The Minimal Model Program for Orders over Surfaces

Daniel Chan and Colin Ingalls
danielch@maths.unsw.edu.au, cingalls@unb.ca

Abstract

We develop the minimal model program for orders over surfaces and so establish a noncommutative generalisation of the existence and uniqueness of minimal algebraic surfaces. We define terminal orders and show that they have unique étale local structures. This shows that they are determined up to Morita equivalence by their centre and algebra of quotients. This reduces our problem to the study of pairs (Z, α) consisting of a surface Z and an element α of the Brauer group $\operatorname{Br} k(Z)$. We then extend the minimal model program for surfaces to such pairs. Combining these results yields a noncommutative version of resolution of singularities and allows us to show that any order has either a unique minimal model up to Morita equivalence or is ruled or del Pezzo.

Contents

1 Introduction

The existence and uniqueness of minimal models of surfaces with non-negative Kodaira dimension is an important result of classical algebraic geometry and it is conjectured by Artin in [A4] that similar results should hold for noncommutative algebras. In this paper we prove such results for orders, certain noncommutative algebras which are finite rank over their centres. We use the ideas of Mori's minimal model program, the modern approach to birational geometry.

An algebraic consequence of the result for surfaces can be stated succinctly as follows. Let k be an algebraically closed field of characteristic zero.

Theorem 1.1 (Enriques, Castelnuovo) *Let K be a field of transcendence degree two over k. Then either there is a unique minimal surface X such that $k(X) \simeq K$, or $K \simeq k(\mathbb{P}^1 \times C)$ where C is a curve.*

As an analogue of this theorem we obtain the following, which is a corollary of the results of this paper. We define a terminal order is given in 2.5.

Theorem 1.2 *Let L be a division algebra with centre $K = Z(L)$ transcendence degree two over k and $\dim_K L < \infty$. Then either there is a minimal terminal order A with $k(A) \simeq L$ that is unique up to Morita equivalence in L, or $Z(L) \simeq k(\mathbb{P}^1 \times C)$.*

The key to this theorem involves analysing the ramification data of an order. This data is composed of cyclic covers of components of the discriminant. When the ramification data satisfies certain constraints (see definition 2.5) we call the order terminal. We classify the étale local structures of terminal orders and show they are determined by their ramification data. As a consequence, we can show that a terminal order is determined by its ring of fractions and centre up to Morita equivalence. This allows us to reduce the minimal model program to the case of pairs (Z, α) consisting of a projective surface Z and an element α of the Brauer group $\mathrm{Br}\, k(Z)$.

In the second part of this paper we carry out the minimal model program for such pairs (Z, α). This part of the paper is purely algebro-geometric. It should be of independent interest to algebraic geometers and can be read completely independently of the other sections. We use the results and methods of the minimal model program for log surfaces. The ramification data of a maximal order A depends only on its centre Z and the Brauer class $k(A)$ and so allows us to define the ramification data of a pair (Z, α). As for orders, we say that (Z, α) is terminal if its ramification satisfies certain constraints (definition 2.5). We also use the ramification data to attach a log surface (Z, Δ) to (Z, α). Terminal pairs yield klt log surfaces and so we may run the log surface minimal model program on a terminal pair.

There are several points of departure with the usual situation of log surfaces. Firstly, if we blow up a model Z, the Brauer class α determines a coefficient for the exceptional curve in the boundary. This allows us to define discrepancy of (Z, α) and we show that the terminal condition can also be defined in terms of this discrepancy. However, this discrepancy is not the discrepancy of the log surface (Z, Δ). Indeed, one can find Brauer classes $\alpha, \alpha' \in \mathrm{Br}\, k(Z)$ such that (Z, α) and (Z, α') have the same associated log surface, but one is terminal whilst the other is not. Hence, the Mori category of terminal pairs (Z, α) is not some Mori category of log surfaces.

Running the log MMP on a terminal pair (Z, α) preserves the terminal condition and hence the Mori category of terminal pairs is closed under log surface contractions. We arrive at a terminal pair (Z, α) where Z is smooth and Δ has normal crossings. As usual, we have a dichotomy, either i) $K_Z + \Delta$ is nef or ii) Z is a Mori fibre space with $-(K_Z + \Delta)$-ample fibres. However, in the first case, we obtain stronger uniqueness results than is usual for log surfaces since we can resolve singularities before starting the minimal model program. In the second case, we obtain del Pezzo and ruled pairs. We classify these pairs in terms of their ramification data.

In the last part of the paper, we combine the results of the previous sections to determine the existence and uniqueness of minimal models for orders. It follows that any order has a resolution of singularities of global dimension two. We may then run the minimal model program to obtain minimal models of an order which are either unique up to Morita equivalence, or are del Pezzo or ruled.

Since the Brauer group is also used to classify objects such as trivially banded μ_r-gerbes and $\mathbb{P}^n$-bundles, our MMP for pairs (Z, α) can similarly be applied to minimal models of these structures.

Throughout, all schemes are assumed to be noetherian and separated.

Acknowledgements. We would like to thank our supervisor Michael Artin for suggesting this problem and for teaching us the necessary background material. We would also like to thank János Kollár for suggesting extensions of the results and improvements in the exposition. We also thank Alessio Corti and Miles Reid for explaining connections with rational threefold conic bundles.

2 Maximal Orders on Surfaces

2.1 Normal Orders

Let Z be an integral normal scheme with quotient field K. An order A over Z is a coherent torsion free sheaf of $\mathcal{O}_Z$ central algebras such that $A \otimes_Z K$ is a central simple K-algebra. The degree of A is defined to be $\sqrt{\operatorname{rank} A}$. We will assume that the degree of A is prime to the characteristic of k. We write $k(A) = A \otimes_Z K$ and $Z(A) = Z$ for the centre of A.

The set of orders over Z in the central simple algebra $k(A)$ forms a partially ordered set under inclusion. A maximal object in this poset is called a maximal order. The proof of the following proposition is in [R] Section 10.

Proposition 2.1 *Every order is contained in a maximal order.*

Passing to a maximal order is similar to normalising a variety, although there is generally not a unique choice of maximal order. We next need to define the dualising sheaf of an order. Let Z be an integral normal Cohen-Macaulay scheme with dualising sheaf ω_Z, and let A be an order over Z. Then we define the canonical sheaf for A to be the A-bimodule

$$\omega_A = \mathcal{H}om_Z(A, \omega_Z).$$

Since A is finite over Z, and Z is Cohen-Macaulay, A is Cohen Macaulay as well. We interpret ω_A as the dualising sheaf since it is used in formulating Serre duality for A. The statement and proof can be found in [YZ], which will not be used in this paper. The next result is proved in [AG], which is also a good general reference for maximal orders on rings.

Proposition 2.2 *Let A be a maximal order over an integral normal scheme Z. Then A satisfies the following two conditions.*

- $\mathbf{R_1}$: *Let $p \in Z^1$ be a codimension one prime in Z. Then $A_p = A \otimes_Z \mathcal{O}_{Z,p}$ is a hereditary order and $A_p \simeq \omega_{A,p}$ on the left and on the right, (but not necessarily as bimodules).*

- $\mathbf{S_2}$: *A is reflexive as a sheaf over Z.*

Being a maximal order is Morita invariant but the condition R_1 is not. For a hereditary order on a d.v.r., the condition R_1 is equivalent to the radical being a principal ideal. Consequently, Hijikata and Nishida call such orders principal [HN]. Since the property of being a maximal order is not preserved under henselisation or completion at a point of codimension ≥ 2, we will use the conditions above to a define a notion of normal orders.

Definition 2.3 Let A be an order over an integral normal scheme Z. We say that A is *normal* if A satisfies the conditions R_1 and S_2 and, for all codimension one points p in Z such that $k(p)$ has transcendence degree one over k, we also have that A_p is a maximal order.

It is clear that the notions of maximal order and normal order coincide when the centre is of finite type. Also, it follows easily that the property of being a normal order is preserved under henselisation or completion at a point of codimension ≥ 2.

2.2 Ramification Data

In this section we define the ramification data of an order, and we define a class of orders called terminal orders via conditions on the ramification data. The connection with the usual notion of terminal as used in the minimal model program will appear in section 3.4.

Proposition 2.4 *Let R be a d.v.r with maximal ideal p and let Λ be a normal order over R. Then Λ has radical $\operatorname{rad}\Lambda \supseteq p\Lambda$, and $\Lambda/\operatorname{rad}\Lambda = \prod_{i=1}^n A$ where A is a central simple L-algebra, and L is a cyclic extension of the residue field $k(p)$.*

Proof. Embed Λ in a maximal order Λ'. Since the fibres at the residue field do not change after completion, we may assume that Λ, Λ' are complete. We have the following inclusions

$$\operatorname{rad} \Lambda' \subseteq \operatorname{rad} \Lambda \subseteq \Lambda \subseteq \Lambda'$$

from the proof of 39.14 in [R]. It follows from the arguments of [S] Chapter IV, Section 1, Appendix, that L is a cyclic extension. It also follows from the proof of 39.14 in [R] and the R_1 condition that the subalgebra $\Lambda/\operatorname{rad}\Lambda'$ of $L^{n\times n}$ is congruent to a block upper triangular subalgebra with equal size blocks. The quotient $\Lambda/\operatorname{rad}\Lambda$ of this algebra is clearly of the desired form. □

Let A be a normal order over an integral normal scheme Z. The above proposition associates to every codimension one point p of Z, a product of cyclic extensions of $k(p)$ given by

$$\tilde{k}(p) := Z(A_p/\operatorname{rad} A_p) \simeq L^n.$$

Denote its degree by

$$e(p) := \dim_{k(p)} \tilde{k}(p).$$

We will call $e(p)$ the ramification index of A at p and say A ramifies at p if $e(p) > 1$. Let the discriminant divisor D be the union of prime divisors $D_i \subset Z$ where A ramifies. We note that if a codimension one point p is not in D then A_p is Azumaya over the d.v.r. $\mathcal{O}_{Z,p}$.

Using the above proposition we define the ramification data of A to be $R(A) = (\tilde{D} \twoheadrightarrow D \hookrightarrow Z)$ where $\tilde{D}, D, Z$ are described as follows. The centre of A is Z and $D = \cup D_i$ is the discriminant curve. We let $\tilde{D}$ be the disjoint union of the curves $\tilde{D}_i$ where $\mathcal{O}_{\tilde{D}_i}$ is the integral closure of $\mathcal{O}_D$ inside the product of cyclic extensions $\tilde{k}(p)$. So for each component D_i of D we have that $\tilde{D}_i$ is a union of cyclic ramified covers of the normalisation of D_i.

The class of normal orders we are most interested in are those whose ramification data satisfies the conditions below.

Definition 2.5 Let A be a normal order with ramification data $R(A) = (\tilde{D} \twoheadrightarrow D \hookrightarrow Z)$. We say that A is *terminal* if

- Z is smooth,
- the discriminant curve D has normal crossings,
- the cyclic covers $\tilde{D}_i$ ramify at the nodes of D,

- at a node p, one cover $\tilde{D}_1$ is totally ramified at p of degree e and the other cover $\tilde{D}_2$ ramifies at p with index e and has degree ne for some positive integer n.

2.3 Etale Local Models

The main goal of this section is to classify the possible étale local structures of terminal orders. We begin by defining an algebra that was suggested by Artin as an example of a possible local structure for a terminal order. We will see that this algebra is actually the only possible étale local structure of a terminal order up to isomorphism. This will give us the corollary that resolutions of global dimension two exist. In particular, we will get the fact that the ramification data determines a terminal order up to Morita equivalence within its algebra of quotients (see corollary 2.13).

For the rest of this section, we restrict to k being an algebraically closed field of characteristic prime to the degree of the order A. Let $k\{u,v\} \subset k[[u,v]]$ denote the subalgebra of power series that satisfy an algebraic equation. In this section, we will work over this strictly hensel local algebra.

Definition 2.6 Let $R = k\{u,v\}$, and let n, e be a positive integers and let ζ be a primitive e^{th} root of unity. Let $S = R\langle x, y\rangle$ with the relations $yx = \zeta xy, x^e = u, y^e = v$. Note that S is finite over R, $Z(S) = R$ and $k(S)$ is a division ring. We define an R-order in $k(S)^{n\times n}$ by

$$\Lambda(n,\zeta) := \begin{pmatrix} S & \cdots & \cdots & S \\ xS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ xS & \cdots & xS & S \end{pmatrix}.$$

We can also describe $\Lambda(n,\zeta)$ as follows.

Proposition 2.7 *1. Let B be the hereditary order*

$$\begin{pmatrix} k[x] & \cdots & \cdots & k[x] \\ (x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ (x) & \cdots & (x) & k[x] \end{pmatrix} \subset k[x]^{n\times n}$$

and σ be the automorphism of $k[x]$ defined by $\sigma(x) = \zeta x$. If we extend the automorphism σ to B, then étale locally, $\Lambda(n, \zeta)$ is isomorphic to $A = B[y; \sigma]$.

2. *Let η be a primitive n-th root of unity and $B = k\langle x, t\rangle/(xt - \eta tx, t^n - 1)$. Consider the automorphism σ of B which maps $t \mapsto t$ and*

$$x \mapsto x + \frac{1}{n}(\zeta - 1)x(1 + t + \dots t^{n-1}).$$

Then, étale locally, $\Lambda(n, \zeta)$ is isomorphic to $A = B[y; \sigma]$.

Proof. In part 1, the centre of A is $k[u, v]$ where $u = x^e, v = y^e$. There is a natural inclusion $B \to \Lambda(n, \zeta)$ which extends to a map $A \to \Lambda(n, \zeta)$ on mapping y to diagonal matrix $(y, \dots, y)$. This becomes an isomorphism on tensoring with $k\{u, v\}$.

For part 2, we note that B is isomorphic to the hereditary order in part 1 via the map which sends x to the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ x & 0 & \dots & \dots & 0 \end{pmatrix}$$

and t to the diagonal matrix $(1, \eta, \dots, \eta^{n-1})$. Conjugating the matrix for x above with the diagonal matrix $(y, \dots, y)$ gives

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ \zeta x & 0 & \dots & \dots & 0 \end{pmatrix}.$$

Consequently, part 2 follows from part 1. $\square$

We list some important properties of $\Lambda(n, \zeta)$. Once we have shown that terminal orders are étale locally of this form then these will become local properties of terminal orders.

Proposition 2.8 *Let $\Lambda = \Lambda(n, \zeta)$, then*

1. *Λ has global dimension two.*
2. *Λ satisfies the conditions R_1 and S_2.*
3. *if $n = e = 1$ then Λ is unramified.*
4. *if $e = 1$ and $n > 1$ then L is ramified on $D = V(u)$ and the cyclic cover $\tilde{D}$ of D is unramified with degree n.*
5. *if $e > 1$ then Λ is ramified on $D = V(uv)$ and the cyclic cover of $V(u)$ has degree ne and the cover of $V(v)$ has degree e and both are ramified with ramification index e.*

Proof. The first statement follows from Hibert's syzygies theorem and the description of $\Lambda(n, \zeta)$ given in the previous proposition. The other properties all follow from direct calculations. □

So the above proposition shows that Λ is actually a terminal order. We will now show that any terminal order is étale locally isomorphic to $\Lambda(n, \zeta)^{m\times m}$, a matrix algebra with entries in $\Lambda(n, \zeta)$. We first need the following lemma.

Lemma 2.9 *Let $\Lambda = \Lambda(n, \zeta)^{m\times m}$ and let P be a projective right Λ-module. Let B be the strict henselisation of the local ring $k\{u, v\}_{(u)} = R_{(u)}$. If $P \otimes_R B \simeq \Lambda \otimes_R B$ then $P \simeq \Lambda$.*

Proof. We may reduce the case $m = 1$ by Morita equivalence. Note that

$$S \otimes B \simeq \begin{pmatrix} B & \cdots & \cdots & B \\ uB & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ uB & \cdots & uB & B \end{pmatrix} \subset B^{e\times e}$$

and

$$xS \otimes B \simeq \begin{pmatrix} uB & B & \cdots & B \\ \vdots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & B \\ uB & \cdots & \cdots & uB \end{pmatrix} \subset B^{e\times e}.$$

Let M^n denote the row vector $(M, \ldots, M)$ taken e times. The indecomposable right projectives of Λ are all of the form

$$\begin{pmatrix} xS^{n-q} & S^q \end{pmatrix}$$

by Krull-Schmidt. So P must be a sum of such modules. Let $1 \leq i \leq ne$ and write $i = qe + r$ where $1 \leq r \leq e$ and $0 \leq q \leq n-1$. The indecomposable right projectives of $\Lambda \otimes B$ are of the form

$$P_i := \begin{pmatrix} \begin{pmatrix} uB^{r-1} & B^{e-r+1} \end{pmatrix}^{n-q} & \begin{pmatrix} uB^r & B^{e-r} \end{pmatrix}^q \end{pmatrix}.$$

Note that there is a distinct number of uB appearing in each P_i. Also

$$\begin{pmatrix} xS^{n-q} & S^q \end{pmatrix} \otimes B \simeq \bigoplus_{r=1}^{e} P_{qe+r}.$$

So $P \otimes \Lambda$ will have summands as above. These can only come from the henselisation of (xS^{q-i}, S^q) and so the statement follows. □

This next proposition is a well known result on the structure of orders, which describes the étale local structure at a generic point of the discriminant curve. It follows from [R] theorem 39.14 and [HN] theorem 0.1.

Proposition 2.10 *Let A be a normal order. Let p be a generic point in the discriminant divisor D. Then there exists a finite étale extension R' of the discrete valuation ring $R = \mathcal{O}_{Z,p}$ so that*

$$A \otimes R' \simeq \begin{pmatrix} R' & \cdots & \cdots & R' \\ uR' & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ uR' & \cdots & uR' & R' \end{pmatrix}^{l \times l}$$

where u is a parameter of the discrete valuation ring R'.

Let $R = k\{u, v\}$ and let $Z = \operatorname{Spec} R$. We let p be the closed point of Z and let $U = Z - p$ be the punctured spectrum. Let Λ be a reflexive R-order, so Λ is determined by its restriction to U. A form of Λ over U is an order Γ over U such that for all codimension one points p in U we can find an étale extension R' of $\mathcal{O}_{U,p}$ so that $\Gamma \otimes R' \simeq \Lambda \otimes R'$. By taking $H^0(U, \Gamma)$ we get an extension of Γ to an order over Z that will be reflexive. Abusing notation we will also call this order Γ. Any normal order Γ that has the same ramification data and rank as Λ will be a form of Λ by 2.10, and further we will have that $k(\Gamma) \simeq k(\Lambda)$ by 3.2.

We need another result before we can prove the main theorem.

Proposition 2.11 *Let $\Lambda = \Lambda(n, \zeta)$. Let Γ be a form of Λ that is trivial at the generic point, i.e. $k(\Gamma) = k(\Lambda)$. Let p be the prime ideal uR. Then $\Lambda_p \simeq \Gamma_p$.*

Proof. Since the question is local at p we will relabel $\Lambda_p, \Gamma_p, R_p, S_p$ with Λ, Γ, R, S. Let R' be an étale extension of R that splits Γ and S. Note the isomorphisms

$$S \otimes R' \simeq \begin{pmatrix} R' & \cdots & \cdots & R' \\ uR' & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ uR' & \cdots & uR' & R' \end{pmatrix} \quad \operatorname{rad} S \otimes R' \simeq \begin{pmatrix} uR' & R' & \cdots & R' \\ \vdots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & R' \\ uR' & \cdots & \cdots & uR' \end{pmatrix}.$$

We know that $\Lambda \otimes R' \simeq \Gamma \otimes R'$ and we will first show that $\hat{\Lambda} \simeq \hat{\Gamma}$. Note that

$$\hat{\Gamma} \otimes R' \simeq \hat{\Lambda} \otimes R' \simeq \begin{pmatrix} \hat{R}' & \cdots & \cdots & \hat{R}' \\ u\hat{R}' & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ u\hat{R}' & \cdots & u\hat{R}' & \hat{R}' \end{pmatrix}.$$

We also know that $\hat{\Gamma}$ is a hereditary order in $\hat{S}^{n\times n}$ and so by the classification of hereditary orders over complete discrete valuation rings [R] we know that

$$\hat{\Gamma} \simeq \begin{pmatrix} \hat{S}^{n_1\times n_1} & \cdots & \cdots & \hat{S}^{n_1\times n_r} \\ \operatorname{rad} \hat{S}^{n_2\times n_1} & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \operatorname{rad} \hat{S}^{n_r\times n_1} & \cdots & \operatorname{rad} \hat{S}^{n_r\times n_{r-1}} & \hat{S}^{n_r\times n_r} \end{pmatrix}$$

for some n_i where $n = n_1 + \cdots + n_r$. In order for this isomorphism to yield the isomorphism above upon tensoring with R', we must have that $r = n$ and all $n_i = 1$. This allows only a unique possibility for $\hat{\Gamma}$ and so we conclude that $\hat{\Gamma} \simeq \hat{\Lambda}$.

We will now show that $\Gamma \simeq \Lambda$. First note that $\operatorname{rad} \hat{S}^{n \times n} \subseteq \operatorname{rad} \hat{\Gamma} \subseteq \hat{\Gamma}$ by the beginning of the proof of theorem 39.14 in [R]. Hence $\hat{\Gamma}/\operatorname{rad} \hat{S}^{n \times n} \subset (\hat{S}/\operatorname{rad} \hat{S})^{n \times n} \simeq k((y))^{n \times n}$. It is also shown in the proof of theorem 39.14 of [R] that there is a partial flag

$$F = (0 \subset V_1 \subset \cdots \subset V_r = k((y))^n)$$

such that $\hat{\Gamma}/\operatorname{rad} \hat{S}^{n \times n} \simeq \operatorname{End} F$, the matrices in $k((y))^{n \times n}$ that stabilise the flag F. Let $n_i = \dim V_i/V_{i-1}$. Then $\hat{\Gamma}/\operatorname{rad} \hat{\Gamma} \simeq k((y))^{n_1 \times n_1} \times \cdots \times k((y))^{n_r \times n_r}$. However, we also know that $\hat{\Gamma}/\operatorname{rad} \hat{\Gamma} \simeq \hat{\Lambda}/\operatorname{rad} \hat{\Lambda} \simeq k((y)) \times \cdots \times k((y))$. So we conclude that all $n_i = 1$ and $r = n$ and F is a complete flag. So there is a matrix c in $k((y))^{n \times n}$ whose first i columns generate the spaces V_i in the flag. Using the injection $k((y)) \subset \hat{S}$ we may consider c as a matrix in $\hat{S}^{n \times n}$. We now replace $\hat{\Gamma}$ with its conjugation $c^{-1}\hat{\Gamma}c$ so that the quotient

$$\hat{\Gamma}/\operatorname{rad} \hat{S}^{n \times n} \simeq \begin{pmatrix} k((y)) & \cdots & \cdots & k((y)) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & k((y)) \end{pmatrix}$$

stabilises the standard flag. So now we have that $\hat{\Gamma}/\operatorname{rad} \hat{S}^{n \times n} \simeq \hat{\Lambda}/\operatorname{rad} \hat{S}^{n \times n}$, so we may conclude that $\hat{\Gamma} = \hat{\Lambda}$. Now since $\Gamma = \hat{\Gamma} \cap K\Lambda$ by [R] theorem 5.2, we may conclude that $\Gamma = \Lambda$. $\square$

The following theorem was shown by Artin in [A1] for $\Lambda(n, \zeta)$ is the cases where either $n = 1$ or $\zeta = 1$. It also follows from [Ra] theorem 5.4 in the case that $n = 1$.

Theorem 2.12 *Let Λ be a reflexive R-order. Suppose there exists a height one prime p of R such that for any projective left Λ-module P, if $P_p \simeq \Lambda_p$ then $P \simeq \Lambda$. Suppose Γ is a form of Λ such that $k(\Lambda) = k(\Gamma)$. Then Λ is isomorphic to Γ. In particular, any terminal order has global dimension two.*

Proof. Let Γ be a form of Λ so that $k(\Lambda) = k(\Gamma)$. Lemma 2.11 allows us to adjust Γ by conjugation so that we have $\Lambda_p = \Gamma_p$ for the special prime p above. Let $(-)^{**}$ denote the reflexive hull of an R-module. We will show that the bimodule $P = (\Lambda\Gamma)^{**}$ gives a Morita equivalence between Λ and Γ. Since P is reflexive, the Auslander-Buchsbaum formula [SZ] implies that P is a projective Λ-module. Note that $P_p = \Lambda_p \Gamma_p = \Lambda_p$. So we must have that the left module ${}_\Lambda P \simeq {}_\Lambda\Lambda$ by our assumption. Since $\Gamma \to \operatorname{End}_\Lambda P = \Lambda$, we get an inclusion $\Gamma \to \Lambda$. Let q be an arbitrary height one prime of R. So we have that Γ_q is contained in Λ_q. We wish to show this injection is an isomorphism and it suffices to do this after tensoring with $\tilde{R}_q$, the strict henselisation which we will denote by $(-)'$. Since Γ is a form of Λ we also have that $\Gamma' \simeq \Lambda'$ as R'-algebras. So by Skolem-Noether there is an invertible element c in $k(\Lambda')$ such that $\Gamma' \subset c^{-1}\Gamma' c \subset c^{-2}\Gamma' c^2 \subseteq \cdots$ the union of this chain is contained in the finitely generated R'-module Γ'^* by the arguments of [R] 10.3,10.4. So this chain must terminate and we may conclude that $\Lambda' = \Gamma'$. □

The next corollary allows us to reduce questions about terminal orders to studying the geometry of the ramification data. It will allow us to use the results of the next part to carry out the minimal model program for orders.

Corollary 2.13 *Let A be a terminal order over a projective surface Z. Then A is uniquely determined up to Morita equivalence by $k(A)$ and Z. Also, any order Morita equivalent and generically isomorphic to A is of the form* $\operatorname{End} P$ *where P is a Zariski locally principal A-module. Moreover, such Morita equivalent orders are classified by $H^1(Z, A^*/\mathcal{O}_Z^*)$.*

Proof. Let B be another maximal order in the central simple algebra $k(A)$ with centre Z. Since $R(B) = R(A)$ by the Artin-Mumford sequence (see theorem 3.1), we know B is terminal and so has global dimension two by theorem 2.12. Let P be the reflexive hull of the bimodule $AB \subset k(A)$. The Auslander-Buchsbaum formula implies that P is locally projective on both sides. Since A and B are maximal we get that the two inclusions

$$B \to \operatorname{End}_A P \qquad A \to \operatorname{End} P_B$$

must be isomorphisms. So P gives the desired Morita equivalence. The fact that P is Zariski locally principal follows from [Ra] theorem 5.4. □

3 Minimal Model Program relative to a Brauer Class

We will work over an algebraically closed field k with characteristic that is prime to the order of α in $\operatorname{Br} K$, we extend the results to non closed fields in section 3.6. All cohomology groups are assumed to be étale. Furthermore, we only consider the prime to char k part of any such cohomology group and similarly for all exact sequences in étale cohomology. All surfaces considered will be projective.

3.1 Ramification Data

Let K be a field of transcendence degree two over k, and let α be an element of $\operatorname{Br} K$. In this section we define the ramification data of α on a normal model Z of K. There is a sequence of Artin-Mumford which gives restrictions on the possible ramification data. We also discuss how the ramification data changes under blowing up a smooth model Z.

In this paper, a pair (Z, α) will always mean a normal surface Z and a Brauer class $\alpha \in \operatorname{Br} k(Z)$.

The following theorem gives ramification information from a Brauer group element and also constrains possible ramification on surfaces. It was proved by Artin and Mumford in [AM]. Let μ be the group scheme of roots of unity, and let $\mu^{-1} = \cup_n \operatorname{Hom}(\mu_n, \mathbb{Q}/\mathbb{Z})$ where μ_n is the group of n-th roots of unity.

Theorem 3.1 (Artin, Mumford) *Let Z be a normal surface with codimension one points Z^1. Then there is a natural morphism*

$$\operatorname{Br} k(Z) \xrightarrow{a} \bigoplus_{C \in Z^1} H^1(k(C), \mathbb{Q}/\mathbb{Z}).$$

If Z is a smooth projective surface then this map extends to a complex

$$0 \to \operatorname{Br} Z \to \operatorname{Br} k(Z) \xrightarrow{a} \bigoplus_{C \in Z^1} H^1(k(C), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} \bigoplus_{p \in Z} \mu^{-1} \to \mu^{-1} \to 0.$$

The cohomology is $H^3(Z, \mu)$ at $\oplus H^1(k(C), \mathbb{Q}/\mathbb{Z})$ and zero elsewhere. So if Z is simply connected the sequence is exact.

The map a in the theorem arises from the Leray spectral sequence for the inclusion $j : \operatorname{Spec} k(Z) \to Z$, and exact sequence $0 \to \mathbb{G}_m \to j_* \mathbb{G}_m \to \oplus_C \mathbb{Z}$, and the Kummer exact sequence.

We will use this map to define ramification data of α over Z. Firstly, we say that α ramifies at an irreducible curve $C \subset Z$ if the component of $a(\alpha)$ indexed by C is non-zero. The discriminant curve of (Z, α), denoted D, is the union of irreducible curves D_i where α ramifies. Recall that $H^1(k(C), \mathbb{Q}/\mathbb{Z})$

classifies cyclic field extensions of $k(C)$ so $a(\alpha)$ gives cyclic field extensions $\tilde{k}(D_i)$ of $k(D_i)$. We define $\tilde{D}_i$ by letting $\mathcal{O}_{\tilde{D}_i}$ be the integral closure of $\mathcal{O}_{D_i}$ in $\tilde{k}(D_i)$. Thus $\tilde{D}_i$ is a ramified cyclic cover of the desingularisation of D_i. Let $\tilde{D}$ be the disjoint union of the $\tilde{D}_i$ and define *ramification data* for (Z, α) to be

$$R(Z, \alpha) = (\tilde{D} \twoheadrightarrow D \hookrightarrow Z).$$

The morphism r in the Artin-Mumford sequence maps the cover $\tilde{D}_i$ to its ramification at the point p. In particular, if $p \in Z$ and $c \in H^1(k(D_i), \mathbb{Q}/\mathbb{Z})$ defines $\tilde{D}_i$ then the component of $r(c)$ indexed by p has order equal to the ramification index of $\tilde{D}_i \to D_i$ above p. Since the Artin-Mumford sequence is a complex the ramification must cancel at these points.

This next proposition is a local version of the Artin-Mumford sequence and is shown in [A3].

Proposition 3.2 *Let R be a normal local k-algebra of dimension 2 and residue field k. Assume also that R is complete or is the henselisation of a finitely generated k-algebra. Let K be the field of fractions of R. If R has rational singularities, then there is an exact sequence*

$$0 \to \operatorname{Br} K \to \bigoplus_p H^1(k(p), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} \mu^{-1} \to 0.$$

where the sum is over height one primes p of R.

As in the projective case, the map r in the above sequence is given by the ramification of the cyclic covers. Since each curve $V(p)$ is henselian, we have canonical isomorphisms $H^1(k(p), \mathbb{Q}/\mathbb{Z}) \simeq \mu^{-1}$ and the map r is the sum of these.

Remark 3.3 Let $R(Z, \alpha)$ be the ramification data of α on a smooth surface Z. Then the cyclic covers $\tilde{D}_i \to D_i$ can ramify only at the singular points of D. The ramification of the various covers must cancel at these singular points. If $f : \mathbb{P}^1 \to D$ is a non-constant map, then $f^{-1}(\operatorname{Sing} D)$ contains at least two points.

After we change the model Z by blowing up, the Artin-Mumford sequence tells us whether or not α is ramified on the exceptional curve. That is the content of the next result, a key lemma which will be used throughout.

Lemma 3.4 *Let (Z, α) be a pair with Z smooth and discriminant curve D. Consider the blow up $f : Z' \to Z$ at a point p with exceptional curve E. Then the ramification of (Z', α) above E is uniquely determined by the ramification of the cyclic covers $\tilde{D}_i$ at the point p.*

Proof. The ramification of (Z', α) above E is given by a ramified cyclic cover $\tilde{E} \to E$ which can be described as follows. Note first that $E \simeq \mathbb{P}^1$ so $\tilde{E}$ is determined by ramification data of $\tilde{E} \to E$. More precisely, we have the following exact sequence,

$$0 \to H^1(k(\mathbb{P}^1), \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{q \in \mathbb{P}^1} \mu^{-1} \to \mu^{-1} \to 0$$

so we need only give an element of μ^{-1} for each point $q \in E$. We fix such a q and use the Artin-Mumford sequence to show that the ramification of the $\tilde{D}_i$ at p determine a unique element of μ^{-1}. Let D_i be a ramification divisor whose strict transform $f_*^{-1} D_i$ passes through q. Then (Z', α) ramifies on $f_*^{-1} D_i$ and this ramification is also given by the cyclic cover $\tilde{D}_i$. The ramification of $\tilde{D}_i \to f_*^{-1} D_i$ at q, let us denote it by η_i, is of course the same as that of $\tilde{D}_i \to D_i$ at p. Now the only ramification divisors of (Z', α) passing through q are these $f_*^{-1} D_i$ and possibly E. For the Artin-Mumford sequence to be a complex, we must have that the ramification of $\tilde{E} \to E$ at q is given by $-\sum \eta_i \in \mu^{-1}$. □

3.2 Terminal Resolution of Singularities

In this section we consider resolving singularities of (Z, α) to a terminal model. We seek a modification $Z' \to Z$ which improves the singularities of $R(Z, \alpha)$ as far as possible. Firstly, it is clear that we can ensure that Z is smooth and the discriminant curve has only normal crossings by repeatedly blowing up the centre. Suppose we have normal crossings so locally at a node p, the discriminant is say $D_1 \cup D_2$. The Artin-Mumford sequence ensures that the ramified cyclic covers $\tilde{D}_1, \tilde{D}_2$ have equal ramification index e at p, and we may assume their degrees to be ne and me for some $n, m \in \mathbb{N}$. If we blow up the node we obtain a degree e cyclic cover of the exceptional curve that is totally ramified at the two new nodes. So we have obtained ramification data that fits the description in 2.5.

Definition 3.5 We say that a pair (Z, α) is *terminal* if R satisfies the conditions in 2.5.

We will run the MMP on the category of terminal pairs (Z, α) by doing log MMP contractions on an associated log surface. We have already observed the following result which is an essential starting place for the minimal model program.

Corollary 3.6 (Resolution of Singularities) *Let (Z, α) be a pair. Then there exists a finite sequence of blowups and normalisations $Z' \to \cdots \to Z$ such that (Z', α) is terminal.*

We will show in 3.10 that log MMP contractions stay inside this category of terminal pairs.

We will also show later that this condition on the ramification data coincides with the usual notion of terminal used in the minimal model program in proposition 3.14. This usual definition of positive discrepancy is exactly the condition needed to show that minimal models are unique in section 3.4.

The following remark and theorem follow immediately from the definition and the Artin-Mumford sequence.

Remark 3.7 Let (Z, α) be terminal and let $Z' \to Z$ be a blow up at a point. Then (Z', α) is terminal.

Theorem 3.8 (Zariski's Factorisation) *Let $f : Z' \to Z$ be a birational morphism where (Z, α) is terminal. Then f factors as a series of blow downs of (-1)-curves*

$$Z' \to Z_1 \to \cdots \to Z_n = Z$$

and (Z', α) is terminal.

Unlike for surfaces, the terminal condition is not an étale local condition, even though it can be checked étale locally on ramification data. To see this, consider a pair (Z, α) where Z is a projective surface and let Z^h be the étale localisation of Z at some closed point p. Let $\phi : \operatorname{Br} k(Z) \to \operatorname{Br} k(Z^h)$ be the map on Brauer groups. Let D_i be a smooth ramification curve of (Z, α) and $\tilde{D}_i$ the associated ramified cyclic cover. To determine if the localisation of D_i at p, denoted D_i^h is also in the discriminant of $\phi(\alpha)$ we need only localise $\tilde{D}_i$ at p to obtain a cyclic cover $\tilde{D}_i^h \to D_i^h$. This cyclic cover splits up into a number of copies of some irreducible curve $\widetilde{D_i^h}$ and this is the ramification of $\phi(\alpha)$ at D_i^h. In particular, if p is a node of D and the cyclic covers are unramified above p, then α is not terminal but $\phi(\alpha) = 0$ so étale locally, α appears to be unramified and hence terminal at the node.

3.3 Castelnuovo Contraction

In this section we introduce a log surface associated to the ramification data $R(Z, \alpha)$. The canonical divisor of the log surface will act as the canonical divisor of the pair (Z, α). By using this log surface we can show a form of Castelnuovo's theorem of contraction of (-1)-curves. It can be interpreted as saying that the category of terminal pairs is stable under log MMP contractions. Using this result we will show the existence of minimal models of pairs.

Consider a pair (Z, α) and its ramification data $R(Z, \alpha) = (\tilde{D} \twoheadrightarrow D \hookrightarrow Z)$ with $D = \cup D_i$ and $\tilde{D} = \cup \tilde{D}_i$ the decompositions into irreducible components. Let e_i be the degree of the cover $\tilde{D}_i \to D_i$. We write

$$\Delta_\alpha = \sum \left(1 - \frac{1}{e_i}\right) D_i$$

and drop the subscript α whenever it is clear. The log surface $LR(Z, \alpha)$ associated to R is (Z, Δ_α).

The motivation for this particular log surface comes from the following adjunction formula for orders. The following proposition shows that the log canonical divisor is numerically equivalent to the canonical divisor of the order A. This result is not used in this paper. It is stated in [A2] and is also proved in [CK] and [AdJ].

Proposition 3.9 *Let A be a maximal order of degree n over a surface Z. If α is the Brauer class represented by $k(A)$ then*

$$\omega_A^{\otimes n} \simeq A \otimes \mathcal{O}_Z(n(K_Z + \Delta_\alpha))$$

in codimension one.

Theorem 3.10 (Castelnuovo's Contraction) *Let (Z, α) be a terminal pair with associated log surface $LR(Z, \alpha) = (Z, \Delta)$. Suppose there is an irreducible curve E in Z such that $E^2 < 0$ and $(K_Z + \Delta).E < 0$. Then there is a map $Z \to Z'$ that contracts exactly E, and the pair (Z', α) is terminal.*

Proof. We will show that $E^2 = -1$ and $E.K_Z = -1$ so we will have a map $\pi : Z \to Z'$ contracting E. We will show that $\pi_*\Delta$ with its cyclic covers satisfies the conditions of definition 2.5.

If E is not contained in D then $E.\Delta \geq 0$ so $E.K_Z < 0$. Since we also have $E^2 < 0$ we must have that $p_a(E) = 0$ by the genus formula. So E is a smooth rational curve. We also get that $E.K_Z = E^2 = -1$, so E is a (-1)-curve. Since $(K_Z + \Delta).E < 0$ we must have that $E.\Delta < 1$. Since the coefficients of Δ are of the form $1 - 1/e_i \geq 1/2$ we see that E can meet D in at most one point. So we contract E via the map $\pi : Z \to Z'$ and we see that $\pi_*\Delta \simeq \Delta$ so the ramification data on Z' satisfies the conditions in definition 2.5.

Suppose now that E is a component of Δ. We write $\Delta = \Delta' + (1 - \frac{1}{e})E$. Since $E.\Delta' - \frac{1}{e}E^2 > 0$, and $E.(K_Z + \Delta' + (1 - \frac{1}{e})E) < 0$ we must have that $E.(K_Z + E) < 0$. So $p_a(E) = 0$ and again we conclude that E is a smooth rational curve. Since $E.(K_Z + E) = -2$ we have that $-2 + E.\Delta' - \frac{1}{e}E^2 < 0$. Solving we get that $-E^2 < e(2 - E.\Delta')$. So $2 - E.\Delta' > 0$. Since D has normal crossings, $E.\Delta' = \sum(1 - \frac{1}{a_i})$ for some integers $a_i \geq 2$. So we immediately have that $E.\lceil\Delta'\rceil$ is at most three points. We wish to show that the case of three points does not occur so suppose that $E.\lceil\Delta'\rceil = 3$. The possible triples (a_1, a_2, a_3) that satisfy the inequality $2 - E.\Delta' > 0$ are the Platonic triples $(2, 3, 5), (2, 3, 4), (2, 3, 3)$ and $(2, 2, a)$ for any $a \geq 2$. We also have the following exact sequence

$$0 \to H^1(k(\mathbb{P}^1), \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{p \in \mathbb{P}^1} \mu^{-1} \to \mu^{-1} \to 0$$

which maps injectively into the sequence of Artin and Mumford 3.1 in the natural way. The last map of this injection is the Poincaré dual of the natural isomorphism $H^0(Z, \mu) \to H^0(\mathbb{P}^1, \mu)$. So the ramification of the cyclic cover of $\mathbb{P}^1$ would have to be given by three nontrivial roots of unity whose order divides the given numbers, and whose product is one. This is not possible for the given triples. Of course there are ramified covers of $\mathbb{P}^1$ with ramification given by Platonic triples but they are not cyclic.

This leaves the possibility of $E.\lceil\Delta'\rceil = 2$. In this case we have a ramified cover of $\mathbb{P}^1$ of degree e that is ramified at two points. The cover must be totally ramified at the two points. So we must have that the ramification index of $\lceil\Delta'\rceil$ at the two points is ne and me for some integers $n, m \geq 1$. So $E.\Delta' = (1 - \frac{1}{ne}) + (1 - \frac{1}{me})$. Combining this with the above inequality gives $-E^2 < \frac{1}{n} + \frac{1}{m}$. Since $-E^2 \geq 1$ we conclude that n or m must be one.

These inequalities also tell us that $E^2 = -1$ and so we also see that E is a (-1)-curve. So now we contract E to a point p under the map $\pi : Z \to Z'$ and we see that $\pi_* \Delta$ at the point p satisfies the conditions of definition 2.5. $\square$

The following result occurs in the proof above and we note it separately for future reference.

Corollary 3.11 *Let (Z, α) be a terminal pair and let E be an irreducible curve such that $E.(K_Z + \Delta) < 0$ and $E^2 < 0$. Then either E is not in the discriminant D and $E.D \leq 1$, or E is in D and E intersects the curve $D - E$ transversely in two distinct points.*

We define (Z, α) to be a minimal model if (Z, α) is terminal and it has no curve such as E above. So (Z, α) is a minimal model if for any irreducible curve E such that $E.(K_Z + \Delta) < 0$ we must have that $E^2 \geq 0$. Let $\rho(Z)$ denote the Picard number of Z. The following theorems follow immediately.

Theorem 3.12 *Let (Z, α) be terminal. Then there is a series of n blow downs of (-1)-curves*

$$Z \to Z_1 \to Z_2 \to \ldots \to Z_n$$

where $n < \rho(Z)$ and (Z_n, α) is minimal.

3.4 Unique Minimal Models

In this section, we study the Mori category of terminal pairs (Z, α). We use the log surface (Z, Δ) to run the log minimal model program on a terminal pair (Z, α) to arrive at a unique minimal model if we achieve $K_Z + \Delta$ is nef. In contrast to the situation for log surfaces, we can always resolve singularities to a terminal pair (Z, α) before starting the MMP. Consequently, we obtain a stronger uniqueness result which parallels the usual uniqueness of minimal models of surfaces.

For example if Z is a germ of a point on a smooth surface with boundary Δ a nodal curve with coefficients $1/2$, then (Z, Δ) is already a minimal klt log surface from the point of view of the MMP, but such a log surface may or may not come from a pair (Z, α) which is terminal. Hence, we would only start the MMP in this situation if (Z, α) was terminal whereas the usual log MMP would allow starting at this log surface (Z, Δ).

To be more explicit, we define discrepancy for pairs (Z, α) which is not the usual discrepancy of the log surface (Z, Δ_α). The precise relation with the usual discrepancy is proved in 3.15.

Definition 3.13 Let (Z, α) be a pair and $f : Z' \to Z$ be a birational morphism. Write the associated log surfaces as $LR(Z, \alpha) = (Z, \Delta)$ and $LR(Z', \alpha) = (Z', \Delta')$. Let $\{E_i\}$ be the exceptional curves for the birational morphism $Z' \to Z$, and let e_i be the ramification index of α on E_i. Write the equation

$$K_{Z'} + \Delta' \equiv f^*(K_Z + \Delta) + \sum a_i E_i.$$

We define the *discrepancy* of (Z, α) to be $\mathrm{discrep}(Z, \alpha) = \inf\{a_i e_i\}$ where the minimum is taken over all birational maps to Z and all exceptional curves.

Note that the coefficient of E_i in Δ' depends on α so the discrepancy is not just a function of the associated log surface (Z, Δ).

Proposition 3.14 *Let Z be a normal model of K, then the pair (Z, α) is terminal if and only if* $\mathrm{discrep}(Z, \alpha) > 0$.

Proof. We first prove that if (Z, α) is terminal than the discrepancy is postive. Let $f : Z' \to Z$ be any birational morphism. It may be factored into a series of blow ups at points by theorem 3.8, so we only need to check the case of a single blow up at a point p. We write the equation

$$K_{Z'} + \Delta' \equiv f^*(K_Z + \Delta) + aE$$

where E is the exceptional curve. If p is not in D then $a = 1$. So suppose first that p is a smooth point of D. So we have that $\Delta' = f_*^{-1}\Delta$, the strict transform of Δ. After localising at p we may suppose that $\Delta = (1-1/e)D$ and we write F for the proper transform of the smooth curve D through p, recall that $F \equiv -E$. So we have that $K_{Z'} \equiv E$, $\Delta' = (1 - 1/e)F \equiv -(1 - 1/e)E$ and $f^*(K_Z + \Delta) = 0$. So $ae = 1$.

Now consider the case that p is a node of D and the cyclic covers have ramification index e and degrees e and ne. Let F, F' be the proper transform of the branches. Then α will ramify on the exceptional curve with degree e. Since $K_{Z'} \equiv E$, and

$$\Delta' = (1 - 1/e)F + (1 - 1/ne)F' + (1 - 1/e)E \equiv (1/ne - 1)E$$

we get that $ae = 1/n$.

Next we prove the converse, so we assume that the pair $\mathrm{discrep}(Z,\alpha) > 0$. Let $Z' \to Z$ be a terminal resolution of (Z,α). Write the equation

$$K_{Z'} + \Delta' \equiv f^*(K_Z + \Delta) + \sum a_i E_i$$

and take the intersection product with $\sum a_i E_i$. Since the matrix $(E_i.E_j)$ is negative definite and $E_i.f^*C = 0$, we get

$$(K_{Z'} + \Delta').\sum a_i E_i = \left(\sum a_i E_i\right)^2 < 0.$$

So some $(K_{Z'} + \Delta').E_j < 0$ and $E_j^2 < 0$. So we can apply the contraction theorem 3.10 to contract E_j to get something smaller than (Z',α) that is still terminal. So we are done by induction. □

Let Z be a normal model of K. We say that (Z,α) is *canonical, log terminal, log canonical,* if $a = \mathrm{discrep}(Z,\alpha)$ satisfies $a \geq 0$, $a > -1$ $a \geq -1$ respectively.

We will not use canonical, log terminal, or log canonical pairs in this paper. However, we will need the fact that terminal pairs yield klt log surfaces, which follows from the next more general proposition. We first recall the standard definition of discrepancy for log surfaces.

Let (Z,Δ) be a log surface and let $Z' \to Z$ be a birational morphism with exceptional curves $\{E_i\}$. Write the equation

$$K_{Z'} + f_*^{-1}\Delta \equiv f^*(K_Z + \Delta) + \sum b_i E_i,$$

where $f_*^{-1}\Delta$ is the strict transform of Δ. The discrepancy of (Z,Δ) is $\mathrm{discrep}(Z,\Delta) = \inf\{b_i\}$ where the minimum is taken over all birational morphisms $Z' \to Z$ and all exceptional curves. If $\lfloor\Delta\rfloor = 0$ and $\mathrm{discrep}(Z,\Delta) > -1$ then (Z,Δ) is called klt (Kawamata log terminal).

Proposition 3.15 *The log surface (Z,Δ_α) is klt (log canonical) if and only if (Z,α) is log terminal (log canonical).*

Proof. Corollary 2.32 (2) of [KM] says that there exists a log resolution of the log surface (Z,Δ) which can be used to compute its discrepancy. Let $f : Z' \to Z$ be a resolution, and let $LR(Z',\alpha) = (Z',\Delta')$. Write the equations

$$K_{Z'} + \Delta' \equiv f^*(K_Z + \Delta) + \sum a_i E_i,$$

$$K_{Z'} + f_*^{-1}\Delta \equiv f^*(K_Z + \Delta) + \sum b_i E_i,$$

where the $f_*^{-1}\Delta$ is the strict or proper transform of Δ, and the $\{E_i\}$ are the exceptional curves of the morphism $Z' \to Z$. The numbers a_i measure the discrepancy as a pair while the numbers b_i give the usual discrepancy of the log surface (Z, Δ). Cancelling like terms yields the equation

$$\Delta' - f_*^{-1}\Delta = \sum (a_i - b_i)E_i.$$

On any given curve E_i the coefficient of Δ' is of the form $(1 - 1/e_i)$ for some number $e_i \geq 1$. So we have that $1 - 1/e_i = a_i - b_i$. So we have $a_i + 1/e_i = b_i + 1$ and so $a_i > -1/e_i$ if and only if $b_i > -1$ and $a_i \geq -1/e_i$ if and only if $b_i \geq -1$. □

We need the condition $\mathrm{discrep}(Z, \alpha) > 0$ to show the existence of unique minimal models.

Lemma 3.16 *Let (Z, α) be a terminal pair and let E, F be irreducible curves in Z such that $F.(K_Z + \Delta) < 0$ and $E.(K_Z + \Delta) < 0$ and $E^2 < 0$. Let $\pi : Z \to Z'$ be the contraction of E. If $E \neq F$ then $\pi_* F.(K_{Z'} + \Delta') < 0$, and $(\pi_* F)^2 \geq F^2$.*

Proof. We have that $K_Z + \Delta = \pi^*(K_{Z'} + \Delta') + aE$ where $a > 0$. We simply compute

$$\pi_* F.(K_{Z'} + \Delta') = F.\pi^*(K_{Z'} + \Delta) = F.(K_Z + \Delta - aE)$$

However $F.E \geq 0$ and $F.(K_Z + \Delta) < 0$. We also can compute

$$\pi_* F.\pi_* F = F.\pi^*\pi_* F = F.(F + (E.F)E) = F^2 + (E.F)^2 \geq F^2.$$

□

Proposition 3.17 *Let $\pi : Z \to Z'$ be a birational morphism of surfaces such that (Z', α) is minimal and $K_{Z'} + \Delta_\alpha$ is nef. Then Z' is uniquely determined by (Z, α).*

Proof. Let $Z \to Y$ be the blow down that contracts the curve F. Let E be the exceptional curve of the map $Z \to Z'$. If $F \not\subseteq E$ then we would have that $\pi_* F.(K_{Z'} + \Delta) < 0$ by the above lemma. So we must have that $F \subseteq E$ and so F will be contracted by π. So that contraction π will factor through Y so we are done by induction on the number of components of E. □

3.5 Minimal Model Program

In this section we prove the results of the minimal model program in our situation. We use some important theorems from surface theory. We use the cone theorem for klt surfaces [KM] and Castelnuovo's characterisation of $\mathbb{P}^2$. In this subsection, all surfaces will be projective.

The following corollary of Castelnuovo contraction says that a $(K + \Delta)$-negative extremal curve for the log surface (Z, Δ) is also K-negative for the surface Z.

Corollary 3.18 *Let (Z, α) be a terminal pair with $LR(Z, \alpha) = (Z, \Delta)$ and let E be an extremal irreducible curve such that $(K_Z + \Delta).E < 0$. Then $K_Z.E < 0$ and E generates an extremal ray of the cone of curves of Z.*

Proof. If $E^2 < 0$ then by theorem 3.10 we have that $E.K_Z < 0$. If $E^2 \geq 0$ then since E is irreducible and Δ is effective we have $E.\Delta \geq 0$. So if $E.(K_Z + \Delta) < 0$ we get that $E.K_Z < 0$. □

The following theorem describes the three possible types of contractions. In the proof we use a description of the possible contractions for an extremal curve on a surface as in [KM] 1.28. The isomorphism to $\mathbb{P}^2$ in the del Pezzo case relies on Castelnuovo's characterisation of $\mathbb{P}^2$ as used in [KM] 1.28. The theorem below includes definitions of the types of contractions.

Theorem 3.19 *Let (Z, α) be a terminal pair. Then either $K_Z + \Delta$ is nef or there exists an extremal curve E such that $E.(K_Z + \Delta) < 0$ and one of the following occurs.*

$E^2 < 0$ **: Blowing down:** *E is a (-1)-curve and $\pi : Z \to Z'$ contracts exactly E.*

$E^2 = 0$ **: Ruled:** *$\pi : Z \to C$ is a ruled surface with E a fibre, and $-(K_Z + \Delta)$ is relatively ample for the map π.*

$E^2 > 0$ **: del Pezzo:** *$Z \simeq \mathbb{P}^2$ and $-(K_Z + \Delta)$ is ample.*

Proof. Suppose that $K_Z + \Delta$ is not nef. So by the cone theorem for klt log surfaces we can find an irreducible curve E so that $(K_Z + \Delta).E < 0$ and E generates an extremal ray in the cone of curves of (Z, Δ). By corollary 3.18 we know that E is an extremal curve for Z and $E.K_Z < 0$. So we may apply theorem [KM] 1.28 to obtain a contraction of Z. Theorem 3.10 gives us the first statement, and the rest follow immediately. □

Corollary 3.20 *Suppose that* (Z', α) *is a terminal pair. Then there is a sequence of blow downs of* (-1)*-curves*

$$Z' \to Z_1 \to \ldots \to Z_n = Z$$

so that one of the following holds:

- $K_Z + \Delta$ *is nef and* (Z, α) *is the unique minimal model,*
- (Z, α) *is ruled with* $\rho(Z) = 2$*,*
- (Z, α) *is del Pezzo with* $\rho(Z) = 1$*.*

We shall classify all possible pairs in the last two cases by classifying their ramification data. First consider the case of minimal del Pezzo pairs. We will be studying del Pezzo pairs that are terminal, but a more general class of del Pezzo pairs with singularities are classified in [CK].

Proposition 3.21 *Let* (Z, α) *be a minimal del Pezzo pair. Then* $Z \simeq \mathbb{P}^2$ *and all the cyclic covers of the components of* D *have the same degree* e*. The numbers* $d = \deg D$ *and* e *satisfy the inequality* $3 > (1 - 1/e)d$*.*

Proof. This result can be obtained by reading off the terminal orders described in Proposition 24 of [CK]. However, the proof simplifies immensely under the terminal condition so we shall give this proof here. We know that $Z \simeq \mathbb{P}^2$ and that $-(K + \Delta)$ is ample. Let H be the class of a line on $\mathbb{P}^2$ and write d_i for the degrees of the components D_i of D. So we can write $-(K + \Delta) \equiv 3H - \sum(1 - 1/e_i)d_i H$, so

$$3 > \sum(1 - 1/e_i)d_i \qquad (*).$$

The proposition thus follows if we can show that all the ramification indices e_i are all equal to e. Now the terminal condition and the fact that any two curves in $\mathbb{P}^2$ intersect show that we may re-index the e_i so that $e_1 \geq e_2 \geq \ldots$ and that further e_{i+1} divides e_i. Arguing by way of contradiction, let j be the smallest index for which $e_j < e_1$. Now if $i \geq i'$ then the cyclic covers of D_i and $D_{i'}$ ramify at their points of intersection with ramification index $e_{i'}$.

Furthermore, if D_i is rational then the associated cyclic cover is ramified at at least two points and e_i is the lowest common multiple of the ramification indices. In fact, in view of the divisibility relationship amongst the e_l's and the exact sequence

$$0 \to H^1(k(\mathbb{P}^1), \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{p \in \mathbb{P}^1} \mu^{-1} \to \mu^{-1} \to 0,$$

we see that there are at least two points in D_i above which the cyclic cover of D_i ramifies with ramification index e_i. It follows that $d_1 + \ldots + d_{j-1} \geq 3$. Now $e_1 \geq 4$ since it is a composite number so (*) forces $d_1 + \ldots + d_{j-1} = 3$. In fact, (*) shows that there can only be one ramification curve whose ramification index is not e_1, namely, D_j and that furthermore, $d_j = 1, e_j = 2$ or 3. We arrive at a contradiction when $e_j = 2$ since there are no double covers of $\mathbb{P}^1$ ramified at three points. Also, $e_j = 3$ contradicts (*) as then $d_1 \geq 2e_j = 6$. The proposition now follows.

□

So we may classify the minimal del Pezzo pairs according to the data of Z, $d = \deg D$ and the ramification index e. The possible types are all defined in the table below, and in the case where $\deg D = 5$ we separate the pairs into two possible cases depending on whether the theta characteristic of the double cover $\tilde{D} \to D$ is even or odd. The $+/-$ indicates the parity of the theta characteristic.

type	$\deg D$	e
F_3^e	3	≥ 2
F_4^3	4	3
F_4^2	4	2
F_5^{2+}	5	2
F_5^{2-}	5	2

In the ruled case we can break the analysis into two cases. Let (Z, α) be a minimal ruled pair with ruling map $\pi : Z \to C$. Since $(K + \Delta).E < 0$ and $E.K = -2$ we get that $E.\Delta < 2$, so we may conclude that $E.D = 2$ or $E.D = 3$. If $E.D = 3$ then all the ramification indices must be $e = 2$. If $E.D = 2$ then D is either a union of fibres and an irreducible bisection of π, or a union of fibres and two sections of π. In the latter case, we claim that the ramification indices along the two sections is the same. Indeed, if C is rational then this follows as the ramification indices of the associated cyclic covers of these sections is just the ramification indices of α along fibres. When C is irrational, then the Artin-Mumford sequence (see Theorem 3.1) has cohomology $H^3(Z,\mu) \simeq H^1(C,\mu)$. Now the ramification of α along the two sections must cancel each other in $H^1(C,\mu)$ so their associated cyclic covers must certainly have the same degree. Let e be the ramification index of the curve D with any fibres removed. We have the following possibilities for the types of ruled orders.

type	$D.F$	e
R_2^3	2	≥ 2
R_3^2	3	2

3.6 Non Closed Base Fields

In this section we consider the minimal model program for pairs (Z,α) over a general field k. We assume as before that the order of α is prime to the characteristic of k. We will use the usual notation of $\bar{k}, \bar{Z}, \bar{E}$ and so on when passing to the algebraic closure. We will note how the results of the previous sections change when we pass to the algebraic closure. We say that (Z,α) is terminal if $(\bar{Z},\bar{\alpha})$ is terminal. Note that the ramification data $R(Z,\alpha)$ is well defined over k. However, the Artin-Mumford sequence only holds over $\bar{k}$ so we will only use cancellation of ramification of cyclic covers over $\bar{k}$. It is clear that we can resolve singularities to a terminal pair over k.

So if we start with a terminal pair (Z,α), the same arguments as above show that we have a unique minimal model when $K_Z+\Delta$ is nef. When $K_Z+\Delta$ is not nef, the cone theorem for log surfaces over general fields shows us that we have a k-irreducible extremal curve E.

Theorem 3.22 *Let (Z,α) be a terminal pair and suppose that we have a k-irreducible curve E such that $E.(K_Z+\Delta)<0$. Then one of the following occurs.*

- *If $E^2<0$ then $\bar{E}$ is a disjoint union of (-1)-curves and there is a contraction $Z \to Z'$ that blows down exactly E and the pair (Z',α) is terminal.*

- *If $E^2 = 0$ then there is a morphism $Z \to C$, and Z is a conic bundle over C and E is a fibre.*
- *If $E^2 > 0$ then Z is a del Pezzo surface with $-(K_Z + \Delta)$ ample.*

Proof. In the first case we write $\bar{E} = \sum \bar{E}_i$, and by 3.10 we know that each $\bar{E}_i$ is a (-1)-curve. So since $E^2 = (\sum \bar{E}_i)^2 < 0$, and each $\bar{E}_i^2 = -1$, we can use the action of the Galois group Gal $\bar{k}/k$ to show that the curves must be disjoint as in the proof of Theorem 2.7 of [Mo]. The last two cases follow since when $E^2 > 0$ we see that $(K_Z + \Delta).E < 0$ yields $K_Z.E < 0$, so we have an extremal curve E and we may apply loc. cit. to obtain the result. □

We use Iskovskih's classification of minimal models of surfaces over arbitrary fields [I1].

Proposition 3.23 *Let (Z, α) be a minimal del Pezzo pair. Then the cyclic covers of the components of D have the same degree e. Write $-K_Z = nH$ for an ample divisor H that generates* $\operatorname{Pic} Z$*, and let $D = dH$. Then the numbers d, n, e satisfy the inequality $n > d(1 - 1/e)$.*

Proof. It is shown in [I1] that minimal del Pezzo surfaces Z with $\rho(Z) = 1$ are either $\mathbb{P}^2$, or a quadric in $Q \subset \mathbb{P}^3$, or $\operatorname{Pic} Z$ is generated by $-K_Z$. We study each of these cases. In the case that $Z \simeq \mathbb{P}^2$ the argument proceeds as in the proof of theorem 3.21. If $Z \simeq Q \subset \mathbb{P}^3$, a quadric, we have that $n = 2$. In this case one may check that there is no possible ramification curve such that $D \equiv H$. An analysis of possible component degenerations shows that all ramification indices are equal when $D = 3H$.

Lastly, if $n = 1$ we must have that $D \equiv -K$ and so no analysis of special cases is necessary. □

The results of the above analysis are summarised in the following table.

$Z \simeq \mathbb{P}^2$		$Z \simeq Q \subset \mathbb{P}^3$		$\operatorname{Pic} Z \simeq \mathbb{Z}K_Z$	
$\deg D$	e	$\deg D$	e	$\deg D$	e
3	≥ 2	2	≥ 2	1	≥ 2
4	2,3	3	2		
5	2				

Similarly, if Z is a conic bundle with no section, we see that $1 > d(1-1/e)$ where $D = -dK_Z$ in the relative Picard group. So we have the following possibilities, where F is a fibre.

Z ruled		Z conic bundle	
$D.F$	e	$D.F$	e
2	≥ 2	2	≥ 2
3	2		

4 Minimal Model Program for Orders

In this section we tie together the previous two parts to obtain the minimal model program for orders over surfaces. We will apply the results of the first part to obtain resolution of singularities for orders. Then we apply the second part to carry out the MMP for orders. As before, all surfaces in this section are assumed to be projective.

We begin by rephrasing our definitions in geometric language.

Definition 4.1 Let Z be a normal surface with function field K. Let A be a central simple K-algebra. An order $\mathcal{O}_X$ over Z is a coherent sheaf of central $\mathcal{O}_Z$-algebras with an injective algebra map $\mathcal{O}_X \hookrightarrow A$ such that $\mathcal{O}_X \otimes_Z K$ is naturally isomorphic to A. We will write $X = \mathcal{S}pec_Z \mathcal{O}_X$ for the ringed space $(\mathcal{O}_X, Z)$ and often refer to X as an order too. Let $k(X) = A$ and let $Z(X) = Z$ denote the centre of X. In this section we will work over an algebraically closed field of characteristic prime to the degree of $\mathcal{O}_X$.

Let $\operatorname{Mod} X$ denote the category of right $\mathcal{O}_X$-modules. We will usually consider orders up to Morita equivalence within their quotient rings. By this we just mean, a Morita equivalence between rings with isomorphic rings of fractions. Let X, X' be orders in A. There is an equivalence of categories $\operatorname{Mod} X \simeq_{\mathrm{M}} \operatorname{Mod} X'$ if and only if there exists a $X-X'$ bimodule that is locally a finitely generated projective generator on both sides [AZ] proposition 8.

Given two orders X, Y, we will write $f : X \to Y$ for a morphism of ringed spaces given by a map $f_Z : Z(X) \to Z(Y)$ and a morphism of sheaves of algebras $f^* : f_Z^* \mathcal{O}_Y \to \mathcal{O}_X$. Any order has a canonical structure map to its centre, $\pi : X \to Z$ where f_Z is the identity and $f^* : \mathcal{O}_Z \to \mathcal{O}_X$ is just the algebra structure map. More generally we say that $X \to Y$ is a morphism of orders if there are Morita equivalences $X' \simeq_{\mathrm{M}} X$ and $Y' \simeq_{\mathrm{M}} Y$ and a map of ringed spaces $X' \to Y'$. A morphism of orders is a birational morphism if the map f_Z is a birational morphism, and the Morita equivalences and f^* induce an isomorphism $k(X) \simeq k(Y)$.

We record the following statement which equates the ramification data of a maximal order X with the data $R(Z, \alpha)$ considered in the second part. It is stated in [AM], and proved in [AdJ], section 2.7.

Proposition 4.2 *Let X be a maximal order over a surface Z. Let α be the Brauer class represented by $k(X)$. Then $R(X) = R(Z, \alpha)$.*

Now that we know the ramification data coincide we have the equivalence of three different characterisations of terminal.

Corollary 4.3 *Let X be a maximal order over a surface Z and let α be the Brauer class respresented by $k(X)$. Then X is terminal if it satisfies one of the three equivalent conditions*

- *X has ramification data satisfying the conditions in definition 2.5.*
- *X is étale locally isomorphic to $\Lambda(n, \zeta)^{m \times m}$.*
- $\mathrm{discrep}(Z, \alpha) > 0$*.*

The above corollary fails for canonical, log terminal or log canonical orders. Such orders are not determined étale locally by their ramification data.

Proposition 4.2 allows us to transfer concepts and definitions about pairs to orders. For example, an order X on a projective surface Z is minimal if the associated pair (Z, α) is minimal.

We now can reinterpret the results of the minimal model program in terms of orders. Let K be a field of transcendence degree two and let A be a central simple K-algebra with Brauer class α in $\mathrm{Br}\, K$. Given any model Z of K, we may choose a maximal order $\mathcal{O}_X$ in A. We will carry out the minimal model program starting with $\mathcal{O}_X$.

Consider the following operation on an order.

Definition 4.4 (Blowing up) Let X be a maximal order over a normal surface Z. Let p be a point in Z. Let $f : Z' \to Z$ be the blow up of Z at the point p. We define X' to be any maximal order containing the order $f^*\mathcal{O}_X$. We will allow a Morita equivalence in $k(X)$ at the beginning and in $k(X')$ at the end. So a *blow up at a point* $Y' \to Y$ consists of the composition of two Morita equivalences in the same ring of quotients $Y' \simeq_{\mathrm{M}} X'$ and $Y \simeq_{\mathrm{M}} X$ and the morphism $X' \to X$.

Note that this procedure is compatible with blowing up the associated pair. We also note that the choice of which maximal order contains the pull back is equivalent to choosing an irreducible representation of the order and carrying the blowing up construction of Van den Bergh as described in [VdB1].

Theorem 2.12 and Corollary 3.6 yield the following result.

Corollary 4.5 *Given any order X on a normal surface, there is a resolution of singularities $Y \to X$ to a terminal order Y obtained by blowing up as above. In particular, Y has global dimension two.*

We are now in a position to begin the minimal model program. Let X be a terminal order over a surface Z. We run the log minimal model program on the associated log surface (Z, Δ) to obtain an minimal model $\pi : (Z, \Delta) \to (Z', \Delta')$. We let $\mathcal{O}_{X'}$ be the reflexive hull of $\pi_* \mathcal{O}_X$. This is a minimal model of $k(X)$. We get the following corollary.

Corollary 4.6 *Any terminal order has a terminal minimal model, which is unique up to Morita equivalence in $k(X)$ if $K + \Delta$ is nef. Otherwise the minimal model is a ruled order or a del Pezzo order.*

References

[A1] M. Artin, Local structure of maximal orders on surfaces. Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), pp. 146–181, Lecture Notes in Math., 917, Springer, Berlin-New York, 1982.

[A2] M. Artin, Geometry of quantum planes. Azumaya algebras, actions, and modules (Bloomington, IN, 1990), 1–15, Contemp. Math., 124, Amer. Math. Soc., Providence, RI, 1992.

[A3] M. Artin, Two-dimensional orders of finite representation type, Amnuscripta Math. 58 (1987), 445-471

[A4] Artin, M. Some problems on three-dimensional graded domains. Representation theory and algebraic geometry (Waltham, MA, 1995), 1–19, London Math. Soc. Lecture Note Ser., 238, Cambridge Univ. Press, Cambridge, 1997.

[AdJ] M. Artin, A. J. deJong, Stable Orders over Surfaces, preprint.

[AM] M. Artin, D. Mumford, Some elementary examples of unirational varieties which are not rational. Proc. London Math. Soc. (3) 25 (1972), 75–95. Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[AZ] M. Artin, J. J. Zhang, Noncommutative projective schemes. Adv. Math. 109 (1994), no. 2, 228–287.

[AG] Auslander, Goldman, Maximal Orders. Trans. Amer. Math. Soc. 97 1960 1–24.

[HN] H. Hijikata, K. Nishida, When is $\Lambda_1 \otimes \Lambda_2$ hereditary? Osaka J. Math. 35 (1998), no. 3, 493–500.

[I1] V. A. Iskovskih, Minimal Models of rational surfaces over arbitrary fields. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 1, 19–43, 237.

[CK] D. Chan, R. Kulkarni, Del Pezzo Orders on Projective Surfaces, Advances in Mathematics 173 (2003), 144-177.

[KM] J. Kollár, S. Mori, Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998.

[Mo] S. Mori, Threefolds whose canonical bundles are not numerically effective, Annals of Mathematics 116 (1982), 133-176.

[Ra] M. Ramras, Orders with finite global dimension. Pacific J. Math. 50 (1974), 583–587.

[R] I. Reiner, Maximal orders. London Mathematical Society Monographs, No. 5. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], London-New York, 1975.

[S] J.-P. Serre, Local class field theory. 1967 Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)

[SZ] J. T. Stafford, J. J. Zhang, Homological properties of (graded) Noetherian PI rings. J. Algebra 168 (1994), no. 3, 988–1026.

[VdB1] M. Van den Bergh, Blowing up of non-commutative smooth surfaces. Mem. Amer. Math. Soc. 154 (2001), no. 734,

[YZ] A. Yekutieli, J. J. Zhang, Serre duality for noncommutative projective schemes. Proc. Amer. Math. Soc. 125 (1997), no. 3, 697–707.

Daniel Chan
University of New South Wales

Colin Ingalls
University of New Brunswick