot f \in \operatorname{rad}^2(M,N)$, so the action on elements of $\operatorname{Irr}(M,N)$ is equivalent to scalar multiplication by elements in k. As such, we may define the k-vector space dimension

$$\operatorname{irr}(M,N) = \dim_k \operatorname{Irr}(M,N).$$

Note that $\operatorname{irr}(M,N)$ is finite since $\operatorname{rad}(M,N)$ is finitely generated as an R-module.

We are now in a position to define the AR quiver.

Definition 5.1.2. The *AR quiver* Γ of $\mathfrak{C}(R)$ for a simple singularity R is a directed graph where:

- each vertex corresponds to a non-isomorphic, indecomposable CM module.
- the number of arrows from vertex $[M]$ to vertex $[N]$ corresponds to the integer $\operatorname{irr}(M,N) \geq 1$.

Also, to encode the information of the AR translation $\tau(M)$, we connect the vertex $[M]$ to the vertex $[N]$ with a dotted line if $N = \tau(M)$ such that there is an AR sequence $0 \to \tau(M) \to E \to M \to 0$ for some E; see [17, 1] for details of this theory.

Definition 5.1.3. We say that R is of *finite representation type* (FRT) if the number of vertices in the AR quiver of $\mathfrak{C}(R)$ is finite.

We now have enough to construct a naive, but original, example of the quiver of type A_n. We will obtain the vertices via matrix factorisations combined with an exhaustivity argument that we have all of them. The line of reasoning used combines elements from Yoshino (*cf.* [17]) in a completely original manner, to illustrate the use of matrix factorisations in visualising the torsion-free modules. To obtain the arrows, we will compute the necessary radicals directly. While it is easy using matrix factorisations to obtain the torsion-free modules for both even and odd n, the arrows will only be obtained in the case that n is even; the case that n is odd requires more complicated localisation arguments (*cf.* [9]). The reader will need to develop the proof for n odd on one's own if a naive approach without the theory of AR sequences (*cf.* [1]) is required.

5.2 The Quiver of type A_n

5.2.1 Construction of some Cohen-Macaulay Modules

We now wish to obtain the indecomposable torsion-free modules of $R = k[\![x,y]\!]/(f)$, where $f = x^2 + y^{n+1}$. We will begin with the case that n is odd, say $n = 2\ell - 1$, for some positive integer ℓ. Observe that f has factorisation into irreducibles

$$f = x^2 + y^{2\ell} = (y^\ell + ix)(y^\ell - ix),$$

so the modules given by $N_+ = R/(y^\ell + ix)$ and $N_- = R/(y^\ell - ix)$ are torsion-free over R. Note also that these have matrix factorisations of f given by $(y^\ell + ix, y^\ell - ix)$ and $(y^\ell - ix, y^\ell + ix)$ respectively.

Now returning to arbitrary n (even or odd), consider the 2×2 matrices

$$\varphi_j = \begin{pmatrix} x & y^j \\ y^{n+1-j} & -x \end{pmatrix}, \quad 0 \le j \le n+1.$$

Then we have the matrix factorisations of f given by (φ_j, φ_j), since, for each j, the matrix product

$$\varphi_j{}^2 = \begin{pmatrix} x^2 + y^{n+1} & xy^j - xy^j \\ xy^{n+1-j} - xy^{n+1-j} & y^{n+1} + x^2 \end{pmatrix} = f \cdot I_2.$$

Thus, let $M_j = \operatorname{coker} \varphi_j$ be torsion-free modules over R by Eisenbud's theorem 4.1.4.

Now $M_0 \simeq R$. To see this, let $S = k[\![x,y]\!]$, so that we have

$$0 \to S^2 \xrightarrow{\varphi_j} S^2 \to M_j \to 0.$$

In addition, let $\alpha_j = \binom{x}{y^{n+1-j}}$, $\beta_j = \binom{y^j}{-x}$, the generators of $\varphi_j(S^2)$. For M_0, we investigate the nature of $S^2/\operatorname{im}\varphi_0 = \operatorname{coker}\varphi_0 = M_0$. Let $\tilde{S} = (0) \oplus S$ and see that $\tilde{S} + \operatorname{im}\varphi_0 = S^2$, since we can obtain a 1 in the first direct summand via the generator β_0. As well as this, an element $\binom{0}{h} \in \tilde{S}$ also belongs to $\varphi_0(S^2)$ precisely when $\binom{0}{h} = g_1\alpha_0 + g_2\beta_0$ such that the power series $g_1, g_2 \in S$ satisfy $g_2 = -xg_1$ (ensuring that the first coordinate is zero). Thus, we have that

$$\begin{pmatrix} 0 \\ h \end{pmatrix} = g_1\alpha - xg_1\beta = \begin{pmatrix} 0 \\ g_1(y^{n+1} + x^2) \end{pmatrix} \in (\tilde{f}), \quad \text{where } (\tilde{f}) = (0) \oplus (f).$$

Since g_1 can be taken arbitrarily, we actually have $\tilde{S} \cap \operatorname{im}\varphi_0 = (\tilde{f})$. Now $\tilde{S}$ is naturally isomorphic to S, so this induces the isomorphism $\tilde{S}/(\tilde{f}) \simeq S/(f)$ since

(f) must map to $(\tilde{f})$ under the isomorphism $\tilde{S} \to S$; $(0, g) \mapsto g$. Thus the second isomorphism theorem (*cf.* [14]) yields

$$M_0 \simeq S^2/\operatorname{im}\varphi_0 = \frac{\tilde{S} + \operatorname{im}\varphi_0}{\operatorname{im}\varphi_0} \simeq \tilde{S}/(\tilde{f}) \simeq R.$$

The reasoning above relies on the fact that β_0 has a 1 in the 1-st coordinate to obtain 0 in the first direct summand. Observe that for $j = n+1$ we then have 1 in the second coordinate of α_{n+1}, and the proof follows similarly that $M_{n+1} \simeq R$. In fact, $M_j \simeq M_{n+1-j}$ for all $0 \le j \le n+1$ because $-\varphi_j = U^{-1}\varphi_{n+1-j}U$, where $U = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$; it suffices to see that these matrices are similar for then $(-U, U)$ is an isomorphism of (φ_j, φ_j) and $(\varphi_{n+1-j}, \varphi_{n+1-j})$.

Thus we have a complete picture of the M_j for n even, but returning to the case that $n = 2\ell - 1$, we must investigate the nature of the distinct middle module M_ℓ. Consider the torsion-free module $N_+ \oplus N_-$. This has matrix factorisation, which we will denote $(\psi_\ell^+, \psi_\ell^-)$, given by the direct sum (defined in section 4.3)

$$(y^\ell + ix, y^\ell - ix) \oplus (y^\ell - ix, y^\ell + ix) = \left(\begin{pmatrix} y^\ell + ix & 0 \\ 0 & y^\ell - ix \end{pmatrix}, \begin{pmatrix} y^\ell - ix & 0 \\ 0 & y^\ell + ix \end{pmatrix} \right).$$

This is equivalent to $(\varphi_\ell, \varphi_\ell)$ via the isomorphism $(\eta, \zeta) : (\varphi_\ell, \varphi_\ell) \to (\psi_\ell^+, \psi_\ell^-)$ given by

$$(\eta, \zeta) := \left(\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} \right).$$

A simple calculation reveals that

$$\eta\varphi_\ell = \begin{pmatrix} x - iy^\ell & y^\ell + ix \\ y^\ell - ix & -iy^\ell - x \end{pmatrix} = \psi_\ell^+\zeta,$$

$$\eta\psi_\ell^- = \begin{pmatrix} y^\ell - ix & -iy^\ell + x \\ -iy^\ell - x & y^\ell + ix \end{pmatrix} = \varphi_\ell\zeta,$$

and this suffices via remark 4.3.3 to ensure that the following diagram commutes:

$$\begin{array}{ccccc}
S^2 & \xrightarrow{\varphi_\ell} & S^2 & \xrightarrow{\varphi_\ell} & S^2 \\
{\scriptstyle\eta}\downarrow & & {\scriptstyle\zeta}\downarrow & & {\scriptstyle\eta}\downarrow \\
S^2 & \xrightarrow{\psi_\ell^+} & S^2 & \xrightarrow{\psi_\ell^-} & S^2 \\
{\scriptstyle\eta}\downarrow & & {\scriptstyle\zeta}\downarrow & & {\scriptstyle\eta}\downarrow \\
S^2 & \xrightarrow{\varphi_\ell} & S^2 & \xrightarrow{\varphi_\ell} & S^2
\end{array}$$

Thus $M_\ell \simeq N_+ \oplus N_-$ and we have a complete picture of the torsion-free modules arising from these matrix factorisations for any n. Before investigating the morphisms between them we must verify that these are in fact all of the torsion-free modules which correspond to vertices in the quiver of A_n.

5.2.2 Verification of Exhaustivity

We will now show that these torsion-free modules M_j, $0 \leq j \leq \lfloor \frac{n+1}{2} \rfloor$, are all indecomposable and that there are no more non-isomorphic torsion-free modules. For any n and any j ($\neq \ell$ in the case that $n = 2\ell - 1$), note that for the generators $\alpha = (1, 0)$ and $\beta = (0, 1)$ of S^2, modulo $\varphi_j(S^2)$, we can produce the following relations

$$\begin{aligned} x\alpha &= \begin{pmatrix} x \\ 0 \end{pmatrix} - \alpha_j = \begin{pmatrix} 0 \\ -y^{n+1-j} \end{pmatrix} = -y^{n+1-j}\beta, \\ x\beta &= \begin{pmatrix} 0 \\ x \end{pmatrix} + \beta_j = \begin{pmatrix} y^j \\ 0 \end{pmatrix} = y^j\alpha. \end{aligned} \tag{5.1}$$

If $n = 2\ell - 1$ and $j = \ell$ then we are in the special case of $M_\ell = N_+ \oplus N_-$. We will show that any torsion-free module that does not contain $N_+ \oplus N_-$ as a direct submodule has a direct submodule with the relations (5.1) for some generators α, β, thus our M_j are all the indecomposable torsion-free modules over R. Note that N_+ and N_- are clearly indecomposable.

Let $M \neq 0$ be a torsion-free module over R; in the case that n is odd we also stipulate that M does not contain a submodule isomorphic to $N_+ \oplus N_-$. If R were a principal domain then we could say that M is free via appeal to the structure theorem for finitely generated modules over principal domains. However, R is clearly not principal, and in fact, not necessarily a domain if, say, n is odd. Nevertheless, the subring $T = k[\![y]\!]$ of R satisfies all the necessary conditions and thus M is free as a T-module (so certainly torsion-free over T) for any $n \geq 1$. Recall remark 2.2.7 and note that any factorisation of f requires x, but T has only even powers of x via $(n+1)$-th powers of powers of y.

Now, regarding M as a T-module, let μ be the maximum integer with the property that $xM \subseteq y^\mu M$. Note that $\mu < n + 1$, for if $\mu \geq n + 1$, then $xM \subseteq y^{n+1}M = x^2M$ so we must have $M = 0$ contrary to assumption. We also observe that $y^{n+1}M = x^2M \subseteq xy^\mu M$ and then since y is a non-zerodivisor we have $y^{n+1-\mu}M \subseteq xM \subseteq y^\mu M$. Thus, this inclusion tells us that $n + 1 - \mu \geq \mu$, so we obtain that $\mu \in \mathbb{N}$ varies over $0, 1, \ldots, \frac{n+1}{2}$, and this is made sharp depending on the parity of n by

$$0 \leq \mu \leq \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Actually, if n is even, we need not consider $\mu = \lfloor \frac{n+1}{2} \rfloor$ as it will become apparent that this corresponds to the submodule $N_+ \oplus N_-$ which we have already excluded from our reasoning.

We will now begin to construct our generators α and β to see that M has a submodule isomorphic to one of the M_j obtained previously. We may take $\beta \in M$ for which $x\beta \in y^\mu M - y^{\mu+1}M$ to obtain $x\beta = y^\mu \alpha$ for some $\alpha \in M$. These are not in yM, for if $\alpha \in yM$, then $x\beta \in y^{\mu+1}M$ though we have chosen it not to be, and if $\beta \in yM$, then $x\beta = y^\mu\alpha \in xyM \subseteq y^{\mu+1}M$ hence $\alpha \in yM$ since y is a non-zerodivisor (so cycle back to the start of the sentence). Then the T-submodule N of M generated by α and β may also be regarded as an R-submodule with the action of y defined by

$$\begin{aligned} x\alpha &= -y^{n+1-\mu}\beta \\ x\beta &= y^\mu \alpha. \end{aligned} \tag{5.2}$$

Compare (5.2) with (5.1). These show that any CM R-module M has a T-submodule N isomorphic to one of the M_j regarded as T-modules.

We next proceed to show that N is a direct summand of M as a T-module. We shall do this by proving that α and β are part of a T-free base of M; i.e. it suffices to show that $\alpha \notin yM + T\beta$. Suppose that α can be written $\alpha = y\gamma + t\beta$ for some $\gamma \in M$, $t \in T$. Then (5.2) has that $-y^{n+1-\mu}\beta = x\alpha = xy\gamma + tx\beta = xy\gamma + ty^\mu\alpha$, thus $ty^\mu\alpha = -y^{n+1-\mu}\beta - xy\gamma \in y^{\mu+1}M$ since $x\gamma \in y^\mu M$ and $n+1-\mu \geq \frac{n+1}{2} \geq \mu$ (using the largest bound case for n). Since y is a non-zerodivisor in M, and M is torsion-free, it must be that $t\alpha \in yM$ after cancellation. Thus $t \in yT$ because we know that $\alpha \notin yM$. Hence returning to the supposition $\alpha = y\gamma + t\beta$, since each summand is in yM we obtain the contradiction $\alpha \in yM$.

It remains to see that N may be extended to a direct summand of M as an R-module and we will have proved that every torsion-free module over R can be decomposed into the irreducible $M_j \simeq \operatorname{coker} \varphi_j$ (including M_ℓ for the case that $n = 2\ell - 1$). Let $\{\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_\ell\}$ be a T-free base of M, existent by the argument in the preceding paragraph. Using $xM \subseteq y^\mu M$, we have

$$x\gamma_i = y^\mu(a_i\alpha + b_i\beta + \sum_{j=1}^{\ell} c_{ij}\gamma_j) \tag{5.3}$$

for each $1 \leq i \leq \ell$ where $a_i, b_i, c_{ij} \in T$. For $\gamma_i{}' = \gamma_i - a_i\beta$, it is clear that $\{\alpha, \beta, \gamma_1{}', \gamma_2{}', \ldots, \gamma_\ell{}'\}$ is also a T-free base of M, with each

$$x\gamma_i{}' = x\gamma_i - a_i y^\mu \alpha = y^\mu(b_i\beta + \sum_{j=1}^{\ell} c_{ij}(\gamma_i{}' + a_i\beta)) = y^\mu((b_i + \sum_{j=1}^{\ell} c_{ij}a_i)\beta + \sum_{j=1}^{\ell} c_{ij}\gamma_i{}').$$

Thus we may assume that each $b_i = 0$, $1 \leq i \leq \ell$ in (5.3). Then we have that $-y^{n+1}\gamma_i = x^2\gamma_i = y^\mu(b_i x\beta + \sum_{j=1}^{\ell} c_{ij}(x\gamma_j))$, so

$$\begin{aligned} -y^{2\mu}b_i\alpha - y^{n+1}\gamma_i &= y^\mu \sum_{j=1}^{\ell} c_{ij}x\gamma_j \\ &= y^\mu \sum_{j=1}^{\ell} c_{ij}(b_j\beta + \sum_{k=1}^{\ell} c_{jk}\gamma_k) \\ &= (y^\mu \sum_{j=1}^{\ell} c_{ij}b_j)\beta + y^\mu \sum_{j,k=1}^{\ell} c_{ij}c_{jk}\gamma_k. \end{aligned} \tag{5.4}$$

Since $\{\alpha, \beta, \gamma_1, \gamma_2, \ldots, \gamma_\ell\}$ is a free base over T, comparing coefficients of α in (5.4) shows that we must have each $b_i = 0$, $1 \leq i \leq \ell$. Thus, the T-free submodule N' generated by $\{\gamma_1, \gamma_2, \ldots, \gamma_\ell\}$ is closed under the action of y, hence it is also an R-submodule of M; i.e. $M = N \oplus N'$ as an R-module.

We have just proven the following proposition, which is an original compilation and extension of results found in Yoshino (*cf.* [17]) applied to our particular case.

Proposition 5.2.1. *Let $R = k[\![x, y]\!]/(x^2 + y^{n+1})$ be a simple singularity of type A_n. Then the R-modules $M_j = \operatorname{coker}\varphi_j$ given by the matrix factorisations*

$$\varphi_j = \begin{pmatrix} x & y^j \\ y^{n+1-j} & -x \end{pmatrix}, \quad 0 \leq j \leq \left\lfloor \frac{n+1}{2} \right\rfloor,$$

together with N_+ and N_- if n is odd, is a complete list of non-isomorphic indecomposable torsion-free modules. In particular, R is of finite representation type with the number of vertices in its AR quiver given by

$$1 + \left\lceil \frac{n+1}{2} \right\rceil + (-1)^{n+1}.$$

❑

We now proceed with determining the arrows between the M_j. We will focus on the case that n is even. We apply ideas on localisation which were omitted from discussion in Yoshino (*cf.* [17]).

5.2.3 The Morphisms between the Cohen-Macaulay Modules for even n

We now move on to determining the arrows of the AR quiver for R for even n by obtaining the numbers $\operatorname{irr}(M_i, M_j)$. Let $n = 2m$ for some positive integer m. Then in fact, R is always an integral domain and we may identify R with the subring $k[\![t^2, t^{n+1}]\!]$ of the power series ring $k[\![t]\!]$ by taking $t = \frac{x}{y^m}$. Observe that in R, we have $t^2 = \frac{x^2}{y^n} = \frac{-y^{n+1}}{y^n} = -y$ and $t^{n+1} = \frac{x^{n+1}}{y^{m(n+1)}} = \frac{x^{n+1}}{(-1)^m x^n} = (-1)^m x$, which is convincing enough of the isomorphism up to a change in sign of the variables. Also, each M_j may be identified with the fractional ideal $(1, t^{n+1-2j})R = R + t^{n+1-2j}R$, and so when $j = m$ we have $M_m = (1, t)R = k[\![t]\!]$. Thus we have the series of inclusions

$$M_0 = R \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = k[\![t]\!]. \tag{5.5}$$

Consider M_i, M_j where $0 \leq i, j \leq m$. We will visualise the homomorphisms between M_i and M_j as certain multiplication maps in $k[\![t]\!]$ in order to easily deduce results about $\operatorname{irr}(M_i, M_j)$. To do this, we must consider the localisation of R. For the multiplicative set $U = R - 0$, we may formally adjoin U^{-1} to R to create the ring of fractions $R[U^{-1}]$ which allows multiplicative inverses of any non-zero element (*cf.* [14]). In fact, this is the field of fractions of R, which we will denote $K := R[U^{-1}] = k((t))$; K consists of k-linear combinations of powers (including negatives) of t. Formally, elements of K are the equivalence classes of pairs $(r, u) \in R \times U$ where two such pairs $(r_1, u_1) \sim (r_2, u_2)$ if and only if there exists $v \in U$ for which

$$v(u_1 r_2 - u_2 r_1) = 0. \tag{5.6}$$

Note in particular that, for our torsion-free modules M_i, $M_i[U^{-1}] = K$ for any $0 \leq i \leq n + 1$ as a K-module. Thus, given an R-homomorphism $\zeta : M_i \to M_j$, we have a K-linear map

$$\zeta[U^{-1}] : K = M_i[U^{-1}] \to M_j[U^{-1}] = K.$$

Thus since $\zeta[U^{-1}] \in \operatorname{End}_K(K)$, it is simply defined by multiplication of some $g \in K$. This is because $\operatorname{End}_K(K) = K$ for any commutative ring K, via $g \mapsto gh$ for $g, h \in K$, under the natural K-module action. Thus we have a map

$$\operatorname{Hom}_R(M_i, M_j) \to \operatorname{Hom}_K(M_i[U^{-1}], M_j[U^{-1}]);$$
$$\zeta \mapsto \zeta[U^{-1}].$$

This is injective because M_i, M_j are torsion-free over R so that we are assured to recover (r_1, u_1), (r_2, u_2) for distinct r_1, r_2 in (5.6) after cancellation of the non-zerodivisor $v \in U$. Hence we have that $\operatorname{Hom}_R(M_i, M_j) \subseteq K$ via the following commutative diagram:

$$\begin{array}{ccc} M_i & \xrightarrow{\zeta} & M_j \\ \big\downarrow & & \big\downarrow \\ M_i[U^{-1}] & \xrightarrow{\zeta[U^{-1}]} & M_j[U^{-1}] \end{array}$$

This means that an element $\zeta \in \operatorname{Hom}(M_i, M_j)$ can be computed if and only if, upon multiplication by some $g \in K$ via $\zeta[U^{-1}]$, we have $g(R + t^{n+1-2i}R) \subseteq R + t^{n+1-2j}$. This is equivalent to asking that multiplication by g on the generators of M_i is closed inside M_j; i.e. $g, gt^{n+1-2i} \in R + t^{n-2j}R$. In fact, we need only consider multiplication by some non-negative power e of the regular parameter t for which $t^e M_i \subseteq M_j$. This is because any linear combination of these will belong to $\operatorname{rad}^2(M_i, M_j)$ and is consequently factored out during calculation of $\operatorname{irr}(M_i, M_j)$ (note that e must be non-negative because after all we are interested in recovering R-homomorphisms).

Now for such multiplication maps $t^e \in \operatorname{Hom}(M_i, M_j)$, we must have t^e and $t^{n+1-2i+e}$ among the elements of $R + t^{n+1-2j}R = k[\![t^2, t^{n+1-2j}]\!]$. So, considering the form of such elements, we may write

$$t^e = a_0 + a_2 t^2 + a_4 t^4 + \cdots + a_{n-2j} t^{n-2j} + a_{n+1-2j} t^{n+1-2j} + \ldots, \qquad (5.7)$$

$$t^{n+1-2i+e} = b_0 + b_2 t^2 + b_4 t^4 + \cdots + b_{n-2j} t^{n-2j} + b_{n+1-2j} t^{n+1-2j} + \ldots, \qquad (5.8)$$

for coefficients $a_m, b_m \in k$. Thus, if e is odd, (5.7) shows that we must have $e \geq n + 1 - 2j$ (and (5.8) yields a trivial result $e \geq 0 \geq 2i - n - 1$). If e is even, (5.7) has that $e \geq 0$, but also $n+1-2i+e$ is odd and so (5.8) has that $e \geq 2(i-j)$. Hence, we sum up this information in the following.

$$e \geq \begin{cases} n + 1 - 2j & e \text{ even}, \\ \max\{2(i-j), 0\} & e \text{ odd}. \end{cases} \qquad (5.9)$$

Now we can proceed to investigate the numbers $\operatorname{irr}(M_i, M_j)$. Firstly, if $i \neq j$ then $\operatorname{rad}(M_i, M_j) = \operatorname{Hom}(M_i, M_j)$; we have seen that M_i and M_j are not isomorphic for all distinct $0 \leq i, j \leq m$, so all homomorphisms between them are non-invertible. Also, if $|i - j| \geq 2$, then (5.5) shows that any homomorphism $M_i \to M_j$ may be decomposed into a composition of the inclusion map $M_i \hookrightarrow M_{i+1}$ and a map $M_{i+1} \to M_j$. Specifically, $M_{i+1} \to M_j$ exists since if $i+1 \geq j+3$, then we can always choose a non-zero e satisfying (5.9), and if $i+1 < j+3$, then we may continue along the chain of inclusion maps until we can again choose an appropriate non-zero e. Thus $\operatorname{rad}^2(M_i, M_j) = \operatorname{Hom}(M_i, M_j)$ and so $\operatorname{Irr}(M_i, M_j) = 0$. We have just shown that any two non-adjacent vertices do not share an arrow; i.e.

$$|i-j| \geq 2 \Rightarrow \operatorname{irr}(M_i, M_j) = 0. \tag{5.10}$$

Now, whenever we have a homomorphism $f : M_i \to M_{i+1}$, f is of the form $f = u\iota$ where ι is the inclusion map derived from (5.5) and u is an automorphism of M_{i+1} restricted to $\iota(M_i)$. It thus stands to reason that ι is the unique irreducible homomorphism in $\operatorname{Irr}(M_i, M_{i+1})$. Similarly, if a retraction of $M_{i-1} \to M_i$ exists it must be unique $\operatorname{Irr}(M_i, M_{i-1})$ and so we have shown that there are at most two arrows between any two adjacent vertices, one in each direction; i.e.

$$|i-j| = 1 \Rightarrow \operatorname{irr}(M_i, M_j) \leq 1 \tag{5.11}$$

Finally, for the case that $i = j$, see that if $t^e M_i \subseteq M_i$, then the bound on odd e from (5.9) becomes trivial and we actually have that t^e can take on any value for the powers of t in $k[\![t^2, t^{n+1}]\!]$. This means that the set G of all powers t^e is multiplicative. Thus if $n + 1 - 2i \neq 1$ (equivalently, $i \neq m$), then G has more than one element of which any such multiplication map t^e is a composition of factors from G. Thus $\operatorname{rad}^2(M_i, M_i) = \operatorname{rad}(M_i, M_i)$, $i \neq m$, hence $\operatorname{irr}(M_i, M_i) = 0$. Alternatively, if $i = m$ then we have the unique non-invertible multiplication map t (if it exists) and hence we have shown the result

$$\begin{aligned} i = m &\Rightarrow \operatorname{irr}(M_i, M_i) \leq 1 \\ i \neq m &\Rightarrow \operatorname{irr}(M_i, M_i) = 0 \end{aligned} \tag{5.12}$$

Actually, all the inequalities in the results (5.10), (5.11) and (5.12) are equalities. We can see this by considering the matrix factorisation formulation of the M_i. We are able to obtain the short exact sequences

$$0 \to M_i \to E \to M_i \to 0,$$

for some rank 2 submodule E of S^2, and an AR theoretical argument regarding the dimension of $\operatorname{Irr}(M_i, M_j)$ for any other of the M_j (*cf.* [17]) shows that $\operatorname{irr}(M_i, M_j) = 2$, the rank of E. Then it is not too difficult to see that, for each $1 \leq j \leq m$,

$$\sum_{i=0}^{m} \operatorname{irr}(M_i, M_j) = \sum_{i=0}^{m} \operatorname{irr}(M_j, M_i) = 2.$$

Consequently, we deduce equality in the results (5.11) and (5.12).

Thus we are able to construct the quiver of type A_n for n even. In figure 5.1, each vertex M_j is thought of as a representative from the isomorphism class $[M_j] \supseteq \{M_j, M_{n+1-j}\}$.

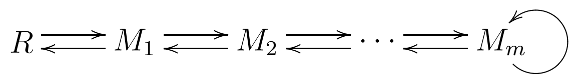


Figure 5.1: The quiver of type A_n for even $n = 2m$

5.2.4 The Morphisms between the Cohen-Macaulay Modules for odd n

It is an easy exercise to adapt the argument on page 46 above for all $i \leq \ell - 1$ (except for the loop at $M_{\ell-1}$) when $n = 2\ell - 1$. The maps between $M_{\ell-1}$ and M_ℓ are more complicated, as one can see from figure 5.2. Note that M_ℓ decomposes as $M_\ell = N_+ \oplus N_-$, so whenever we have a map $f \in \mathrm{Irr}\,(M_{\ell-1}, M_\ell)$, we may obtain a map $g \in \mathrm{Irr}\,(M_{\ell-1}, N_\pm)$ via $g = \pi f$ where $\pi : M_\ell \longrightarrow\!\!\!\!\!\rightarrow N_\pm$ is the canonical projection map. We can likewise do this with $\mathrm{Irr}\,(N_\pm, M_{\ell-1})$ using the canonical inclusion map. The rest of the proof is left to the reader as an exercise in the above logic.

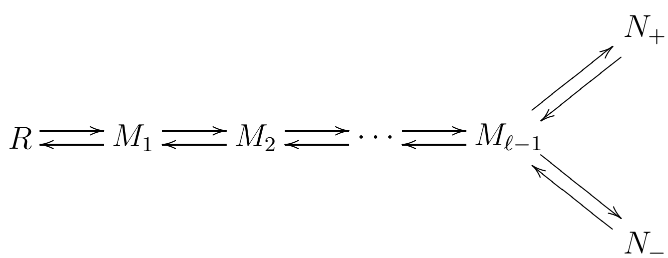


Figure 5.2: The quiver of type A_n for odd $n = 2\ell - 1$

We now have a comprehensive treatment of the quiver of type A_n, for the case that n is even, and a picture of how the case that n is odd differs from out result. To make this thesis complete as reference material for the shape of the quivers, in what remains of this thesis we will look towards the quiver of type D_n. In addition, appendix A contains the AR quivers of all simply singularity types, as well as the exceptional quivers E_6, E_7 and E_8, all of which (less the AR translations) can be obtained by similar methods as we have just exhibited. What we have just seen should provide a very useful naive demonstration of the techniques and ideas of one-dimensional simple singularity theory. The reader can always find the remaining constructions implementing deeper AR theory in Yoshino (*cf.* [17]).

5.3 The Quiver of type D_n

We now concern ourselves with visualising the quiver of CM modules of $R = k[\![x,y]\!]/(f)$ where $f = x^2y + y^{n-1}$. We will obtain the result for n even; n odd is obtained by a small tweak to be discussed later.

Let $n = 2\ell$ for some positive integer ℓ and observe that

$$f = y(x^2 + y^{2\ell-2}) = y(x + iy^{\ell-1})(x - iy^{\ell-1}).$$

Then the following are clearly matrix factorisations of f:

$$\begin{array}{rcll} (\alpha,\beta) & = & (y, x^2 + y^{2\ell-2}), & (\beta,\alpha), \\ (\gamma_+,\delta_+) & = & (y(x+iy^{\ell-1}), x - iy^{\ell-1}), & (\delta_+,\gamma_+), \\ (\gamma_-,\delta_-) & = & (y(x-iy^{\ell-1}), x + iy^{\ell-1}), & (\delta_-,\gamma_-). \end{array}$$

Denote the cokernels

$$A = \operatorname{coker}\alpha,\ B = \operatorname{coker}\beta,\ \Gamma_\pm = \operatorname{coker}\gamma_\pm,\ \Delta_\pm = \operatorname{coker}\delta_\pm.$$

Now, for the 2×2 matrices

$$\varphi_j = \begin{pmatrix} x & y^j \\ y^{n-j-2} & -x \end{pmatrix}, \qquad \psi_j = \begin{pmatrix} xy & y^{j+1} \\ y^{n-j-1} & -xy \end{pmatrix},$$

$$\xi_j = \begin{pmatrix} x & y^j \\ y^{n-j-1} & -xy \end{pmatrix}, \qquad \eta_j = \begin{pmatrix} xy & y^j \\ y^{n-j-1} & -x \end{pmatrix},$$

where $0 \le j \le n-3$, we see that (φ_j,ψ_j) and (ξ_j,η_j) are matrix factorisations of f via directly multiplying the matrices, e.g.

$$\varphi_j\psi_j = \begin{pmatrix} x^2y + y^{n-1} & xy^{j+1} - xy^{j+1} \\ xy^{n-j-1} - xy^{n-j-1} & y^{n-1} + x^2y \end{pmatrix} = f \cdot I_2.$$

The check $\psi_j\varphi_j$ is similar and we can do the same for ξ_j and η_j. Thus set

$$\begin{array}{rclrcl} M_j & = & \operatorname{coker}\varphi_j, & N_j & = & \operatorname{coker}\psi_j, \\ X_j & = & \operatorname{coker}\xi_j, & Y_j & = & \operatorname{coker}\eta_j \end{array}$$

and see that

$$M_0 \simeq B, \quad N_0 \simeq A \oplus R, \quad X_0 \simeq R, \quad Y_0 \simeq R.$$

Thus far, all our work holds in the case that n is odd, but now we note that the relations

$$M_j \simeq M_{n-j-2}, \quad N_j \simeq N_{n-j-2}, \quad X_j \simeq Y_{n-j-1}, \quad Y_j \simeq X_{n-j-1}$$

for $1 \leq j \leq n-3$ will pose a problem for odd n, namely for the unique middle term $j = (n-1)/2$. For the case that $n = 2m+1$ we have that $X_m \simeq Y_m$, otherwise for the even case $n = 2\ell$ we have $M_{\ell-1} \simeq \Delta_+ \oplus \Delta_-$ and $N_{\ell-1} \simeq \Gamma_+ \oplus \Gamma_-$.

These are in fact all of the vertices of the quiver by invoking Yoshino's Brauer-Thrall type theorem (*cf.* [16]) and so we obtain the following.

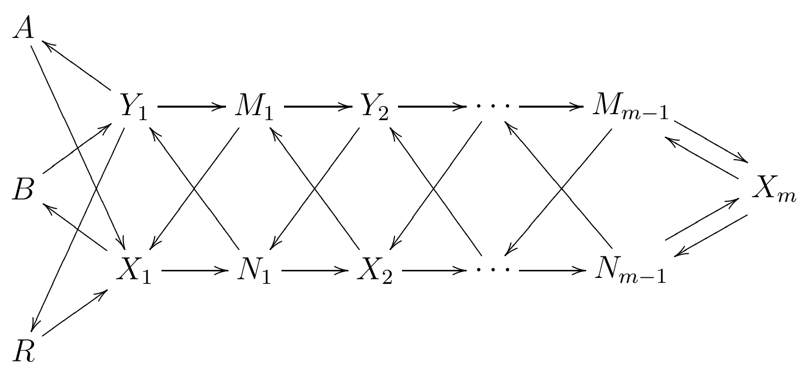


Figure 5.3: The quiver of type D_n for odd $n = 2m + 1$

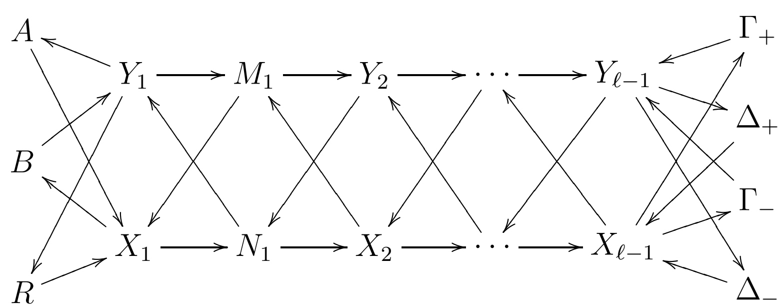


Figure 5.4: The quiver of type D_n for even $n = 2\ell$

Chapter 6

Closing Remarks

The reader should now have a good understanding of simple singularities of finite representation type in dimension one. The exceptional quivers of types E_6, E_7 and E_8 are an easy exercise to draw from the methods used for the case A_n, and can serve as an adequate consolidation of this material. Their drawings can be found in appendix A for reference. We were unable to compute the *AR translations* between our CM modules in this text; a brief explanation as well as the full quivers with AR information is also included in appendix A.

If one were to ask, *"where to from here?"*, the logical next step would be to study the AR theory of almost split sequences (*cf.* [1]) in order to complete the quivers treated in this thesis by including their AR translations of each vertex.

To extend this theory from plane curves to hypersurfaces of arbitrary degree, one must begin with a result due to Artin-Verdier, Auslander and Esnault in 1985 for the dimension two case (*cf.* [15]). It states that the singularity $R = k[\![x, y, z]\!]/(f)$, $f \in (x, y, z)$ has FRT if and only if it is fixed by a finite subgroup G of the special linear group acting on two parameters $\mathrm{SL}(ku \oplus kv)$; i.e. $R \simeq k[\![u, v]\!]^G$. Here we see how the theory of classification of these simple singularities is much more strongly correlated with the classical Dynkin types we have used in the thesis, whereby SL_2 has a strong connection with the Lie theory classification into these types.

Finally, there is a result by Knörrer in 1987 (*cf.* [11]) which shows that the ring $k[\![x, y]\!]/(f)$ has FRT if and only if $k[\![x, y, z]\!]/(f)$ has FRT, via the skew group ring $S \oplus S\sigma$ where σ is an order two permutation which fixes x and y and conjugates z; i.e. $\sigma(z) = -z$. This shows how extension to higher dimensions provides classification in the one-dimensional case also.

In this thesis, we have attempted to avoid this much more profound (and more powerful in terms of the number of case it deals with) theory by providing a special classification theory in the case of dimension one. We hope that this has helped the naive reader comprehend a small but wholly connected part of classification of simple plane curve singularities and perhaps provided the inspiration to pursue deeper meaning in this subject.

Appendix A

The AR Quivers — Complete with AR Translations

For a brief explanation of some AR theory, for an indecomposable CM module $M \in \mathfrak{C}(R)$, we define the set of non-split exact sequences

$$S(M) = \{s : 0 \to N_s \to E_s \to M \to 0\},$$

where $N_s \in \mathfrak{C}(R)$ is indecomposable. Then an *AR sequence for* M is given by the minimum element in $S(M)$ partially ordered by inclusion on N_s; note that if $S(M)$ is non-empty there will exist an AR sequence. In such a case that we have the AR sequence s of M, we denote by $\tau(M)$ the unique N_s which yields this AR sequence s, and this is called the *AR translation* of M.

In the following figures, we have re-drawn the AR quivers of each simple singularity type and included a broken line between two vertices M and N if there exists and AR sequence $0 \to \tau(M) \to E \to M \to 0$ with $\tau(M) \simeq N$.

A.0.1 The AR Quiver of Type A_n

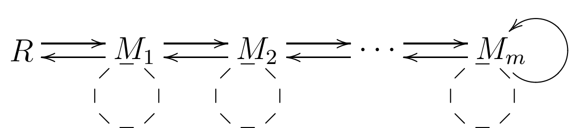


Figure A.1: The AR quiver of type A_n for even $n = 2m$

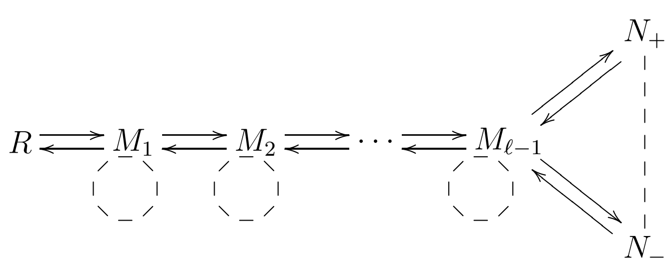


Figure A.2: The AR quiver type A_n for odd $n = 2\ell - 1$

A.0.2 The AR Quiver of Type D_n

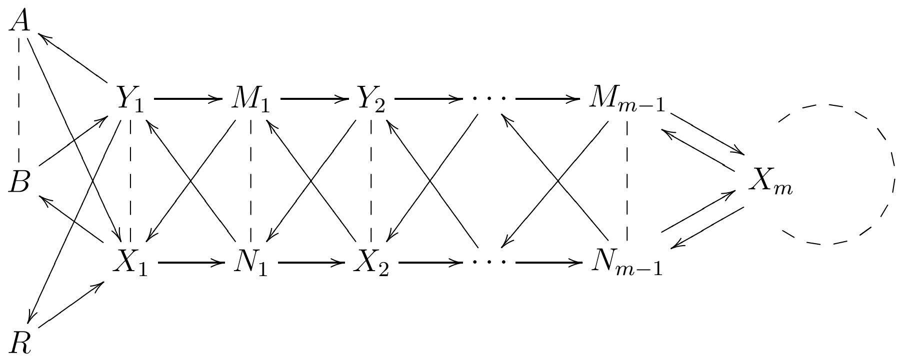


Figure A.3: The AR quiver of type D_n for odd $n = 2m + 1$

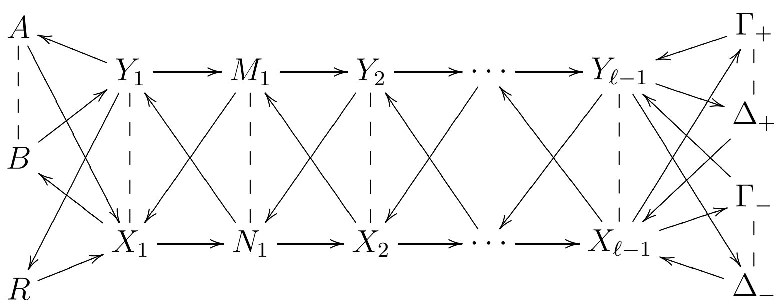


Figure A.4: The AR quiver of type D_n for even $n = 2\ell$

Below is a reference for the quivers E_6, E_7, E_8. Their computation is an easy exercise based on similar concepts as in the case A_n, but nonetheless can be found in [3, 17].

A.0.3 The Exceptional AR Quivers

The AR Quiver of Type E_6

Let $R = k[\![x, y]\!]/(f)$ where $f = x^3 + y^4$. Consider the matrices

$$\varphi_1 = \begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} x^2 & y \\ y^3 & -x \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x & y^2 \\ y^2 & -x^2 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} x^2 & y^2 \\ y^2 & -x \end{pmatrix},$$

$$\alpha = \begin{pmatrix} y^3 & x^2 & xy^2 \\ xy & -y^2 & x^2 \\ x^2 & -xy & -y^3 \end{pmatrix}, \qquad \beta = \begin{pmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & -y \end{pmatrix}.$$

Then (φ_j, ψ_j), (ψ_j, φ_j), (α, β), (β, α) for $j = 1, 2$ are all matrix factorisations of f, but we also have a non-isomorphic indecomposable matrix factorisation given by $(\xi, \eta) \simeq (\eta, \xi)$ where

$$\xi = \begin{pmatrix} \varphi_2 & \gamma_1 \\ 0 & \psi_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \psi_2 & \gamma_2 \\ 0 & \varphi_2 \end{pmatrix},$$

and $\gamma_1 = \begin{pmatrix} 0 & y \\ -xy & 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & xy \\ y & 0 \end{pmatrix}$. We define the CM modules

$$\begin{array}{rclrcl} M_j & = & \operatorname{coker} \varphi_j, & N_j & = & \operatorname{coker} \psi_j, \\ A & = & \operatorname{coker} \alpha, & B & = & \operatorname{coker} \beta, \end{array}$$

as well as $X = \operatorname{coker} \xi \simeq \operatorname{coker} \eta$. By similar methods as in the case A_n it is easy to obtain the isomorphisms

$$N_1 \simeq \mathfrak{m}, \quad M_1 \simeq (x^2, y)R, \quad N_2 \simeq M_2 \simeq (x^2, y^2)R, \quad B \simeq (x, y)^2 R.$$

The AR quiver for the case E_6 is as in figure A.5 below.

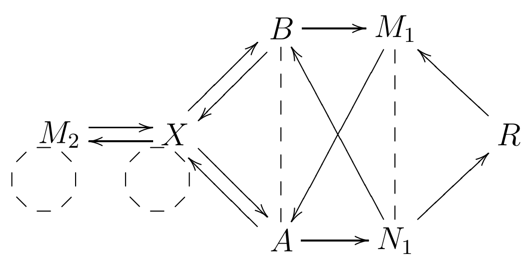


Figure A.5: The AR quiver of type E_6

The AR Quiver of Type E_7

Let $R = k[\![x, y]\!]/(f)$ where $f = x^3 + xy^3$. Consider the matrices

$$\alpha = (x), \qquad \beta = (x^2 + y^3),$$

$$\gamma = x \begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix}, \qquad \delta = \begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix},$$

$$\varphi_1 = \begin{pmatrix} x & y \\ xy^2 & -x^2 \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x & y^2 \\ xy & -x^2 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} x^2 & y^2 \\ xy & -x \end{pmatrix},$$

$$\xi_1 = \begin{pmatrix} xy^2 & -x^2 & x^2y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{pmatrix}, \qquad \eta_1 = \begin{pmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} x^2 & -y^2 & -xy \\ xy & x & -y^2 \\ xy^2 & xy & x^2 \end{pmatrix}, \qquad \eta_2 = \begin{pmatrix} x & 0 & y \\ -xy & x^2 & 0 \\ 0 & -xy & x \end{pmatrix}.$$

Then (α, β), (β, α), (γ, δ), (δ, γ), (φ_j, ψ_j), (ψ_j, φ_j), (ξ_j, η_j), (η_j, ξ_j) for $j = 1, 2$ are all matrix factorisations of f, but we also have non-isomorphic indecomposable matrix factorisations given by (ξ_3, η_3) and (η_3, ξ_3) where

$$\xi_3 = \begin{pmatrix} \gamma & \varepsilon \\ 0 & \delta \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} \delta & -\varepsilon \\ 0 & \gamma \end{pmatrix},$$

and $\varepsilon = y \cdot I_2$ (I_2 the 2×2 identity matrix). We define the CM modules

$$\begin{aligned} A &= \operatorname{coker}\alpha, & B &= \operatorname{coker}\beta, & \Gamma &= \operatorname{coker}\gamma, & \Delta &= \operatorname{coker}\delta, \\ M_i &= \operatorname{coker}\varphi_i, & N_i &= \operatorname{coker}\psi_i, & X_j &= \operatorname{coker}\xi_j, & Y_j &= \operatorname{coker}\eta_j. \end{aligned}$$

for $i = 1, 2$ and $j = 1, 2, 3$. The AR quiver for the case E_7 is as in figure A.6 below.

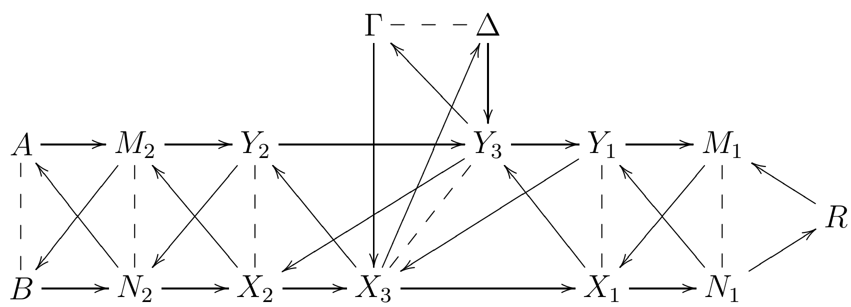


Figure A.6: The AR quiver of type E_7

The AR Quiver of Type E_8

Let $R = k[\![x, y]\!]/(f)$ where $f = x^3 + y^5$. Consider the matrices

$$\varphi_1 = \begin{pmatrix} x & y \\ y^4 & -x^2 \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} x^2 & y \\ y^2 & -x \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} x & y^2 \\ y^3 & -x^2 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} x^2 & y^2 \\ y^3 & -x \end{pmatrix},$$

$$\alpha_1 = \begin{pmatrix} y & -x & 0 \\ 0 & y & -x \\ x & 0 & y^3 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} y & -x & 0 \\ 0 & y^2 & -x \\ x & 0 & y^2 \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} y^4 & xy^2 & x^2 \\ -x^2 & y^3 & xy \\ -xy^2 & -x^2 & y^3 \end{pmatrix},$$

$$\gamma_1 = \begin{pmatrix} y & -x & 0 & y^3 \\ x & 0 & -y^3 & 0 \\ -y^2 & 0 & -x^2 & 0 \\ 0 & -y^2 & -xy & -x^2 \end{pmatrix}, \qquad \delta_1 = \begin{pmatrix} 0 & x^2 & -y^3 & 0 \\ -x^2 & xy & 0 & -y^3 \\ 0 & -y^2 & -x & 0 \\ y^2 & 0 & y & -x \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} x & y^2 & 0 & y \\ y^3 & -x^2 & -xy^2 & 0 \\ 0 & 0 & x^2 & y^2 \\ 0 & 0 & y^3 & -x \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} x^2 & y^2 & 0 & xy \\ y^3 & -x & -y^2 & 0 \\ 0 & 0 & x & y^2 \\ 0 & 0 & y^3 & -x^2 \end{pmatrix},$$

$$\xi_1 = \begin{pmatrix} y^4 & xy^2 & x^2 & 0 & 0 & xy \\ -x^2 & y^3 & xy & -x & 0 & 0 \\ -xy^2 & -x^2 & y^3 & 0 & -xy & 0 \\ 0 & 0 & 0 & y & -x & 0 \\ 0 & 0 & 0 & 0 & y^2 & -x \\ 0 & 0 & 0 & x & 0 & y^2 \end{pmatrix}, \qquad \eta_1 = \begin{pmatrix} y & -x & 0 & 0 & 0 & -x \\ 0 & y^2 & -x & xy & 0 & 0 \\ x & 0 & y^2 & 0 & xy & 0 \\ 0 & 0 & 0 & y^4 & xy^2 & x^2 \\ 0 & 0 & 0 & -x^2 & y^3 & xy \\ 0 & 0 & 0 & -xy^2 & -x^2 & y^3 \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} y^4 & x^2 & 0 & -xy^2 & 0 \\ -x^2 & xy & 0 & -y^3 & 0 \\ 0 & -y^2 & -x & 0 & y^3 \\ -xy^2 & y^3 & 0 & x^2 & 0 \\ -y^3 & 0 & -y^2 & xy & -x^2 \end{pmatrix}, \qquad \eta_2 = \begin{pmatrix} y & -x & 0 & 0 & 0 \\ x & 0 & 0 & y^2 & 0 \\ -y^2 & 0 & -x^2 & 0 & -y^3 \\ 0 & -y^2 & 0 & x & 0 \\ 0 & 0 & y^2 & y & -x \end{pmatrix}.$$

Then (φ_j, ψ_j), (ψ_j, φ_j), (α_j, β_j), (β_j, α_j), (γ_j, δ_j), (δ_j, γ_j), (ξ_j, η_j), (η_j, ξ_j) for $j = 1, 2$ are all matrix factorisations of f. We define the CM modules

$$
\begin{array}{rclrcl}
M_j & = & \operatorname{coker}\varphi_j, & N_j & = & \operatorname{coker}\psi_j, \\
A_j & = & \operatorname{coker}\alpha_j, & B_j & = & \operatorname{coker}\beta_j, \\
\Gamma_j & = & \operatorname{coker}\gamma_j, & \Delta_j & = & \operatorname{coker}\delta_j, \\
X_j & = & \operatorname{coker}\xi_j, & Y_j & = & \operatorname{coker}\eta_j,
\end{array}
$$

for $j = 1, 2$. The AR quiver for the case E_7 is as in figure A.7 below.

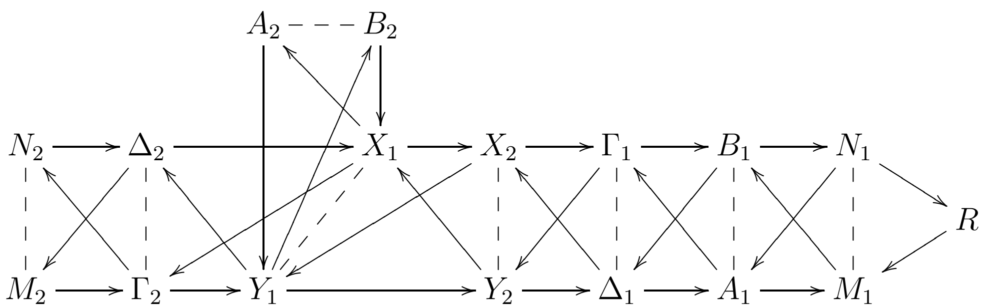


Figure A.7: The AR quiver of type E_8

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