Noncommutative Coordinate Rings and Stacks

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Abstract

Let $s, t : Y \rightrightarrows X$ be a finite flat groupoid scheme with X a quasi-projective variety and let S be its coarse moduli scheme. We associate to the groupoid scheme a coherent sheaf of algebras $\mathcal{O}_{X/Y}$ on S which we call the noncommutative coordinate ring of the groupoid scheme. We show that when X is a smooth curve and the groupoid action is generically free, the noncommutative coordinate rings which can occur are, up to Morita equivalence, the hereditary orders on smooth curves. This gives a bijective correspondence between smooth one dimensional Deligne-Mumford stacks of finite type and Morita equivalence classes of hereditary orders on smooth curves.

Throughout, k denotes some base field. All objects and maps are assumed to be defined over k. All schemes are assumed to be noetherian.

1 Introduction

In noncommutative differential geometry as developed by Connes and others (see [Connes]), one associates a noncommutative algebra to a groupoid $s, t : Y \rightrightarrows X$. This algebra can be thought of as the ring of functions on the quotient object associated to the groupoid. In algebraic geometry, these quotient objects are known as algebraic stacks. They have been studied intensely as a result of their many applications to moduli problems.

It seems desirable to have a version of Connes' construction in the algebraic setting and that is the goal of this paper. Unfortunately, as a result of the paucity of algebraic functions, we are led to impose fairly strong conditions on our groupoid $s, t : Y \rightrightarrows X$, namely, we insist that s, t are finite and flat and that X is quasi-projective. The resulting noncommutative coordinate ring $\mathcal{O}_{X/Y}$ that we associate to this data is an $\mathcal{O}_X$-bimodule algebra, or, more naively, a coherent sheaf of algebras on the coarse moduli scheme of the stack X/Y. The construction comes from dualising the $\mathcal{O}_X$-bimodule coalgebra associated to the groupoid as defined, for example in [Del, §1.14] or [KR, §2.1.1].

As occurs in the coalgebra setting, the category of quasi-coherent sheaves on the stack X/Y is equivalent to the category of $\mathcal{O}_{X/Y}$-modules. It is thus hoped that the study of noncommutative algebra may be useful to the study of algebraic stacks and vice versa. Though there are many questions one can ask, we content ourselves in this paper to studying examples. The main results are in Section 7 where the case of curves is disposed of. We show that, up to Morita equivalence, the noncommutative coordinate rings $\mathcal{O}_{X/Y}$ which can occur when X is a smooth quasi-projective curve and the groupoid action is generically free, are precisely the hereditary orders on smooth curves. This correspondence was suggested by Van den Bergh and can be reinterpreted in terms of smooth noncommutative curves. For the rest of the introduction, we shall assume that our base field k is algebraically closed of characteristic zero. In [RVdB], Reiten and Van den Bergh studied saturated connected noetherian Ext-finite hereditary categories (see [RVdB, Introduction] or [SV, Section 7] for definitions). We will view these as noncommutative curves since they behave like the category of coherent sheaves on a smooth projective curve. Reiten and Van den Bergh classify all such noncommutative curves ([RVdB, Theorem C, page 8]) into two types:

i. $A -$ mod where A is an indecomposable finite dimensional hereditary algebra.

ii. $A-\text{mod}$ where A is an hereditary order on a smooth projective curve.

Our result thus shows that the smooth noncommutative curves which do not possess a projective generator correspond precisely to the smooth proper 1 dimensional Deligne-Mumford stacks of finite type over k which are generically schemes.

In Section 5 are more examples. In particular, much of our construction does not depend on the inverse of the groupoid. Quivers give examples of semigroupoids and, when they are finite and have no loops, our construction may be applied to yield path algebras. Since these are precisely the finite dimensional hereditary algebras (see [ARS, Proposition III.1.13]), we see that all the other noncommutative curves arise in this fashion.

Section 6 contains some general results on the relationship between stacks and their noncommutative coordinate rings. For example, we show that fibre products of stacks correspond to tensor products of coordinate rings. From the point of view of noncommutative algebraic geometry, this is rather reassuring.

The idea of studying noncommutative algebraic geometry via stacks is not new. Kontsevich and Rosenberg (see [KR]) have developed a fairly general theory in this regard. Our viewpoints however are rather different. They are interested in building up a general theory of noncommutative algebraic geometry whereas our goal is to study the relationship between algebraic stacks and noncommutative algebra.

Finally, it is hoped that this paper may be of interest both to noncommutative algebraists and algebraic geometers interested in stacks. As a result, more detail has been put into proofs than would otherwise have been the case.

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2 Background on Bimodules

In this section, we review Van den Bergh's definition of a bimodule and review an alternate description in the simple case where the bimodule has a large centraliser. The bimodules we consider will all be what Van den Bergh calls sheaf bimodules. We will thus drop the adjective sheaf.

We need a technical

Definition 2.1 *([VdB96, Definition 2.1]) Let $f : Y \longrightarrow X$ be a morphism of finite type between two schemes Y and X. A quasi-coherent $\mathcal{O}_Y$-module $\mathcal{B}$ is said to be* relatively locally finite for f *if for every coherent $\mathcal{B}' \subseteq \mathcal{B}$ the restriction of f to Supp$\mathcal{B}'$ is finite.*

Let S be a scheme and X an S-scheme of finite type. We define

Definition 2.2 *([VdB96, Definition 2.3]) Let $\mathcal{B}$ be an $\mathcal{O}_{X\times_S X}$-module. We say that $\mathcal{B}$ is an* $\mathcal{O}_S$-central $\mathcal{O}_X$-bimodule *if $\mathcal{B}$ is relatively locally finite for the two projection maps $X \times_S X \longrightarrow X$.*

The simplest example of an $\mathcal{O}_X$-bimodule is $\mathcal{O}_\Delta$ where $\Delta \subset X \times_S X$ is the diagonal. Abusing notation, we will usually denote this bimodule by $\mathcal{O}_X$.

For most of the $\mathcal{O}_X$-bimodules we are interested in in this paper, they will be defined to be k-central, but there will exist a scheme S such that X is an S-scheme and the support of the bimodule actually lies in the subscheme $X \times_S X$ of $X \times X$. Hence, $\mathcal{B}$ can be construed as an $\mathcal{O}_S$-central bimodule. Accordingly, we will say that these bimodules are also *$\mathcal{O}_S$-central* or that *$\mathcal{O}_S$ acts centrally.*

From [VdB96], we know that $\mathcal{O}_X$-bimodules form an abelian category and there is a tensor product $-\otimes_{\mathcal{O}_X}-$ on the category. Furthermore, $\mathcal{O}_X$ is the unit of this tensor product in the sense that there are natural isomorphisms $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B} \simeq \mathcal{B} \simeq \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{O}_X$. We may thus talk about algebra objects or monads in the category. These are called *$\mathcal{O}_X$-bimodule algebras.* Given such an $\mathcal{O}_X$-bimodule algebra $\mathcal{B}$, Van den Bergh (in [VdB96, Definition 3.1.2]) defines a (right) $\mathcal{B}$-*module* M to be an $\mathcal{O}_X$-module equipped with a multiplication map $\mu : M \otimes_{\mathcal{O}_X} \mathcal{B} \longrightarrow M$ satisfying the usual compatibilities. We let $\mathcal{B} - \mathrm{Mod}$ denote the category of $\mathcal{B}$-modules. For the reader unfamiliar with these concepts, we offer an alternate description when the bimodules involved satisfy an additional hypothesis.

This additional hypothesis is defined in,

Definition 2.3 *Let $\mathcal{B}$ be an $\mathcal{O}_X$-bimodule. Suppose there exists a scheme S such that $\mathcal{O}_S$ acts centrally on $\mathcal{B}$ and $X \longrightarrow S$ is finite. If further, $\mathcal{B}$ is finite as an $\mathcal{O}_S$-module, then we say that $\mathcal{B}$ is* finite over its centraliser.

In this special case, bimodules can easily be studied via their affine counterparts. Note first that it makes the relatively locally finite condition on the bimodule redundant. Let $\mathcal{B}$ be an $\mathcal{O}_S$-central $\mathcal{O}_X$-bimodule where X is a finite S-scheme. We may push forward $\mathcal{B}$ via the morphism $X \times_S X \longrightarrow S$ to obtain an $\mathcal{O}_S$-module which we denote by $\mathcal{B}_S$. Similarly, we write $(\mathcal{O}_X)_S$ for $\mathcal{O}_X$ considered as a sheaf of algebras on S. Then $\mathcal{B}_S$ is an $(\mathcal{O}_X)_S$-bimodule in the sense that it is an $\mathcal{O}_S$-module such that on every affine open set U of S, it is an $(\mathcal{O}_X)_S(U)$-bimodule. Conversely, given such an $\mathcal{O}_S$-module, one can construct an $\mathcal{O}_S$-central $\mathcal{O}_X$-bimodule. Moreover, this correspondence respects the tensor category structure. The $\mathcal{O}_S$-central $\mathcal{O}_X$-bimodule algebras $\mathcal{B}$ can now be viewed as $(\mathcal{O}_X)_S$-bimodules which locally on S are $\mathcal{O}_X$-algebras where $\mathcal{O}_X$ need not act centrally. There is a well-defined notion of modules over a sheaf of algebras and in fact, $\mathcal{B}$-modules correspond precisely to $\mathcal{B}_S$-modules.

Our main reason for introducing this hypothesis is to simplify the notion of duals of bimodules. Let $\mathcal{B}$ be an $\mathcal{O}_S$-central $\mathcal{O}_X$-bimodule where X is a finite S-scheme. We wish to define an $\mathcal{O}_S$-central $\mathcal{O}_X$-bimodule $\mathrm{Hom}_{\mathcal{O}_X}({}_{\mathcal{O}_X}\mathcal{B}, \mathcal{O}_X)$ called the *left dual* of $\mathcal{B}$. By the correspondence above, we need only define an $\mathcal{O}_S$-module which is an $(\mathcal{O}_X)_S$-bimodule. On each affine open set U in S, let ${}_{\mathcal{O}_X(U)}\mathcal{B}(U)$ denote the left $\mathcal{O}_X(U)$-module structure on $\mathcal{B}(U)$. The group of sections over U of the left dual of $\mathcal{B}$ is given by the $\mathcal{O}_X(U)$-bimodule

$$\mathrm{Hom}_{\mathcal{O}_X}({}_{\mathcal{O}_X}\mathcal{B}, \mathcal{O}_X)(U) := \mathrm{Hom}_{\mathcal{O}_X(U)}({}_{\mathcal{O}_X(U)}\mathcal{B}(U), \mathcal{O}_X(U))$$

where we have dropped the subscript S on the right hand side for the sake of clarity. The right dual is defined similarly and if it is clear which dual we wish to take, we will denote it by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{B}, \mathcal{O}_X)$ or just $\mathcal{B}^\vee$.

In [VdB01], Van den Bergh defines the dual of an $\mathcal{O}_X$-bimodule $\mathcal{B}$ in the case where X is smooth and $\mathcal{B}$ is locally free of finite rank. It agrees with our definition when both are defined.

For the rest of the paper, whenever X is a finite S-scheme, we will confuse the interpretations of $\mathcal{O}_S$-central $\mathcal{O}_X$-bimodules, as sheaves on $X \times_S X$ and as sheaves on S. We will accordingly drop the subscript S for the most part.

3 Groupoids and Coarse Moduli Schemes

A *semigroupoid scheme* consists of schemes X, Y and morphisms

$$Y \times_X Y \xrightarrow{c} Y \underset{t}{\overset{s}{\rightrightarrows}} X\,, \qquad Y \xleftarrow{e} X$$

satisfying the following equalities: $c \circ (c \times \mathrm{id}\,) = c \circ (\mathrm{id}\, \times c)$ as maps from $Y \times_X Y \times_X Y \longrightarrow Y$, $se = \mathrm{id}_X = te$ and $c \circ (e \times \mathrm{id}_Y) = \mathrm{id}_Y = c \circ (\mathrm{id}_Y \times e)$. We call c the *composition* and $e : X \longrightarrow Y$ the *unit*. We will often need the projection maps $Y \times_X Y \longrightarrow Y$, which we denote by π_1 and π_2. For brevity, we will usually denote the semigroupoid scheme by just $s, t : Y \rightrightarrows X$. The semigroupoid scheme is said to be *finite* or *flat* if the maps s, t are.

Note that a semigroupoid scheme differs from a groupoid scheme (as defined for example in [Del, §1.6]) in that there is no inverse map $i : Y \longrightarrow Y$. For most applications though, we will be interested in groupoid schemes. By [A, Theorem 6.1], every flat groupoid scheme gives rise to an algebraic (i.e. Artin) stack X/Y. We will use this notation even when we are dealing with semigroupoid schemes to denote what should morally be the quotient space of X by Y. The interested reader can find the definition of an algebraic stack in [A] or [LM], but we shall have no need of it. We call the natural quotient morphism $X \longrightarrow X/Y$ an *atlas* for the stack X/Y.

By [KM, Theorem 1.1], any finite flat groupoid scheme $Y \rightrightarrows X$ has a coarse moduli space (see [KM, Definition 1.8] for a definition). We will mainly be interested in the case where X (and hence Y) is a quasi-projective scheme. In this case, the coarse moduli space is not just an algebraic space but is an actual scheme. This result is presumably well-known, but lacking a reference, we include the proof based on the corresponding affine result,

Proposition 3.1 *([KM, Proposition 5.1]) Let $s, t : Y \rightrightarrows X$ be a finite flat groupoid scheme with $Y = Spec\ A$ and $X = Spec\ B$ affine. Let*

$$B^Y = \{x \in B | s^* x = t^* x\}.$$

Suppose further that, via s^* or equivalently t^*, B is a free A-module. Then $S := Spec\ B^Y$ is a coarse moduli scheme for X/Y and the natural map $X \longrightarrow S$ is finite.

To apply this result, we need the existence of affine open sets V in X which are *invariant* in the sense that $t^{-1}V = s^{-1}V$. Recall firstly, that the *orbit* of a point $p \in X$ is given by $t(s^{-1}p)$. The following lemma comes from [KM, Lemma 4.8]. We have included a proof, since it was omitted from [KM].

Lemma 3.2 *Let $s, t : Y \rightrightarrows X$ be a finite flat groupoid scheme. Let $p \in X$, $P = t(s^{-1}p)$ and U an open set containing P. Then $V = X - s(Y - t^{-1}U)$ is an invariant open set with $p \in V \subseteq U$.*

Proof. The set V is open because s is finite. We first show that $V \subseteq U$. Let $q \in V$ and e denote the unit of the groupoid scheme. We know $e(q) \notin Y - t^{-1}U$ since $se(q) = q$. Hence $q = te(q) \in U$ as desired.

To show that $p \in V$ we assume the contrary. Then

$$s^{-1}p \cap (Y - t^{-1}U) \neq \varnothing.$$

Examining the image under t we see that

$$\varnothing \neq t(s^{-1}p \cap (Y - t^{-1}U)) \subseteq P \cap (X - U).$$

Our hypothesis on U guarantees that the last set is empty giving the desired contradiction.

It remains now only to show that V is invariant. We wish to show that $s^{-1}V \subseteq t^{-1}V$. The reverse inclusion will follow on applying the inverse of the groupoid scheme since this swaps s and t. We will argue by contradiction and assume that for some $y \in s^{-1}V$, we have $t(y) \notin V$. Hence, there exists $z \in Y - t^{-1}U$ such that $s(z) = t(y)$. Let $(z, y) \in Y \times_X Y$ be a point which projects onto z and y and let $x = c(z, y)$ of Y where c is the composition of the groupoid scheme. Now $t(x) = t(z) \notin U$ by our choice of z. Hence $x \in Y - t^{-1}U$. However, $s(y) = s(x) \notin V$ which contradicts our choice of y. We conclude that V is indeed invariant.

In the next result, we will need the following notation. Given a line bundle L on X and a global section f of L, we let X_f denote the nonzero locus of f.

Theorem 3.3 *Let $s, t : Y \rightrightarrows X$ be a finite flat groupoid scheme where X is a quasi-projective scheme. Then X can be covered by invariant affine open sets. Hence, the groupoid scheme has a coarse moduli scheme S such that the natural quotient map $X \longrightarrow S$ is finite.*

Proof. We wish first to show that there exists an affine open cover of X, consisting of invariant sets over which $s_* \mathcal{O}_Y$ is a free sheaf. To this end, let p be a closed point in X and $P = t(s^{-1}p)$ be its orbit. Since X is quasi-projective, there exists an ample invertible sheaf L on X and $f \in \Gamma(X, L)$ such that $P \subset X_f$. By changing L if necessary, we may assume that $s_* \mathcal{O}_Y$ is free over X_f. Hence, by the previous lemma, we need only show that $V := X - s(Y - t^{-1}X_f)$ is affine. Recall from [EGA II, §6.5] that the norm with respect to s of an invertible sheaf M on Y is an invertible sheaf $N(M)$ on X. Furthermore, there is also a norm map on global sections $N : \Gamma(Y, M) \longrightarrow \Gamma(X, N(M))$. It corresponds to the push-forward of divisors. Now, if we consider $t^*f \in \Gamma(Y, t^*L)$ we see that $V = X - s(Y - Y_{t^*f})$ is just $X_{N(t^*f)}$ by [EGA II, Corollaire 6.5.7]. We know t^*L is ample by [EGA II, Corollaire 6.6.3] so [EGA II, Proposition 6.6.1] ensures that $N(t^*L)$ is too. This guarantees that $V = X_{N(t^*f)}$ is affine as was to be shown.

We construct the coarse moduli scheme as follows. Consider a cover of X consisting of invariant affine open sets over which $s_* \mathcal{O}_Y$ is free. Let V be one such set. We obtain an induced groupoid scheme $s, t : W \rightrightarrows V$ by restricting the original groupoid scheme to $W = t^{-1}V = s^{-1}V$. This new groupoid scheme has a coarse moduli scheme S' by Proposition 3.1 with V finite over S'. It is clear from the formula for this coarse moduli scheme that as V varies, we may glue the results together to obtain a global coarse moduli scheme S for the original groupoid scheme $s, t : Y \rightrightarrows X$ with $X \longrightarrow S$ finite.

4 Noncommutative Coordinate Rings

We return now to the general setup of a semigroupoid scheme $s, t : Y \rightrightarrows X$. Recall that $s_* \mathcal{O}_Y$ and $t_* \mathcal{O}_Y$ are $\mathcal{O}_X$-modules which we can view as right and left module structures on $\mathcal{O}_Y$. More precisely, letting $j = (s, t) : Y \longrightarrow X \times X$, we obtain an $\mathcal{O}_X$-bimodule $j_* \mathcal{O}_Y$ which, abusing notation, we will also denote by $\mathcal{O}_Y$. It is well-known that $\mathcal{O}_Y$ is an $\mathcal{O}_X$-bimodule coalgebra in the sense of [VdB96]. In the case where X is affine, this can be found for example in [Del, §1.14], while the general result is a special case of [KR, §2.1]. The comultiplication on $\mathcal{O}_Y$ is given by $c^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ and the counit is given by $e^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$.

We wish to convert this into an $\mathcal{O}_X$-bimodule algebra by dualising. Accordingly, we will need to assume that the $\mathcal{O}_X$-bimodule $\mathcal{O}_Y$ is finite over its centraliser. A fairly general situation where this hypothesis holds is for finite flat groupoid schemes $s, t : Y \rightrightarrows X$ where X is quasi-projective. In this case, Theorem 3.3 shows that a coarse moduli scheme exists for the groupoid scheme.

Proposition 4.1 *Let $s, t : Y \rightrightarrows X$ be a finite flat groupoid scheme where X is quasi-projective. Let S be the coarse moduli scheme. Then $\mathcal{O}_S$ acts centrally on $\mathcal{O}_Y$ so $\mathcal{O}_Y$ can be viewed as a sheaf of $(\mathcal{O}_X)_S$-bimodules on S (see Section 2).*

Proof. From our construction of coarse moduli schemes, we may reduce to the case where all schemes are affine and so we are in the situation of Proposition 3.1. That proposition shows that $\mathcal{O}_S$ consists of the sections f of $\mathcal{O}_X$ with the property that $s^*f = t^*f$. Since these sections centralise $\mathcal{O}_Y$, $\mathcal{O}_S$ acts centrally. The last statement follows since $X \longrightarrow S$ is finite.

As in Section 2, we let $(-)^\vee$ denote the dual functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$. We recall,

Proposition 4.2 *Let M and N be $\mathcal{O}_S$-central $\mathcal{O}_X$-bimodules which are finite over $\mathcal{O}_S$ and suppose that the left $\mathcal{O}_X$-module ${}_{\mathcal{O}_X}N$ is flat. Then there is a canonical isomorphism $N^\vee \otimes_{\mathcal{O}_X} M^\vee \simeq (M \otimes N)^\vee$ given by,*

$$f \otimes g \mapsto (a \otimes b \mapsto g(af(b))).$$

Proof. By the alternate description of $\mathcal{O}_S$-central $\mathcal{O}_X$-bimodules finite over $\mathcal{O}_S$ given in Section 2, we can reduce to the affine case $X = \mathrm{Spec}\ R$ with ${}_RN$ free. Then the map $\mathrm{Hom}_R(N, R) \otimes_R \mathrm{Hom}_R(M, R) \longrightarrow \mathrm{Hom}_R(M \otimes_R N, R)$ defined by $f \otimes g \mapsto (a \otimes b \mapsto g(af(b)))$ is an isomorphism. Indeed, by adjunction we have a natural isomorphism $\mathrm{Hom}_R(M \otimes_R N, R) \simeq \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, R))$. Furthermore, the functors $\mathrm{Hom}_R(N, -)$ and $\mathrm{Hom}_R(N, R) \otimes_R -$ are exact and isomorphic on free R-modules. Hence they are isomorphic on $\mathrm{Hom}_R(M, R)$ as was to be shown.

Let $\mathcal{O}_{X/Y} := \mathcal{O}_Y{}^\vee$. We assume now that our semigroupoid scheme is flat and that the bimodule $\mathcal{O}_Y$ is finite over its centraliser. In particular, $s, t : Y \rightrightarrows X$ is also finite. The proposition implies that

$$\mathcal{O}_Y{}^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_Y{}^\vee \xrightarrow{\sim} (\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y)^\vee$$

is an isomorphism. Using this isomorphism (which we will view henceforth as an identity), the dual $c^{*\vee}$ of c^* defines a multiplication map

$$\mathcal{O}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_{X/Y} \longrightarrow \mathcal{O}_{X/Y}$$

which is associative because c is. Also, $e^{*\vee} : \mathcal{O}_X \longrightarrow \mathcal{O}_{X/Y}$ is a unit. Again, the axioms follow from the corresponding axioms for the semigroupoid scheme. Hence, $\mathcal{O}_{X/Y}$ is an $\mathcal{O}_X$-bimodule algebra. We call it the *noncommutative coordinate ring* associated to the semigroupoid scheme $s, t : Y \rightrightarrows X$.

Definition 4.3 *Let $s, t : Y \rightrightarrows X$ be a flat semigroupoid scheme. Suppose that the bimodule $\mathcal{O}_Y$ is finite over its centraliser. A* quasi-coherent sheaf *on X/Y consists of a sheaf $F \in QCoh\, X$ and a morphism $\alpha : s^*F \longrightarrow t^*F$ satisfying the cocycle condition $c^*\alpha = \pi_1^*\alpha \circ \pi_2^*\alpha$ and the unit condition $e^*\alpha = id_F$. The morphism α is called the* descent datum. *A* morphism *of quasi-coherent sheaves on X/Y consists of a morphism of the underlying sheaves on X which is compatible with descent data. The category of quasi-coherent sheaves is denoted $QCoh\, X/Y$.*

When $Y \rightrightarrows X$ is actually a groupoid scheme, the unit condition is equivalent to the fact that the descent datum α is an isomorphism. Hence, a quasi-coherent sheaf as defined above is precisely a quasi-coherent sheaf on the stack X/Y.

The Barr-Beck theorem shows that $\mathrm{QCoh}\, X/Y$ is equivalent to the category of $\mathcal{O}_Y$-comodules (see for example [Del, Proposition 4.4] or [KR, Theorem 2.1.2]). There is a dual version of this result.

Proposition 4.4 *Let $s, t : Y \rightrightarrows X$ be a flat semigroupoid scheme and suppose that the bimodule $\mathcal{O}_Y$ is finite over its centraliser. Then $QCoh\, X/Y$ is equivalent to $\mathcal{O}_{X/Y} - Mod$, the category of $\mathcal{O}_{X/Y}$-modules.*

Proof. Since the dual version is well documented, we will only sketch the proof here. For $F \in \mathrm{QCoh}\, X$, we may write $s^*F = \mathcal{O}_Y \otimes_{\mathcal{O}_X} F$ and $t^*F = F \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. First note that by adjunction, giving a morphism $s^*F \longrightarrow t^*F$ of $\mathcal{O}_Y$-modules is equivalent to giving a morphism $F \longrightarrow F \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ of right $\mathcal{O}_X$-modules. If α is a descent datum on F then let α' denote the corresponding morphism $F \longrightarrow F \otimes_{\mathcal{O}_X} \mathcal{O}_Y$.

There are natural adjunction maps $\eta : \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y{}^\vee \longrightarrow \mathcal{O}_X$ and $\delta : \mathcal{O}_X \longrightarrow \mathcal{O}_Y{}^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. Given a descent datum α on F we can construct an $\mathcal{O}_{X/Y}$-module structure on F by defining scalar multiplication to be the composite map

$$F \otimes_{\mathcal{O}_X} \mathcal{O}_Y{}^\vee \xrightarrow{\alpha' \otimes 1} F \otimes_{\mathcal{O}_X} \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y{}^\vee \xrightarrow{1 \otimes \eta} F.$$

We leave it to the reader to verify that the cocycle condition on α yields the associativity of the scalar multiplication and that unit conditions correspond. Conversely, if $m : F \otimes_{\mathcal{O}_X} \mathcal{O}_{X/Y} \longrightarrow F$ furnishes F with an $\mathcal{O}_{X/Y}$-module structure, then the right $\mathcal{O}_X$-module map

$$F \xrightarrow{1 \otimes \delta} F \otimes_{\mathcal{O}_X} \mathcal{O}_Y{}^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_Y \xrightarrow{m \otimes 1} F \otimes_{\mathcal{O}_X} \mathcal{O}_Y$$

induces a descent datum on F. We leave it to the reader to check that these constructions are inverse to each other and that morphisms between objects correspond too.

5 Examples

In this section, we see how skew group rings and path algebras can be realised as noncommutative coordinate rings of semigroupoid schemes. We also construct other examples using a computational tool called a d-matrix.

Let G be a finite group acting on an affine scheme $X = \text{Spec } R$ via $\alpha : G \times X \longrightarrow X$. Recall that the *skew group ring* $G * R$ is a free right R-module with basis $\{\overline{\sigma}\}_{\sigma \in G}$ endowed with a multiplication defined by $\overline{\sigma}\,\overline{\tau} = \overline{\sigma\tau}$ and $r\overline{\sigma} = \overline{\sigma}\sigma(r)$ for $\sigma, \tau \in G$ and $r \in R$.

Associated to the group action is a groupoid scheme $s, t : Y \rightrightarrows X$ with $Y = G \times X$. It is defined as follows. Letting $\pi : G \times X \longrightarrow X$ denote projection, we consider the following isomorphism

$$G \times G \times X \xrightarrow{\sim} (G \times X) \times_{\alpha, X, \pi} (G \times X) : (g', g, x) \mapsto ((g, x), (g', gx))$$

which we will view as an identity. The associated groupoid scheme is

$$G \times G \times X \xrightarrow{c} G \times X \overset{\alpha}{\underset{\pi}{\rightrightarrows}} X$$

where c is given by $(g', g, x) \mapsto (g'g, x)$ and the inverse is given by $i : (g, x) \mapsto (g^{-1}, gx)$. This groupoid scheme is finite and flat.

Definition 5.1 *This groupoid scheme is called the* groupoid scheme induced from α.

In this case, we shall simply write X/G for the associated stack.

Proposition 5.2 *Let G be a finite group acting on an affine scheme $X = \text{Spec } R$ via $\alpha : Y = G \times X \longrightarrow X$. Then $\mathcal{O}_{X/Y} \simeq G * R$.*

Proof. By convention, we shall assume that π^* gives the left R-module structure on $\mathcal{O}_Y$ and α^* the right. Then $\mathcal{O}_Y = \oplus_{\sigma \in G} R1_\sigma$ where 1_σ denotes here the characteristic function on $\sigma \times X \subset Y$. We let $\{\sigma^\vee\}$ be a dual basis to $\{1_\sigma\}$ so that $\mathcal{O}_{X/Y} = {\mathcal{O}_Y}^\vee = \oplus_{\sigma \in G} \sigma^\vee R$. To compute the left R-module structure on $\mathcal{O}_{X/Y}$, note that $r\sigma^\vee$ is the composite map

$$\mathcal{O}_Y \xrightarrow{\alpha^* r} \mathcal{O}_Y \xrightarrow{\sigma^\vee} R.$$

This is just $\sigma^\vee \sigma(r)$. To compute the product, we consider basis elements $\sigma^\vee, \tau^\vee \in \mathcal{O}_{X/Y}$. We view elements of ${\mathcal{O}_Y}^\vee \otimes_{\mathcal{O}_X} {\mathcal{O}_Y}^\vee$ as functions on $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \mathcal{O}_{G \times G \times X}$ which are determined by their values on the R-basis elements $1_g \otimes 1_{g'} =: 1_{(g', g)}$, $g, g' \in G$. Under this description, Proposition 4.2 shows that $\sigma^\vee \otimes \tau^\vee$ is the function defined by

$$1_{(g', g)} = 1_g \otimes 1_{g'} \mapsto \tau^\vee(1_g \sigma^\vee(1_{g'})) = \delta_{\tau g} \delta_{\sigma g'}$$

where δ_{ij} is the Kronecker delta. Hence,

$$\sigma^\vee \tau^\vee : 1_h \mapsto \sum_{g'g = h} 1_{(g', g)} \mapsto \sum_{g'g = h} \delta_{\tau g} \delta_{\sigma g'} = \begin{cases} 1 & \text{if } h = \sigma\tau \\ 0 & \text{if } h \neq \sigma\tau \end{cases}$$

Thus $\sigma^\vee \tau^\vee = (\sigma\tau)^\vee$ and $\mathcal{O}_{X/Y}$ is indeed the skew group algebra.

The removal of the inverse hypothesis allows the study of certain path algebras. Let Q be a finite quiver with no loops. Denote the set of vertices by X and the set of paths by Y. Recall that *the path algebra* kQ, is a vector space over k with basis the elements of Y and multiplication defined by the composition of paths.

The paths of Q form naturally a semigroupoid (in the category of sets) $s, t : Y \rightrightarrows X$. If we endow X and Y with scheme structures by declaring all points to be Spec k, then $s, t : Y \rightrightarrows X$ is a semigroupoid scheme. Note that the semigroupoid scheme is finite flat. Also, the $\mathcal{O}_{\mathrm{X}}$-bimodule $\mathcal{O}_{\mathrm{Y}}$ is finite over its centraliser since it is finite over k. We have

Proposition 5.3 *If $s, t : Y \rightrightarrows X$ is the groupoid scheme associated to a finite quiver Q with no loops then $\mathcal{O}_{\mathrm{X/Y}} \simeq kQ$.*

Proof. Let R denote the coordinate ring of X so $R = \prod_{x \in X} k_x$ where k_x denotes the copy of k sitting over the point x. Then $\mathcal{O}_{\mathrm{Y}} = \oplus_{y \in Y} k e_y$ where the right action on the R-bimodule $k e_y$ is given by projecting R onto $k_{s(y)}$ and the left action by projecting onto $k_{t(y)}$. Let $\{f_y\}$ denote a dual basis to $\{e_y\}$ so that $\mathcal{O}_{\mathrm{X/Y}} = \oplus_{y \in Y} k f_y$. To multiply two basis elements f_y, f_z we view them as functions on $\mathcal{O}_{\mathrm{Y}}$ and compute their values on the basis elements e_w. In the following, we abbreviate $c(u, v)$ to uv.

$$f_y f_z : e_w \overset{c^*}{\mapsto} \sum_{uv=w} e_u \otimes e_v \overset{f_y \otimes f_z}{\mapsto} \sum_{uv=w} f_z(e_u f_y(e_v)) = \sum_{uv=w} \delta_{zu}\delta_{yv} = \begin{cases} 1 & \text{if } w = zy \\ 0 & \text{if } w \neq zy \end{cases}$$

Thus $f_y f_z = f_{zy}$ so $\mathcal{O}_{\mathrm{X/Y}}$ is the path algebra.

For more examples, we recall the representation of bimodules by d-matrices as introduced in [Pat]. Suppose that X is affine with coordinate ring R. Let V be an R-bimodule which is free of rank n on both sides. Right multiplication on V induces a k-algebra map $\rho : R \longrightarrow \mathrm{End}({}_R V)$. Identifying $\mathrm{End}\, V$ with $n \times n$-matrices over R, we see that ρ corresponds naturally to an $n \times n$-matrix M_V^r with entries in $\mathrm{End}_k(R, R)$. We call this the *right d-matrix* of V which acts on the right on V. We define left d-matrices similarly and denote them by M_V^l. Recall from [Pat] that the tensor product of bimodules corresponds to taking the Kronecker composition of d-matrices. Also, the d-matrices of dual bimodules are related via $M_{V^\vee}^l = M_V^r$. Finally, a morphism of bimodules $V \longrightarrow W$ corresponds to an intertwining matrix A, i.e. $M_V^r A = A M_W^r$.

We will use the following proposition, the proof of which is left to the reader.

Proposition 5.4 *Let Y be a closed subscheme of $X \times X$. If there is a groupoid scheme of the form $\pi_1, \pi_2 : Y \rightrightarrows X$ then it is the unique groupoid scheme which makes the following diagram commute,*

$$\begin{array}{ccccc} Y \times_X Y & \xrightarrow{\quad c \quad} & Y & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & X \\ \downarrow & & \downarrow{\scriptstyle (\pi_1, \pi_2)} & & \| \\ X \times X \times X & \xrightarrow{\pi_{13}} & X \times X & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & X \end{array}$$

Such a groupoid scheme exists if and only if the following three conditions hold:

i. The diagonal $\Delta = \{(x, x) \mid x \in X\}$ is contained in Y.

ii. If $\tau \in Aut\, X \times X$ is the flip then $\tau(Y) = Y$.

iii. $\pi_{12}^{-1}(Y) \cap \pi_{23}^{-1}(Y) \subseteq \pi_{13}^{-1}(Y)$.

Note 5.5 *The right hand square in the diagram commutes if the two squares obtained by either deleting both the top arrows π_1 or both the bottom arrows π_2, both commute.*

Example 5.6 *Coarse moduli scheme for $\mathbb{A}^1/(\mathbb{Z}/2)$.*

We consider the group action consisting of $G = \mathbb{Z}/2$ acting on $X = \mathbb{A}^1$ by ± 1. This induces a groupoid scheme as in Definition 5.1. We let Y be the image of $G \times X \xrightarrow{(\pi,\alpha)} X \times X$, that is, the set of all points of the form $(u, \pm u)$. The proposition gives a unique groupoid scheme of the form

$$Y \times_X Y \xrightarrow{c} Y \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X .$$

We describe the groupoid scheme maps explicitly so that we may compute the d-matrices. We choose coordinates so that $\mathcal{O}_{\mathrm{X}} = k[x]$, $\mathcal{O}_{\mathrm{Y}} = k[y_1, y_2]/(y_1^2 - y_2^2)$ and $\mathcal{O}_{Y\times Y} = k[z_1, z_2, z_3]/(z_1^2 - z_2^2, z_2^2 - z_3^2)$. The maps are given by $\pi_1^* : x \mapsto y_1$, $\pi_2^* : x \mapsto y_2$ and $c^* : y_1 \mapsto z_1, y_2 \mapsto z_3$. We choose $e_1 = 1, e_2 = y_1 + y_2$ as our simultaneous left and right bases for $V := \mathcal{O}_{\mathrm{Y}}$ as an $\mathcal{O}_{\mathrm{X}}$-bimodule. Now,

$$e_1 x = 1.\pi_1^*(x) = y_1 = (y_1 + y_2) - y_2 = -y_2 e_1 + e_2 = -\pi_2^*(x) e_1 + e_2 = -x e_1 + e_2.$$

Similarly, one shows $e_2 x = x e_2$. This gives

$$M_V^r(x) = \begin{pmatrix} -x & 1 \\ 0 & x \end{pmatrix}$$

Extending to an algebra homomorphism gives the d-matrix

$$M_V^r = \begin{pmatrix} \alpha & \delta \\ 0 & \mathrm{id} \end{pmatrix}$$

where α is the automorphism of $k[x]$ which maps $x \mapsto -x$ and δ is the α-derivation (i.e. $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$) which maps $x \mapsto 1$. One can check that α and δ commute. Also, $\delta(x^i) = x^{i-1}$ if i is odd and is 0 otherwise. The Kronecker composition gives,

$$M_{V\otimes V}^r = \begin{pmatrix} \alpha^2 & \alpha\delta & \delta\alpha & \delta^2 \\ 0 & \alpha & 0 & \delta \\ 0 & 0 & \alpha & \delta \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix}$$

We wish now to compute the intertwining matrix which gives the map $c^* : V \longrightarrow V \otimes V$. The basis for $V \otimes V$ used to compute the d-matrix is

$$e_1 \otimes e_1 = 1 \ , \ e_1 \otimes e_2 = z_2 + z_3 \ , \ e_2 \otimes e_1 = z_1 + z_2 \ , \ e_2 \otimes e_2 = (z_1 + z_2)(z_2 + z_3).$$

The action of c^* on basis elements is

$$c^*(e_1) = c^*(1) = 1 = e_1 \otimes e_1$$

$$c^*(e_2) = c^*(y_1 + y_2) = z_1 + z_3 = 2z_1 + (z_2 + z_3) - (z_1 + z_2) = 2x e_1 \otimes e_1 + e_1 \otimes e_2 - e_2 \otimes e_1.$$

Thus the intertwining matrix is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2x & 1 & -1 & 0 \end{pmatrix}$$

We let $\{a, b\}$ be a dual basis to $\{e_1, e_2\}$ so that a, b, x generate $\mathcal{O}_{\mathrm{X/Y}}$ as a k-algebra. Since $M_{V^\vee}^l = M_V^r$ (except that M^l now acts on the left), we have the relations

$$xa = -ax \ , \ xb = a + bx.$$

The multiplication map $V^\vee \longrightarrow V^\vee \otimes V^\vee$ is also given by A but now as a matrix acting on the left. Thus multiplying basis elements a, b is given by the columns of A and using Proposition 4.2. This gives the other relations

$$a^2 = a + 2bx \ , \ ba = b = -ab \ , \ b^2 = 0.$$

To compute the identity, note that the groupoid identity $e : X \longrightarrow Y$ is just the diagonal map. Now

$$e^*(e_1) = 1 \ , \ e^*(e_2) = 2x.$$

This gives the identity in $\mathcal{O}_{X/Y}$

$$1 = a + 2bx.$$

Eliminating a gives the algebra

$$\mathcal{O}_{X/Y} = k\langle b, x\rangle/(b^2, xb + bx - 1).$$

Note that this defines a free right $k[x]$-module of rank 2, so there are no other relations. $\mathcal{O}_{X/Y}$ is just the matrix algebra $k[x^2]^{2\times 2}$. In fact, the map $\phi : \mathcal{O}_{X/Y} \longrightarrow k[x^2]^{2\times 2}$ defined by

$$b \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \ x \mapsto \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix}$$

is an isomorphism.

This result is what one expects for the following reason. The stack X/Y in this case is actually the scheme $\mathbb{A}^1$. The easiest way to see this is to consider the double cover $X = \mathbb{A}^1 \longrightarrow X' := \mathbb{A}^1 : x \mapsto x^2$ so X' is naturally Spec $k[x^2]$. The groupoid scheme $s, t : Y \rightrightarrows X$ in this case is just the groupoid scheme $s, t : X \times_{X'} X \rightrightarrows X$ where s and t are just the projection maps. The corresponding stack is just X' so the category of quasi-coherent sheaves on X' is naturally equivalent to $\mathcal{O}_{X/Y}$-Mod. In other words, $\mathcal{O}_{X/Y}$ is Morita equivalent to $k[x^2]$. In Proposition 7.5, we will prove a general result along these lines. The computation has been included here because we will have use of it in the next example, and to show how explicit computations can easily be made using d-matrices.

We recall

Definition 5.7 *A* morphism of semigroupoid schemes $s, t : Y' \rightrightarrows X'$ *and* $s, t : Y \rightrightarrows X$ *is a commutative diagram of the form*

$$\begin{array}{ccccc} Y' \times_{X'} Y' & \xrightarrow{c'} & Y' & \overset{s}{\underset{t}{\rightrightarrows}} & X' \\ \downarrow & & \downarrow & & \downarrow \\ Y \times_X Y & \xrightarrow{c} & Y & \overset{s}{\underset{t}{\rightrightarrows}} & X \end{array}$$

We remind the reader that commutativity of this diagram is as per Note 5.5.

Proposition 5.8 *Suppose there is a morphism of finite flat semigroupoid schemes of the form*

$$\begin{array}{ccc} Y' & \overset{s}{\underset{t}{\rightrightarrows}} & X \\ {\scriptstyle f}\downarrow & & \| \\ Y & \overset{s}{\underset{t}{\rightrightarrows}} & X \end{array}$$

Suppose also that the $\mathcal{O}_X$*-bimodule* $\mathcal{O}_Y$ *is finite over its centraliser. Then the map* $f^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_{Y'}$ *induces a morphism* $f_* := f^{*\vee} : \mathcal{O}_{X/Y'} \longrightarrow \mathcal{O}_{X/Y}$ *of* $\mathcal{O}_X$*-bimodule algebras.*

Proof. Let S be a scheme such that X is a finite S-scheme and $\mathcal{O}_S$ acts centrally on $\mathcal{O}_Y$. Commutativity of the diagrams ensures that $\mathcal{O}_S$ also acts centrally on $\mathcal{O}_{Y'}$. Hence by our alternate description of bimodules in Section 2, we may work locally on S and assume all schemes are affine. We consider first the following commutative diagram

$$\begin{array}{ccccc} Y' \times_{X'} Y' & \xrightarrow{c'} & Y' & \xleftarrow{e'} & X \\ \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} & & \| \\ Y \times_X Y & \xrightarrow{c} & Y & \xleftarrow{e} & X \end{array}$$

The morphisms above can be considered morphisms of X-schemes in two ways, namely, via the source maps s or via the target maps t. This follows from the commutativity of the appropriate diagrams. From this, we obtain a commutative diagram of $\mathcal{O}_X$-bimodules,

$$\begin{array}{ccccc} \mathcal{O}_{Y'} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y'} & \xleftarrow{c'^*} & \mathcal{O}_{Y'} & \xrightarrow{e'^*} & \mathcal{O}_X \\ \uparrow & & \uparrow & & \| \\ \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y & \xleftarrow{c^*} & \mathcal{O}_{Y'} & \xrightarrow{e^*} & \mathcal{O}_X \end{array}$$

Taking the left $\mathcal{O}_X$-dual of this diagram shows that f_* respects the multiplication and the unit. Hence, f_* does indeed yield a morphism of $\mathcal{O}_X$-bimodule algebras.

Example 5.9 *Morphism of stacks* $\mathbb{A}^1/(\mathbb{Z}/2) \longrightarrow \mathbb{A}^1$.

Let $X = \mathbb{A}^1$ and Y be as in Example 5.6. We consider the morphism of groupoid schemes

$$\begin{array}{ccc} \mathbb{Z}/2 \times \mathbb{A}^1 & \rightrightarrows & \mathbb{A}^1 \\ \downarrow{\scriptstyle f} & & \| \\ Y & \rightrightarrows & \mathbb{A}^1 \end{array}$$

where the top groupoid scheme is induced from the action of $\mathbb{Z}/2$ on $\mathbb{A}^1$ by ± 1. We write Y' for $\mathbb{Z}/2 \times \mathbb{A}^1$.

In this case, $\mathcal{O}_{X/Y}$ and $\mathcal{O}_{X/Y'}$ are both bimodules over $\mathcal{O}_X \simeq k[x]$. The morphism $\mathcal{O}_{X/Y'} \longrightarrow \mathcal{O}_{X/Y}$ can thus be computed via d-matrices. We use the notation of Proposition 5.2 and Example 5.6. Let $1, \sigma$ denote the elements of $\mathbb{Z}/2$. The map $\mathcal{O}_Y \longrightarrow \mathcal{O}_{Y'}$ is given by

$$e_1 = 1 \mapsto 1_1 + 1_\sigma \ , \ e_2 = y_1 + y_2 \mapsto 2x1_1.$$

The corresponding d-matrix is

$$\begin{pmatrix} 1 & 1 \\ 2x & 0 \end{pmatrix}$$

Reading off the columns we find

$$f_* : \mathcal{O}_{X/Y'} \longrightarrow \mathcal{O}_{X/Y} : 1^\vee \mapsto a + 2bx = 1 \ , \ \sigma^\vee \mapsto a = 1 - 2bx.$$

Identifying $\mathcal{O}_{X/Y}$ with $k[x^2]^{2\times 2}$ as in Example 5.6, we find that $\mathcal{O}_{X/Y'}$ is the hereditary order in $k[x^2]^{2\times 2}$ generated by $1 - 2bx = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $x = \left(\begin{smallmatrix} 0 & 1 \\ x^2 & 0 \end{smallmatrix}\right)$, that is

$$\mathcal{O}_{X/Y'} = \begin{pmatrix} k[x^2] & k[x^2] \\ x^2 k[x^2] & k[x^2] \end{pmatrix}$$

6 Localisation and Tensor Products

Proposition 4.4 ensures that for every morphism of finite flat semigroupoid schemes, there is a corresponding morphism of the $\mathcal{O}_{X/Y}$-module categories. Unfortunately, there is not necessarily a morphism of the bimodule algebras in general. We have however, seen one example of this type of functoriality in Proposition 5.8. The following proposition provides another example.

Proposition 6.1 *Suppose there is a morphism of finite flat groupoid schemes*

$$\begin{array}{ccccc} Y' \times_{X'} Y' & \xrightarrow{c'} & Y' & \underset{t'}{\overset{s'}{\rightrightarrows}} & X' \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y \times_X Y & \xrightarrow{c} & Y & \underset{t}{\overset{s}{\rightrightarrows}} & X \end{array}$$

where all schemes are quasi-projective and f is affine. Suppose further that the subdiagrams

$$\begin{array}{ccc} Y' & \xrightarrow{s'} & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{s} & X \end{array} \qquad\qquad \begin{array}{ccc} Y' & \xrightarrow{t'} & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{t} & X \end{array}$$

are cartesian squares. Let the induced map on coarse moduli schemes be $h : S' \longrightarrow S$. If we consider $\mathcal{O}_{X/Y}$ and $\mathcal{O}_{X'/Y'}$ as sheaves on S and S' respectively, then the morphism $f^* : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_{X'}$ induces a morphism $\mathcal{O}_{X/Y} \longrightarrow h_*\, \mathcal{O}_{X'/Y'}$ of sheaves of algebras on S.

Proof. If $\pi : X \longrightarrow S$ and $\pi' : X' \longrightarrow S'$ denote the natural maps, then for any affine open set $U \subset S$ Chevalley's theorem [EGA II, Théorème 6.7.1] implies that $h^{-1}(U) = \pi' f^{-1}\pi^{-1}(U)$ is also affine so h is an affine morphism. This allows us to restrict to the case where all schemes are affine.

Now our hypotheses ensure that $\mathcal{O}_{Y'} = \mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. Hence we obtain the following composite morphism

$$\mathcal{O}_{X/Y} = \mathrm{Hom}_{\mathcal{O}_X}({}_{\mathcal{O}_X}\mathcal{O}_Y, \mathcal{O}_X) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{O}_X}({}_{\mathcal{O}_X}\mathcal{O}_Y, f_*\,\mathcal{O}_{X'}) = \mathrm{Hom}_{\mathcal{O}_{X'}}({}_{\mathcal{O}_{X'}}\mathcal{O}_{Y'}, \mathcal{O}_{X'}) = h_*\mathcal{O}_{X'/Y'}.$$

That this actually is a morphism of algebras follows from commutativity of the left hand square in the morphism of groupoid schemes.

We now specialise to the case we are most interested in, for which we need to recall the definition of a free group action. Let $\alpha, \pi : G \times W \rightrightarrows W$ be the groupoid scheme induced from the action of a finite group G on a scheme W as in Definition 5.1. The action is said to be *free* if $(\alpha, \pi) : G \times W \longrightarrow W \times W$ is a closed immersion. Assume from now on that X is an irreducible quasi-projective variety. We will also need to assume that $s, t : Y \rightrightarrows X$ is a finite flat groupoid scheme. Let $K(X)$ and $K(Y)$ be the function fields of X and Y respectively and let X_η and Y_η denote their spectra. Localising $Y \rightrightarrows X$ at the generic point of X yields a groupoid scheme $Y_\eta \rightrightarrows X_\eta$. If we assume further that $K(Y)$ is a separable extension of $K(X)$ (by either s^* or t^*), then this groupoid scheme is induced from a group action and so has the form $G \times X_\eta \rightrightarrows X_\eta$ for some finite group G. Our final assumption will be that this group action is free which here, just means that G acts faithfully on $K(X)$. We say in this case that the groupoid action is *generically free*. Under this assumption, $\mathcal{O}_{X_\eta/Y_\eta} \simeq G * K(X)$ is the trivial cross-product algebra and so by [Jac, §8.4] is isomorphic to the full matrix algebra $K^{n\times n}$ where $K = K(X)^G$ and $n = |G|$. Note that in characteristic zero, generic freeness of the groupoid action is equivalent to a condition on the stack, namely, it means that X/Y is generically a scheme. The above proposition gives a morphism $\mathcal{O}_{X/Y} \longrightarrow \mathcal{O}_{X_\eta/Y_\eta}$ of sheaves of algebras on S and in fact, we have

Proposition 6.2 *Let X be an irreducible quasi-projective variety, $s, t : Y \rightrightarrows X$ a finite flat groupoid scheme and S its coarse moduli scheme. Suppose that the groupoid action is generically free. Then $\mathcal{O}_{X/Y}$ is an $\mathcal{O}_S$-order in $K(S)^{n\times n}$ where n is the degree of s.*

Remark: For a groupoid scheme, the degrees of s and t are the same since the inverse of the groupoid scheme swaps s and t.

From the category equivalence of Proposition 4.4, one would expect the fibre product construction for stacks to correspond to the tensor product construction for bimodule algebras. This is indeed the case. We first briefly review these two concepts.

We consider two groupoid schemes,

$$Y_i \times_{X_i} Y_i \xrightarrow{c_i} Y_i \underset{t_i}{\overset{s_i}{\rightrightarrows}} X_i$$

where $i = 1$ or 2. Suppose furthermore, that both are groupoid schemes over a scheme S in the sense that there exist morphisms $u_i : X_i \longrightarrow S$ such that $u_i s_i = u_i t_i$. Then if we let $Y = Y_1 \times_S Y_2$ and $X = X_1 \times_S X_2$, we obtain a groupoid scheme

$$Y \times_X Y \xrightarrow{c} Y \underset{t}{\overset{s}{\rightrightarrows}} X$$

by defining all structure morphisms coordinatewise. For example, after identifying $Y \times_S Y$ with $(Y_1 \times_{X_1} Y_1) \times_S (Y_2 \times_{X_2} Y_2)$, one can define the composition map by $c = c_1 \times c_2$. The stack associated to this groupoid scheme is the fibre product stack $X_1/Y_1 \times_S X_2/Y_2$ (see [LM, §2.2.2]).

We now look at tensor products of bimodule algebras. Let X_1 and X_2 be schemes of finite type over S and let $X = X_1 \times_S X_2$. Let $\mathcal{A}$ be an $\mathcal{O}_S$-central $\mathcal{O}_{X_1}$- bimodule and $\mathcal{B}$ be an $\mathcal{O}_S$-central $\mathcal{O}_{X_2}$-bimodule. We consider these as sheaves on $X_i \times_S X_i$ for $i = 1, 2$. Let π_{ij} denote the projection of $X_1 \times_S X_2 \times_S X_1 \times_S X_2$ onto its (i, j)-th factors. We can define the tensor product of $\mathcal{A}$ and $\mathcal{B}$ to be the $\mathcal{O}_{X\times_S X}$-module $\pi_{13}^* \mathcal{A} \otimes_{\mathcal{O}_{X\times_S X}} \pi_{24}^* \mathcal{B}$ which is easily verified to be an $\mathcal{O}_X$-bimodule. We will denote it by $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{B}$. Unravelling definitions, one finds that

$$(\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{B}) \otimes_{\mathcal{O}_{X\times_S X}} (\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{B}) \simeq (\mathcal{A} \otimes_{\mathcal{O}_{X_1}} \mathcal{A}) \otimes_{\mathcal{O}_S} (\mathcal{B} \otimes_{\mathcal{O}_{X_2}} \mathcal{B}).$$

Hence, if $\mathcal{A}$ and $\mathcal{B}$ are bimodule algebras, we see that $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{B}$ inherits a natural bimodule algebra structure. When X_1 and X_2 are finite over S, then $\mathcal{A}$ and $\mathcal{B}$ can be viewed as sheaves of $\mathcal{O}_S$-algebras and the tensor product is the usual tensor product of sheaves of algebras. This will be the case we are most interested in.

The two concepts are related by,

Proposition 6.3 *Let $Y_i \rightrightarrows X_i$ for $i = 1, 2$ be two flat groupoid schemes over a scheme S and $Y \rightrightarrows X$ denote the fibre product groupoid as above. Suppose that Y_i is finite over S. Then the associated $\mathcal{O}_X$-bimodule algebra $\mathcal{O}_{X/Y}$ is isomorphic to $\mathcal{O}_{X_1/Y_1} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_2/Y_2}$.*

Proof. We may work locally on S in which case all schemes become affine.

Consider the natural morphism of $\mathcal{O}_X$-bimodule algebras

$$\mathrm{Hom}_{\mathcal{O}_{X_1}}(\mathcal{O}_{Y_1}, \mathcal{O}_{X_1}) \otimes_{\mathcal{O}_S} \mathrm{Hom}_{\mathcal{O}_{X_2}}(\mathcal{O}_{Y_2}, \mathcal{O}_{X_2}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_1} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_2}}(\mathcal{O}_{Y_1} \otimes_{\mathcal{O}_S} \mathcal{O}_{Y_2}, \mathcal{O}_{X_1} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_2})$$

This is an isomorphism in our case since $Y_i \longrightarrow X_i$ is finite flat. Hence $\mathcal{O}_{X/Y} \simeq \mathcal{O}_{X_1/Y_1} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_2/Y_2}$ at least as bimodules. Let c_i and e_i denote the composition and unit maps of the groupoid scheme $Y_i \rightrightarrows X_i$ and c, e denote the composition and unit of the fibre product. Then, as noted above, $c^* = (c_1 \times_S c_2)^* = c_1^* \otimes_{\mathcal{O}_S} c_2^*$ and $e^* = (e_1 \times_S e_2)^* = e_1^* \otimes_{\mathcal{O}_S} e_2^*$. Dualising these shows that our isomorphism is one of $\mathcal{O}_X$-bimodule algebras.

7 Hereditary Orders On Curves

We will assume from now on, that our base field is algebraically closed of characteristic zero. For brevity, we shall call an irreducible one dimensional Deligne-Mumford stack of finite type over k a Deligne-Mumford curve. Also, when we speak of smooth curves, we shall assume implicitly that they are irreducible.

In this section, we show how the noncommutative coordinate ring construction yields a correspondence between Morita equivalence classes of hereditary orders on smooth curves and smooth Deligne-Mumford curves which are schemes generically.

We refer the reader to [KM; Section 2] for complete definitions of pullbacks and restrictions of groupoid schemes. We content ourselves here with a brief review of these concepts. Let $s, t : Y \rightrightarrows X$ be a finite flat groupoid scheme over a scheme S. Given any morphism $S' \longrightarrow S$, one can construct the *pullback* of $s, t : Y \rightrightarrows X$ along $S' \longrightarrow S$ which is a groupoid scheme of the form $Y \times_S S' \rightrightarrows X \times_S S'$ where all the defining maps of the groupoid scheme are given by base change. This will also be a finite flat groupoid scheme. Given any flat morphism $f : X' \longrightarrow X$, one can construct the *restriction* of $s, t : Y \rightrightarrows X$ to X' which is a groupoid scheme of the form $X' \times_{X,t} Y \times_{s,X} X' \rightrightarrows X'$. All the defining maps of the groupoid scheme are obtained from the corresponding maps for $s, t : Y \rightrightarrows X$ and composing them with projections appropriately. For example, the source and target maps are simply the projection maps while the composition is given by the composite

$$(X' \times_{X,t} Y \times_{s,X} X') \times_{X'} (X' \times_{X,t} Y \times_{s,X} X') = X' \times_{X,t} Y \times_{s,X} X' \times_{X,t} Y \times_{s,X} X'$$

$$\xrightarrow{\mathrm{id} \times f \times \mathrm{id}} X' \times_{X,t} Y \times_{s,X,t} Y \times_{s,X} X' \xrightarrow{\mathrm{id} \times c \times \mathrm{id}} X' \times_{X,t} Y \times_{s,X} X'$$

where c is the composition in the original groupoid scheme. When f is finite, the restriction will also be a finite flat groupoid. There is a criterion given in [KM] for the restriction to give an isomorphic stack. It is stated for sheaves but is also true for stacks. We include a proof seeing as it does not appear in [KM].

Lemma 7.1 *([KM, Lemma 3.1(1)]) Let $s, t : Y \rightrightarrows X$ be a finite flat groupoid scheme and $X' \longrightarrow X$ be a finite flat morphism such that the composite morphism*

$$Y \times_{s,X} X' \xrightarrow{\pi_1} Y \xrightarrow{t} X$$

is surjective. If $Y' \rightrightarrows X'$ denotes the restriction of $s, t : Y \rightrightarrows X$ to X', then X/Y and X'/Y' are isomorphic as algebraic stacks.

Proof. Let $\mathcal{S}$ denote the stack X/Y. It suffices to show that the composite $X' \longrightarrow X \longrightarrow \mathcal{S}$ is surjective and that $Y' \simeq X' \times_{\mathcal{S}} X'$. The second assertion follows from

$$Y' = X' \times_X Y \times_X X' = X' \times_X (X \times_{\mathcal{S}} X) \times_X X' = X' \times_{\mathcal{S}} X'$$

Let $Z := Y \times_X X'$ and let c and i denote the composition and inverse in the groupoid scheme as usual. The first assertion follows from examining the commutative diagram

$$\begin{array}{ccccc} Z \times_{X'} Z & \overset{\pi_2}{\underset{\pi_1}{\rightrightarrows}} & Z & \overset{e\times 1}{\underset{\pi_2}{\rightleftarrows}} & X' \\ \Big\downarrow g & & \Big\downarrow t\circ\pi_1 & & \Big\downarrow \\ Y & \overset{s}{\underset{t}{\rightrightarrows}} & X & \longrightarrow & \mathcal{S} \end{array}$$

where g is defined by $g(y_1, x, y_2, x) = c(y_1, i(y_2))$. Indeed, commutativity of the left hand squares defines a morphism $X' \longrightarrow \mathcal{S}$ while commutativity of the the right hand squares ensures that this is the same as the composite $X' \longrightarrow X \longrightarrow S$. This gives surjectivity of $X' \longrightarrow \mathcal{S}$ since we hypothesised that $Z \longrightarrow X$ was surjective.

Let $k\{u\}$ be the Hensel localisation of $k[u]$ at the prime ideal (u). There is a faithful action of $\mathbb{Z}/n$ on $k\{u\}$ where the generator σ of $\mathbb{Z}/n$ maps $u \mapsto \zeta u$ where ζ is a primitive n-th root of unity. This faithful action is unique up to choice of root of unity. We have the following characterisation of hereditary orders on smooth curves.

Proposition 7.2 *Let $\mathcal{O}$ be an order on a smooth curve and S its centre. Then $\mathcal{O}$ is hereditary if and only if étale locally on S, $\mathcal{O}$ is Morita equivalent to $\mathbb{Z}/n * k\{u\}$ where $\mathbb{Z}/n$ acts faithfully on $k\{u\}$.*

This result is well-known so we shall omit the proof. It follows easily from the arguments in the appendix.

Theorem 7.3 *Let $s, t : Y \rightrightarrows X$ be a finite flat groupoid scheme with X a smooth quasi-projective curve. Suppose that the groupoid action is generically free. Then the coarse moduli scheme S is also a smooth curve and étale locally on S, the stack X/Y is isomorphic to Spec $k\{u\}/(\mathbb{Z}/n)$ where $\mathbb{Z}/n$ acts faithfully on $k\{u\}$. Hence, $\mathcal{O}_{X/Y}$ is an hereditary order on S.*

Proof. The last assertion follows from Propositions 4.4, 5.2 and 7.2 and the other results in the theorem. Also, the coarse moduli scheme of Spec $k\{u\}/(\mathbb{Z}/n)$ is Spec $k\{u^n\}$ so it suffices to show that, étale locally on S, X/Y is Spec $k\{u\}/(\mathbb{Z}/n)$.

Since the construction of coarse moduli schemes commutes with pullback along any affine flat map $S' \longrightarrow S$ by [KM; Lemma 5.3] and our proof of Theorem 3.3, we may assume that S is the spectrum of a Hensel local ring and thus that X is the union of spectra of Hensel discrete valuation rings. Note that the groupoid action is still generically free, so there exists a finite group G acting faithfully on $K(X)$ such that generically, the groupoid scheme is induced from the action of the group G on Spec $K(X)$.

We wish first to reduce to the case where X is the spectrum of a single Hensel discrete valuation ring. To this end, let $\tilde{Y} := G \times X$ which is the normalisation of Y. Since normalisation is functorial, we obtain a morphism of groupoid schemes

$$\begin{array}{ccc} \tilde{Y} & \overset{s}{\underset{t}{\rightrightarrows}} & X \\ \big\downarrow & & \big\| \\ Y & \overset{s}{\underset{t}{\rightrightarrows}} & X \end{array}$$

where the top groupoid scheme is induced from the action of G on X. Now $\mathcal{O}_Y$ embeds in $\mathcal{O}_{Y'}$ so the formula for computing coarse moduli schemes in Proposition 3.1 shows that S is also the coarse moduli scheme for X/G. The only way for S to be connected is if G acts transitively on the components of X. Hence, for every component X' of X, the natural map $t \circ \pi_1 : Y \times_X X' \longrightarrow X$ is surjective since it is surjective generically and finite. By Lemma 7.1, we may restrict to one of the components of X without altering the underlying stack.

Generically, the restricted groupoid scheme is still induced from a group action. However, only a subgroup of the original group G acts. Relabelling, we shall call this subgroup G. Note that this action is still free. Thus G is a finite subgroup of Aut $X_\eta =$ Aut $X = \mu$ the group of roots of unity. This forces G to be cyclic.

We shall now compute $s, t : Y \rightrightarrows X$ explicitly under the new assumption that $X =$ Spec $k\{u\}$. Let x be the closed point of X and consider the two maps

$$j : Y \xrightarrow{(s,t)} X \times X \quad \text{and} \quad \tilde{j} : \tilde{Y} \xrightarrow{(s,t)} X \times X.$$

The morphism of groupoid schemes induces a surjective morphism of stabiliser groups $\tilde{j}^{-1}(x,x) \longrightarrow j^{-1}(x,x)$ (see the beginning of [KM, Section 2] for why these are groups). Now $\tilde{j}^{-1}(x,x) = G$ so $j^{-1}(x,x) = G/K$ for some subgroup K of G. Since $\tilde{j}$ factorises via $\tilde{Y} \longrightarrow Y \longrightarrow \text{im } j \longrightarrow X \times X$, the groupoid scheme $s,t : Y \rightrightarrows X$ is determined completely by $s,t : \tilde{Y} \rightrightarrows X$, and the additional datum of the subgroup K of G.

Conversely, given any subgroup K of G, one has an inverse construction which yields a groupoid scheme $s,t : Y \rightrightarrows X$ with stabiliser group G/K above x and such that there is a morphism of groupoid schemes from $G \times X \rightrightarrows X$ to $s,t : Y \rightrightarrows X$ as before. To construct Y we need some notation. For $g \in G$ let X_g denote a copy of X and, abusing notation, let x denote its closed point. If γ denotes a coset of K in G, let $\gamma_1, \dots, \gamma_r$ denote its elements. We set

$$Y = \coprod_{\gamma \in G/K} (X_{\gamma_1} \coprod_x \cdots \coprod_x X_{\gamma_r})$$

where the subscript x indicates that the direct sum identifies the closed points. We define the map $G \times X \longrightarrow Y$ by identifying $(g, X) \subset G \times X$ with $X_g \subset Y$. This then determines the rest of the groupoid scheme. We will call this the *quotient groupoid scheme of* $s,t : G \times X \rightrightarrows X$ *by* K.

The following result gives another interpretation of quotient groupoid schemes. Below, we let $X//K$ denote the geometric quotient Spec $\mathcal{O}(X)^K$. Note that $X//K$ is naturally equipped with a G/K-action.

Lemma 7.4 *Let G be a cyclic group and K a subgroup. Suppose G acts faithfully on $k\{u\}$ and let $X =$ Spec $k\{u\}$ and $\overline{X} = X//K$. Also, let $s,t : Y \rightrightarrows X$ be the quotient groupoid scheme of $s,t : G \times X \rightrightarrows X$ by K. Then $s,t : Y \rightrightarrows X$ is the restriction of the groupoid scheme $s,t : G/K \times \overline{X} \rightrightarrows \overline{X}$ via the cyclic quotient map $f : X \longrightarrow \overline{X}$.*

Proof. Let $Y' := X \times_{\overline{X},t} (G/K \times \overline{X}) \times_{s,\overline{X}} X$. The restricted groupoid scheme has the form $s,t : Y' \rightrightarrows X$. It suffices to construct a surjective morphism of groupoid schemes

$$\begin{array}{ccc} G \times X & \overset{s}{\underset{t}{\rightrightarrows}} & X \\ \downarrow{\scriptstyle \Phi} & & \| \\ Y' & \overset{s}{\underset{t}{\rightrightarrows}} & X \end{array}$$

where the bottom groupoid scheme has stabiliser G/K above the closed point of X. Let σ be a generator of G and $\overline{\sigma}$ its image in G/K. We define $\Phi : G \times X \longrightarrow Y' : (\sigma^i, z) \mapsto (z, (\overline{\sigma}^i, f(z)), \sigma^i(z))$. It follows immediately from the formula for Φ that Φ is surjective and commutes with the source and target maps. We check here that it commutes with the composition, leaving the reader to show that it also commutes with the inverse and unit maps. To simplify notation, we will use $\circ$ to denote composition.

$$\Phi((\sigma^i, z) \circ (\sigma^j, \sigma^i z)) = \Phi(\sigma^{i+j}, z) = (z, (\overline{\sigma}^{i+j}, f(z)), \sigma^{i+j} z)$$

while

$$\Phi(\sigma^i, z) \circ \Phi(\sigma^j, \sigma^i z) = (z, (\overline{\sigma}^i, f(z)), \sigma^i z) \circ (\sigma^i z, (\overline{\sigma}^j, f(\sigma^i z)), \sigma^{i+j} z) = (z, (\overline{\sigma}^{i+j}, f(z)), \sigma^{i+j} z)$$

ensuring commutativity with c.

Finally, we consider the stabiliser group. Let $x \in X$ denote the closed point. Then the stabiliser group above x is $x \times_{\overline{X}} (G/K \times \overline{X}) \times_{\overline{X}} x = G/K$ as desired.

This completes the proof of Theorem 7.3 since, étale locally on S, Lemmas 7.1 and 7.4 show that X/Y is isomorphic to $\overline{X}/(G/K)$.

The previous lemma shows that the quotient groupoid scheme has a noncommutative coordinate ring Morita equivalent to $(G/K) * k\{u\}$. The following result gives a more precise description of the noncommutative coordinate ring in this case.

Proposition 7.5 *Let $s,t : Y \rightrightarrows X$ be a finite flat groupoid scheme where X is an affine scheme. Let $f : X' \longrightarrow X$ be a finite flat cover such that $\mathcal{O}_{X'}$ is free over $\mathcal{O}_X$ of rank d. If $s,t : Y' \rightrightarrows X'$ is the restriction of $s,t : Y \rightrightarrows X$ via f, then $\mathcal{O}_{X'/Y'}$ is the full matrix algebra ${\mathcal{O}_{X/Y}}^{d\times d}$.*

Proof. We let $A = \mathcal{O}(X)$, $B = \mathcal{O}(Y)$, $A' = \mathcal{O}(X')$ and $B' = \mathcal{O}(Y')$. Note that by the construction of restrictions, $B' = A' \otimes_A B \otimes_A A'$. We will need to use Proposition 4.2 repeatedly. Let $(-)^\vee$ denote the left module dual $\mathrm{Hom}_A(-, A)$. We have the following identifications by Proposition 4.2,

$$\mathcal{O}_{X'/Y'} = \mathrm{Hom}_{A'}(B', A') = \mathrm{Hom}_A(B \otimes_A A', A') = \mathrm{Hom}_A(B \otimes_A A', A) \otimes_A A' = A'^\vee \otimes_A B^\vee \otimes_A A'.$$

There is a natural multiplicative structure on $A'^\vee \otimes_A B^\vee \otimes_A A'$ which makes it the matrix algebra ${\mathcal{O}_{X/Y}}^{d\times d}$; namely, if $\varepsilon : A' \otimes_A A'^\vee \longrightarrow A$ denotes the evaluation map $a' \otimes \xi \mapsto \xi(a')$ and c^* the comultiplication on $\mathcal{O}_{X/Y}$ then the multiplication is the composite map

$$A'^\vee \otimes_A B^\vee \otimes_A A' \otimes_{A'} A'^\vee \otimes_A B^\vee \otimes_A A' \xrightarrow{1\otimes\varepsilon\otimes 1} A'^\vee \otimes_A B^\vee \otimes_A B^\vee \otimes_A A' \xrightarrow{1\otimes c^{*\vee}\otimes 1} A'^\vee \otimes_A B^\vee \otimes_A A'.$$

We wish to show that this multiplication agrees with the usual multiplication on $\mathcal{O}_{X'/Y'}$. We will use Sweedler's notation for comultiplication in B, $c^*(b) = b_{(1)} \otimes b_{(2)}$ suppressing the summation symbol. Let $\xi_1 \otimes f_1 \otimes a_1', \xi_2 \otimes f_2 \otimes a_2' \in A'^\vee \otimes_A B^\vee \otimes_A A'$ which we consider as elements of $\mathrm{Hom}_{A'}(B', A')$ and let $a_1 \otimes b \otimes a_2 \in A' \otimes_A B \otimes_A A'$. Then the above multiplication gives

$$\begin{aligned}(\xi_1 \otimes f_1 \otimes a_1')(\xi_2 \otimes f_2 \otimes a_2')(a_1 \otimes b \otimes a_2) &= (\xi_1 \otimes f_1\xi_2(a_1')f_2 \otimes a_2')(a_1 \otimes b \otimes a_2)\\ &= a_1(f_1\xi_2(a_1')f_2)(b\xi_1(a_2))a_2'\\ &= a_1(f_1 \otimes \xi_2(a_1')f_2)(b_{(1)} \otimes b_{(2)}\xi_1(a_2))a_2'\\ &= a_1(\xi_2(a_1')f_2)(b_{(1)}f_1(b_{(2)}\xi_1(a_2)))a_2'\\ &= a_1 f_2(b_{(1)}f_1(b_{(2)}\xi_1(a_2))\xi_2(a_1'))a_2'.\end{aligned}$$

On the other hand, the usual multiplication is given by

$$\begin{aligned}(\xi_1 \otimes f_1 \otimes a_1')(\xi_2 \otimes f_2 \otimes a_2')(a_1 \otimes b \otimes a_2) &= (\xi_1 \otimes f_1 \otimes a_1')(\xi_2 \otimes f_2 \otimes a_2')(a_1 \otimes b_{(1)} \otimes 1 \otimes 1 \otimes b_{(2)} \otimes a_2)\\ &= (\xi_2 \otimes f_2 \otimes a_2')(a_1 \otimes b_{(1)} \otimes 1((\xi_1 \otimes f_1 \otimes a_1')(1 \otimes b_{(2)} \otimes a_2)))\\ &= (\xi_2 \otimes f_2 \otimes a_2')(a_1 \otimes b_{(1)} \otimes 1(f_1(b_{(2)}\xi_1(a_2))a_1'))\\ &= a_1 f_2(b_{(1)}\xi_2(f_1(b_{(2)}\xi_1(a_2))a_1'))a_2'\\ &= a_1 f_2(b_{(1)}f_1(b_{(2)}\xi_1(a_2))\xi_2(a_1'))a_2'\end{aligned}$$

so the two do in fact agree.

There is a converse result to Theorem 7.3. To prove it we need to introduce ramification data for hereditary orders on a smooth curve. Let $\mathcal{A}$ be an hereditary order on a smooth curve and C its centre. For each closed point $p \in C$, we say that $\mathcal{A}$ *ramifies* at p with ramification index e_p if the étale localisation $\mathcal{A}_p^h$ is Morita equivalent to the skew group algebra $\mathbb{Z}/e_p * k\{u\}$ and $e_p > 1$. This is not the usual definition, but it is not hard to show that it agrees with the usual one. We call the collection of ramification points and their corresponding ramification indices the *ramification data*. The following is presumably a known result whose proof we shall leave to the appendix.

Theorem 7.6 *Two hereditary orders on smooth curves are Morita equivalent if and only if their centres are isomorphic and their ramification data are the same.*

Theorem 7.7 *Let $\mathcal{A}$ be an hereditary order on a smooth quasi-projective curve. Then there exists a finite flat groupoid scheme $s,t : Y \rightrightarrows X$ such that i) X is a smooth curve ii) the groupoid action is generically free and iii) $\mathcal{O}_{X/Y}$ is Morita equivalent to $\mathcal{A}$. Furthermore, X/Y is a Deligne-Mumford stack.*

Proof. We argue by induction on the number of ramification points of $\mathcal{A}$ using Proposition 6.3. Let C be the central curve, $p \in C$ a closed point, $e > 1$ an integer and $D \subset C$ a finite subset. We start by proving the inductive step which consists essentially of constructing a finite flat groupoid scheme $s,t : Y \rightrightarrows X$ with the following properties:

i. $\mathcal{O}_{X/Y}$ ramifies at a single point p with ramification index e and is unramified elsewhere.

ii. The coarse moduli scheme for X/Y is C.

iii. The morphism $X \longrightarrow C$ is unramified over D.

iv. X is a smooth curve.

v. The groupoid action is generically free.

We first construct X as a certain ramified cyclic cover of C which is totally ramified over p. Let H' be a very ample divisor on C and let $H = ep + eH'$. Since H is also very ample, we may choose a linearly equivalent divisor H'' such that $H'' = p + E$ where E is a sum of distinct points, none of which lie in $\{p\} \cup D$. Let $L = \mathcal{O}_C(-p - H')$ and consider the composite map

$$\phi : L^{\otimes e} \xrightarrow{\sim} \mathcal{O}_C(-H'') \longrightarrow \mathcal{O}_C \, .$$

Consider the sheaf of $\mathcal{O}_C$-algebras $\mathcal{B} := \oplus_{i=0}^{e-1} L^{\otimes i}$ where multiplication is defined using the map ϕ. By our choice of H'', $X := \underline{\mathrm{Spec}}_C\, \mathcal{B}$ is a ramified cyclic cover of C of degree e which is totally ramified at p and unramified over D. The action of a generator of $\mathbb{Z}/e$ comes from multiplication on $L^{\otimes i} \subset \mathcal{B}$ by ζ^i where ζ is a primitive e-th root of unity. Furthermore, X is smooth as can be seen either by explicit calculation or [KoM, Lemma 2.5.1].

We now construct the groupoid scheme $s,t : Y \rightrightarrows X$ piecewise over the two open subsets $U := C \setminus p$ and $V := C \setminus \operatorname{Supp} E$. Since X is totally ramified above p, p is a fixed point of the $\mathbb{Z}/e$-action and so $\mathbb{Z}/e$ restricts to an action on V. Over V, we let $s,t : Y \rightrightarrows X$ be the groupoid scheme $s,t : \mathbb{Z}/e \times V \rightrightarrows V$ induced from the group action. Over U, we let $s,t : Y \rightrightarrows X$ be $s,t : U \times_C U \rightrightarrows U$. On $U \cap V$, the cover is Galois so the two groupoid schemes agree and may be glued together. Note that over U, the stack corresponding to $s,t : Y \rightrightarrows X$ is U so $\mathcal{O}_{X/Y}$ is unramified there. Etale locally above p however, it is the stack of a group action so $\mathcal{O}_{X/Y}$ is isomorphic to the skew group algebra $\mathbb{Z}/e * k\{u\}$ and i) holds. By construction, the coarse moduli scheme is C. Finally, the groupoid action is free on $C - \{p\} - D$ and hence generically free. This completes the inductive step.

Suppose that the ramification points of $\mathcal{A}$ are $p_1, \dots, p_r, p$ and their associated ramification indices are $e_1, \dots, e_r, e$. By induction, there exists a smooth curve X' and a finite flat groupoid scheme $Y' \rightrightarrows X'$ with coarse moduli scheme C and such that $\mathcal{O}_{X'/Y'}$ ramifies precisely at $p_1, \dots, p_r$ with ramification indices $e_1, \dots, e_r$. Let $D \subset C$ be the set of points over which $X' \longrightarrow C$ ramifies. By the inductive step, there exists a finite flat groupoid scheme $Y'' \rightrightarrows X''$ satisfying the properties i)-v) above. Let $s,t : Y \rightrightarrows X$ be the fibre product over C of the groupoid schemes $Y' \rightrightarrows X'$ and $Y'' \rightrightarrows X''$. By construction, X/Y is a Deligne-Mumford stack. Condition v) and our result on fibre products Proposition 6.3, show that the groupoid action is generically free and that $\mathcal{O}_{X/Y}$ has the same centre and ramification data as $\mathcal{A}$. Also, conditions iii) and iv) ensure that X is smooth so $s,t : Y \rightrightarrows X$ satisfies the conditions of the theorem.

We can now prove our desired correspondence.

Corollary 7.8 *Let $\mathcal{S}$ be a smooth Deligne-Mumford curve. Then there exists a finite flat atlas $X \longrightarrow \mathcal{S}$ where X is a smooth curve. Let $Y := X \times_{\mathcal{S}} X$ and $s, t : Y \rightrightarrows X$ be the groupoid scheme representing $\mathcal{S}$. Then the assignment $\mathcal{S} \mapsto \mathcal{O}_{X/Y}$ is a bijective correspondence between smooth Deligne-Mumford curves which are generically schemes and Morita equivalence classes of hereditary orders on smooth curves.*

Proof. There is a version of Zariski's main theorem for stacks (see [LM, Théorème 16.6]) which shows that there exists a one dimensional scheme Z and a finite surjective morphism $Z \longrightarrow \mathcal{S}$. Let X be an irreducible component of the normalisation of Z and $U \longrightarrow \mathcal{S}$ be an étale atlas of $\mathcal{S}$. Then $X \times_{\mathcal{S}} U \longrightarrow U$ is flat since $X \times_{\mathcal{S}} U$ is smooth. Hence $X \longrightarrow \mathcal{S}$ is flat. The morphism is also surjective since it is dominant and closed so X is indeed a finite flat atlas. The corollary now follows from the previous theorem.

8 Appendix: Morita Equivalence Classes of Hereditary Orders on Curves

In this appendix, we prove Theorem 7.6 and so classify the Morita equivalence classes of hereditary orders on a smooth curve. The result is known, but does not appear to be in the literature so we have included it here.

We start by recalling the classification of hereditary orders in the étale local case as one might find for example in [Reiner, Section 39]. Let $\mathcal{O} := k\{u\}$, K be its field of fractions and $\mathfrak{m}$ be its maximal ideal. To describe the hereditary $\mathcal{O}$-orders in $K^{n\times n}$ we need to introduce some of Reiner's notation. Suppose given integers $n_1, n_2, \ldots, n_r$ with sum n and ideals $\{\mathfrak{m}_{ij} \,|1 \leq i,j \leq r\}$ of $\mathcal{O}$. We denote by

$$\begin{pmatrix} \mathfrak{m}_{11} & \mathfrak{m}_{12} & \ldots & \mathfrak{m}_{1r} \\ \mathfrak{m}_{21} & \mathfrak{m}_{22} & \ldots & \mathfrak{m}_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ \mathfrak{m}_{r1} & \mathfrak{m}_{r2} & \ldots & \mathfrak{m}_{rr} \end{pmatrix}^{n_1,\ldots,n_r}$$

the set of all $n \times n$ matrices which have a block decomposition of the form (T_{ij}) where T_{ij} are matrices in $\mathfrak{m}_{ij}^{n_i \times n_j}$. The étale local classification result is

Theorem 8.1 *([Reiner, Theorem 39.14]) Let A be an hereditary $\mathcal{O}$-order in $K^{n\times n}$. Then A is isomorphic to*

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \ldots & \mathcal{O} \\ \mathfrak{m} & \mathcal{O} & \ldots & \mathcal{O} \\ \ldots & \ldots & \ldots & \ldots \\ \mathfrak{m} & \ldots & \mathfrak{m} & \mathcal{O} \end{pmatrix}^{n_1,\ldots,n_r}$$

for some choice of integers $n_1, \ldots, n_r$ which sum to n.

We call this the *canonical matrix form* of A. Also, we define the *ramification index* of the order A in the theorem to be r. This agrees with the definition given in the previous section.

Now let C be a smooth quasi-projective curve and $K(C)$ its field of functions. By Tsen's theorem, the only central simple algebras over $K(C)$ are the matrix algebras. Let A be an hereditary order on C in $K(C)^{n\times n}$. At any closed point $p \in C$, let the subscript p denote étale localisation at p. Since A_p is still an hereditary order, it may be described by the previous theorem. We shall call the ramification index of A_p the ramification index of A at p. An hereditary order can ramify at at most a finite number of points. We may now proceed to prove Theorem 7.6.

Let A and B be two hereditary $\mathcal{O}_C$-orders with the same ramification data. By taking matrix algebras in A or B if necessary, we may assume they both are orders in $K(C)^{n\times n}$.

We start by constructing locally projective A-modules of rank n, where by rank, we mean as an $\mathcal{O}_C$-module. Recall that an A-module is locally projective provided it is torsion-free over $\mathcal{O}_C$ (see for example [Reiner, Corollary 10.7]). Etale locally, the A-projectives are easily described. If A_p is written in canonical matrix form as in Theorem 8.1, then the rank n projective A_p-modules correspond to the r different columns that can occur. Explicitly, they are

$$Q_1 = \begin{pmatrix} \mathcal{O}^{n_1} \\ \mathfrak{m}^{n_2} \\ \vdots \\ \mathfrak{m}^{n_r} \end{pmatrix} \quad , \quad \dots \quad , \quad Q_r = \begin{pmatrix} \mathcal{O}^{n_1} \\ \mathcal{O}^{n_2} \\ \vdots \\ \mathcal{O}^{n_r} \end{pmatrix}$$

Note that this projective is unique whenever p is not a ramification point. For each ramification point $p \in C$, we pick a rank n projective A_p-module L_p.

Lemma 8.2 *There exists a locally projective A-module of rank n which is isomorphic étale locally to L_p for every ramification point $p \in C$.*

Proof. We first embed A in a maximal order, which by triviality of the Brauer group Br C, has the form $\mathcal{E}nd_C\, V$ where V is a rank n vector bundle on C. Note that V is a locally projective A-module. We wish to modify it into our desired locally projective. We consider the local question first. Writing A_p in canonical matrix form, we see that the radical of A_p is just

$$\begin{pmatrix} \mathfrak{m} & \mathcal{O} & \dots & \mathcal{O} \\ \mathfrak{m} & \mathfrak{m} & \dots & \mathcal{O} \\ \dots & \dots & \dots & \dots \\ \mathfrak{m} & \mathfrak{m} & \dots & \mathfrak{m} \end{pmatrix}^{n_1,\dots,n_r}$$

Hence multiplication by rad A_p permutes the r rank n projective A_p-modules cyclicly. Let J_p be the ideal sheaf in A, which is A everywhere except at p, where it is the radical of A at p. Multiplying V by the appropriate multiple of J_p gives a locally projective rank n A-module which is L_p étale locally at p and is V elsewhere. We may repeat this process at all the other ramification points to conjure up our desired locally projective.

Our next goal will be to construct a locally projective A-module L and a locally projective B-module L' such that $\mathcal{E}nd_A\, L = \mathcal{E}nd_B\, L'$. To this end, let B'_p be the canonical matrix form of B_p. We seek n rank n locally projective A-modules $L_1, \dots, L_n$ such that their direct sum L satisfies

$$\mathcal{E}nd_A\, L = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{O}(-D_{21}) & \mathcal{O} & \dots & \mathcal{O} \\ \dots & \dots & \dots & \dots \\ \mathcal{O}(-D_{n1}) & \dots & \mathcal{O}(-D_{n,n-1}) & \mathcal{O} \end{pmatrix}$$

where the D_{ij} are sums of distinct ramification points of B chosen so that étale locally at p, the matrix on the right is exactly B'_p. In other words, the divisor D_{ij} contains $p \in C$ precisely when the $(i,j)-th$ entry of B'_p is $\mathfrak{m}$. Our first goal is to compute the endomorphism ring for arbitrary L_i's so it becomes clear how they may be chosen. By examining the generic point, we see that $\mathcal{H}om_A(L_i, L_j)$ is a coherent subsheaf of the constant sheaf $K(C)$ and so has the form $\mathcal{O}(-D)$ for some divisor D. For any closed point $p \in C$, one can determine the multiplicity of p in D by computing étale locally. As before, let

$Q_1, \dots, Q_r$ denote the r rank n projective A_p-modules so that $Q_1 \subset Q_2 \subset \dots \subset Q_r$. One computes that

$$\mathrm{Hom}_{A_p}(Q_a, Q_b) = \begin{cases} \mathcal{O} & \text{if } a \leq b \\ \mathfrak{m} & \text{if } a > b \end{cases}$$

The choice for the L_i's is now clear. By our assumption on the ramification indices, B_p also has r rank n projectives, which we shall denote by $P_1, \dots, P_r$ where, as before, the subscripts are chosen so that we have naturally $P_1 \subset \dots \subset P_r$. Suppose the i-th column of B'_p is P_{a_p}. We repeat this for all ramification points. Then, thanks to Lemma 8.2, we may choose L_i to be a rank n locally projective which, étale locally at p is Q_{a_p}. This choice guarantees that $\mathcal{E}nd_A\, L$ is the desired algebra. We may repeat this procedure for B to obtain a locally projective B-module L' such that $\mathcal{E}nd_B\, L'$ is also the same algebra.

We need now only show that both L and L' induce Morita equivalences. Let $A' := \mathcal{E}nd_A\, L$. The simple A-modules correspond to the projective A_p-modules where p is any closed point of C. In fact they are the modules of the form $P_a/(\text{rad } A_p)P_a$. Hence, L surjects onto any simple A-module and thus, Zariski locally, L is a projective generator. In other words, Zariski locally, the bimodule ${}_AL_{A'}$ induces a Morita equivalence and furthermore, the inverse equivalence is given by $L^* := \mathcal{H}om_A(L, A)$. It follows that L and L^* induce Morita equivalences globally. A similar argument with B gives the desired Morita equivalence with A.

We wish now to show that if A and B are Morita equivalent hereditary orders on smooth curves, then their centres are isomorphic and their ramification data are the same. The first assertion is contained in the proof of [AZ, Proposition 6.8] which also shows that there is a unique isomorphism of the centres compatible with the Morita equivalence. We shall identify the two central curves which we shall call C. Let p be a closed point of C and R its ring of functions. Now the algebras $A \otimes_{\mathcal{O}_C} R$ and $B \otimes_{\mathcal{O}_C} R$ are also Morita equivalent and so, in particular have the same number of simple modules. By the discussion above, this number is precisely ramification index of A and B at p. This completes the proof of the theorem.

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