Splitting Bundles over Hereditary Orders

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Abstract

Let A be a hereditary order on the projective line which ramifies at 2 or fewer points. We show that any locally projective A-module is a direct sum of minimal rank locally projective A-modules. Furthermore, we show that this property fails for all other hereditary orders on smooth projective curves.

Throughout, all objects and maps are assumed to be defined over some algebraically closed base field k.

1 Introduction

The study of the moduli space of vector bundles on curves has been an active field of research over the last few decades. One expects that the corresponding noncommutative analogue should be equally interesting. There are various appoaches one can take to defining an appropriate noncommutative analogue of vector bundles on curves. We will look at locally projective modules over hereditary orders on curves.

A basic result in the commutative theory is the following splitting theorem of Grothendieck.

Theorem 1.1 *Any vector bundle on $\mathbb{P}^1$ decomposes as the direct sum of line bundles.*

In this short note, we investigate the question: In the noncommutative case, when is there a splitting theorem à la Grothendieck? Our answer is

Theorem 1.2 *Let A be a hereditary order on a smooth projective curve C. Then every locally projective A-module splits as a direct sum of minimal rank locally projective A-modules if and only if $C \simeq \mathbb{P}^1$ and A is ramified at two or fewer points.*

There is a notion of a canonical sheaf ω for orders and one can ask if it is anti-ample (see [CK, definitions 4 and 7]). If this is the case, we will say the order is Fano. Any hereditary order on $\mathbb{P}^1$ ramified on two or fewer points is Fano in this sense so the above result is rather reassuring. Curiously though, a hereditary order ramified at three points each with ramification index 2 is also Fano.

The study of locally projective modules is related to the study of parabolic bundles on curves and this is one of the motivations for this note. Indeed, by [CI] there is a category equivalence between modules over a hereditary order A on a smooth projective curve C and quasi-coherent sheaves on a corresponding Deligne-Mumford stack. Also, [Biswas] has shown that there is a corresondence between orbifold bundles and certain parabolic bundles. Alternatively, [RVdB] has shown that hereditary orders on the projective line are Morita equivalent with weighted projective lines and by [Lenzing], modules on the latter are related to parabolic bundles.

The parabolic bundle version of theorem 1.2 seems to be known but I have been unable to find an explicit proof of the result. Regardless, I hope that this short note will encourage interaction between the study of hereditary orders and parabolic bundles. Indeed, there are some differences in the theories which may indicate some interesting interplay. For example, categorical notions are immediate in the theory of modules over hereditary orders whereas it seems that the category of parabolic sheaves is not so obvious and took a while to develop. Also, there are distinct hereditary orders on curves which are Morita equivalent. However, the moduli problem for different orders in the same Morita equivalence class looks different a priori.

2 Positive Genus Case

Let C be a smooth curve and A a hereditary order on C. We shall denote the structure sheaf on C by $\mathcal{O}$. A *locally projective A-module* is a coherent A-module P such that on any affine open set $U \subset C$, $P(U)$ is a projective $A(U)$-module. Since A is hereditary, this is equivalent to the fact that P is a torsion-free $\mathcal{O}$-module by [Reiner; Corollary 10.7].

We define the *degree* and *rank* of an A-module to be its degree and rank as an $\mathcal{O}$-module. Suppose that P is a locally projective A-module. Let $K = K(C)$ be the function field of C. Tsen's theorem shows that $A \otimes K = K^{N \times N}$ for some N. We adopt here the convention that the unadorned tensor symbol means $- \otimes_{\mathcal{O}} -$. Now $P \otimes K$ is a $K^{N \times N}$-module so the rank of P is a multiple of N. Locally projective modules of rank N will be said to be *minimal rank*.

Proposition 2.1 *Let A be a hereditary order on a smooth projective curve C of positive genus. Then there exist indecomposable locally projective A-modules which are not minimal rank.*

Proof. We may embed A in a maximal order B. Since the Brauer group of a curve is trivial by Tsen's theorem, $B = \mathcal{E}nd\, V$ for some vector bundle V over C. Now V is a minimal rank locally projective A-module. At the generic point of C, the sheaf of homomorphisms $\mathcal{H}om_A(V, V)$ is just K so $\mathcal{H}om_A(V, V) = \mathcal{O}(D)$ for some effective divisor D. Since $\mathcal{O}(D)$ is also a sheaf of subalgebras of K, we see that $D = 0$ so $\mathcal{H}om_A(V, V) = \mathcal{O}$.

The local-global ext spectral sequence shows that

$$\mathrm{Ext}^1_A(V, V) = H^1(C, \mathcal{H}om_A(V, V)) = H^1(C, \mathcal{O}) \neq 0.$$

Hence there is a non-split exact sequence of A-modules of the form

$$0 \longrightarrow V \longrightarrow P \longrightarrow V \longrightarrow 0 \tag{1}$$

We wish to show that P is indecomposable. Suppose on the contrary that $P = P_1 \oplus P_2$ where P_1, P_2 are minimal rank locally projective A-modules. If the composite map $P_i \longrightarrow P \longrightarrow V$ is zero then $P_i \longrightarrow P$ factors through $V \longrightarrow P$. Either way P_1, P_2 embed in V. Dually, we see that V embeds in P_1, P_2. This forces $\deg P_i = \deg V$ so the embeddings are in fact isomorphisms and sequence (1) splits giving a contradiction. This completes the proof of the proposition.

3 Locally Projective Modules on Hereditary orders on Curves

Let $e > 2$ be an integer and $p \in C$. Consider the hereditary order

$$A^{ep} := \begin{pmatrix} \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{O}(-p) & \mathcal{O} & \dots & \mathcal{O} \\ \vdots & \ddots & & \vdots \\ \mathcal{O}(-p) & \dots & \mathcal{O}(-p) & \mathcal{O} \end{pmatrix} \subset \mathcal{O}^{e \times e}.$$

We say that a hereditary order A on C is *ramified at p with ramification index e* if étale locally at p, $A \simeq (A^{ep})^{l \times l}$ for some $l \in \mathbb{N}$. a hereditary order ramifies at a finite number of points and the collection of these points with their ramification indices is called the *ramification data*. As is well known, see for example [CI], Morita equivalence classes of hereditary orders on smooth curves correspond precisely to their ramification data.

Having dealt with the positive genus case, we assume from now on that $C = \mathbb{P}^1$. For $i \in \mathbb{Z}$ we write $i = ne - k$ where n, k are integers with $0 \leq k < e$. Let $P(ip)$ be the locally projective A^{ep}-module

$$P(ip) := \begin{pmatrix} \mathcal{O} \\ \vdots \\ \mathcal{O} \\ \mathcal{O}(-p) \\ \vdots \\ \mathcal{O}(-p) \end{pmatrix} \otimes \mathcal{O}(np) \subset K^e$$

where k is the number of $\mathcal{O}(-p)$'s occurring in the column vector above. The notation has been chosen so that $\deg P(ip) = i$. Note that $P((i+e)p) \simeq P(ip) \otimes \mathcal{O}(1)$.

Proposition 3.1 *For $A = A^{ep}$ we have*

$$\mathcal{H}om_A(P(ip), P(jp)) = \mathcal{O}(\lfloor \frac{j-i}{e} \rfloor p).$$

Proof. Away from p, A^{ep} is just the matrix algebra so the hom sheaf is $\mathcal{O}$ on the complement of p. Hence $\mathcal{H}om_A(E_i, E_j) = \mathcal{O}(np)$ for some n and a local computation shows n to be as above.

Suppose A is a hereditary order ramified at $S = \{p_1, \ldots, p_l\}$ with ramification indices $e_1, \ldots, e_l$. Since we are interested in splitting locally projective modules, we may pass to a Morita equivalent order and so assume that $A = A^{e_1 p_1} \otimes \ldots \otimes A^{e_l p_l}$. We shall write $P_A(\sum_m i_m p_m)$ for the minimal rank locally projective A-module

$$P(i_1 p_1) \otimes \ldots \otimes P(i_l p_l).$$

We will drop the subscript A for the most part. A simple induction gives a formula for the degree

$$\deg P_A(\sum_m i_m p_m) = e \sum \frac{i_m}{e_m}, \quad \text{where } e = e_1 e_2 \ldots e_l.$$

We think of $\sum i_m p_m$ as a divisor D on C supported on S. Note that there are isomorphisms between locally projective modules of the form $P(D)$. For example,

$$P((i_1 + e_1)p_1 + i_2 p_2) \simeq P(i_1 p_1 + i_2 p_2) \otimes \mathcal{O}(1) \simeq P(i_1 p_1 + (i_2 + e_2)p_2) \tag{2}$$

Proposition 3.2 *Let P be a minimal rank locally projective A-module. Then $P \simeq P(D)$ for some divisor D supported on S.*

Proof. If P is a minimal rank locally projective A-module, it must be a submodule of $K^{e_1} \otimes \ldots \otimes K^{e_l}$. We argue by induction on l that it has the form described above. Let ε_j for $j = 1, \ldots, e_1$ be the diagonal idempotent of $A_1 := A^{e_1 p_1}$ with a single 1 in the j-th diagonal entry and 0's elsewhere. Note that $\varepsilon_j P$ is a minimal rank locally projective module over $A' := A^{e_2 p_2} \otimes \ldots \otimes A^{e_l p_l}$. Hence we may use induction to see that $\varepsilon_j P = \mathcal{O}(E_j) \otimes P_{A'}(D_j)$ for some divisor D_j supported on $S - p_1$ and some divisor E_j on C. We may alter D_j, E_j using (2) so that E_j is supported away from $S - p_1$.

Since P is closed under multiplication by $A^{e_1 p_1}$, we see that all the D_j are identical, say equal to D', and that for some r, we have

$$E_1 = E_2 = \ldots = E_{r-1} = E_r + p = \ldots = E_e + p.$$

Hence, for some i, $P = P_{A_1}(ip_1) \otimes P_{A'}(D') = P_A(ip_1 + D')$.

Proposition 3.3 *Suppose that A is a hereditary order on $C = \mathbb{P}^1$ which is ramified at three or more points. Then there exists an indecomposable locally projective A-module which is not of minimal rank.*

Proof. As before, we may assume that $A = A^{ep} \otimes A^{fq} \otimes A^{gr} \otimes A^{hs} \otimes \dots$ where $q, p, r, s \in C$ and of course, the A^{hs} term may not exist. We will assume that $e \geq f \geq g \geq h \geq \dots$ so that $\deg P(-p) > \deg P(-q-r)$. By proposition 3.1, we see that $\mathcal{H}om_A(P(-p), P(-q-r)) \simeq \mathcal{O}(-2)$. Hence,

$$\mathrm{Ext}^1_A(P(-p), P(-q-r)) = H^1(C, \mathcal{O}(-2)) = k$$

There exists consequently a non-split exact sequence of the form

$$0 \longrightarrow P(-q-r) \longrightarrow P \longrightarrow P(-p) \longrightarrow 0 \tag{3}$$

We wish to show that P is indecomposable. Suppose to the contrary that $P(D)$ is a direct summand of P where D is a divisor as in proposition 3.2. As before in the positive genus case, we have an embedding of the form $P(-q-r) \hookrightarrow P(D)$ or $P(-p) \hookrightarrow P(D)$. Similarly, $P(D)$ embeds in either $P(-p)$ or $P(-q-r)$. Checking degrees shows that we obtain a contradiction unless $P(-q-r) \hookrightarrow P(D) \hookrightarrow P(-p)$. Modifying D as in equation (2) and applying proposition 3.1 to $\mathrm{Hom}(P(-q-r), P(D))$, we may assume that

$$-q-r \leq D \leq mp + (f-1)q + (g-1)r + hs + \dots$$

for m sufficiently large. This inequality is incompatible with the fact that $\mathrm{Hom}(P(D), P(-p)) \neq 0$. Hence, we conclude that P is indecomposable as desired.

4 Splitting Theorem

In this section we prove

Theorem 4.1 *Let A be a hereditary order on $C = \mathbb{P}^1$ which is ramified at two or fewer points. Then every locally projective A-module is the direct sum of minimal rank locally projective modules.*

Proof. We shall only prove the case where A is ramified at two points since the case where there is one ramification point is easier. We may change A by a Morita equivalence and so assume that $A \simeq A^{ep} \otimes A^{fq}$ where p, q are the ramification points and e, f are the ramification indices.

Let P be a locally projective A-module. We argue by induction on the rank of P. Recall that $H^i(C, -)$ is also the derived functor of $\mathrm{Hom}_A(A, -)$ by adjunction. Pick $n \in \mathbb{N}$ large enough so that $\mathrm{Hom}_A(A, P(n)) = H^0(C, P \otimes \mathcal{O}(n)) \neq 0$. Since A is a direct sum of $P(ip+jq)$'s, there exist i, j and a non-zero morphism $P(ip+jq) \longrightarrow P$. Since $P(ip+jq)$ has minimal rank, this gives rise to an exact sequence of A-modules

$$0 \longrightarrow P(ip+jq) \longrightarrow P \longrightarrow Q \longrightarrow 0. \tag{4}$$

Among all such sequences, pick one such that $fi + ej = \deg P(ip+jq)$ is maximal. This exists since the degree of locally free $\mathcal{O}$-submodules of P is bounded. As in the commutative case, this implies that Q is torsion-free and so by induction is a direct sum of modules of the form $P(i'p+j'q)$. We first prove

Lemma 4.2 *For any direct summand $P(i'p+j'q)$ of Q we have*

$$\deg P(ip+jq) \geq \deg P(i'p+j'q).$$

Proof. Note first that

$$\mathcal{H}om_A(P(rp+sq), P(r'p+s'q)) = \mathcal{O}(\lfloor \frac{r'-r}{e} \rfloor + \lfloor \frac{s'-s}{f} \rfloor)$$

by proposition 3.1. We shall argue by contradiction and assume the lemma is false, that is, $fi' + ej' > fi + ej$ or equivalently,

$$f(i'-i) > e(j-j'). \tag{5}$$

By replacing (i', j') with $(i'+ne, j'-nf)$ as in proposition 3.2, we may assume that $1 \le i'-i \le e$ so that $\lfloor \frac{i-i'}{e} \rfloor = -1$. Let $j'' = \min\{j, j'\}$. Note that

$$\deg P(i'p + j''q) = fi' + ej'' > fi + ej = \deg P(ip + jq)$$

so it suffices to show that $\mathrm{Hom}_A(P(i'p+j''q), P) \neq 0$. We apply $\mathrm{Hom}_A(P(i'p+j''q), -)$ to the exact sequence (4) above. Note first that as $j'' \le j'$ we have $\mathrm{Hom}_A(P(i'p+j''q), P(i'p+j'q)) \neq 0$ and hence $\mathrm{Hom}_A(P(i'p+j''q), Q) \neq 0$. By the long exact sequence arising from (4) it suffices to show that $\mathrm{Ext}^q_A(P(i'p+j''q), P(ip+jq)) = 0$ for $q = 0, 1$ or equivalently, that $\mathcal{H}om_A(P(i'p+j''q), P(ip+jq)) = \mathcal{O}(-1)$ by the local-global Ext spectral sequence. Now (5) and our condition on $i'-i$ shows that $j - j' < f$ so $0 \le j - j'' < f$. Hence

$$\mathcal{H}om_A(P(i'p+j''q), P(ip+jq)) = \mathcal{O}(\lfloor \frac{i-i'}{e} \rfloor + \lfloor \frac{j-j''}{f} \rfloor) = \mathcal{O}(-1).$$

This yields the desired contradiction.

Lemma 4.3 *For any direct summand $P(i'p+j'q)$ of P we have*

$$Ext^1_A(P(i'p+j'q), P(ip+jq)) = 0$$

Proof. By the previous lemma, we know $fi' + ej' \le fi + ej$. Rearranging we find

$$0 \le \frac{i-i'}{e} + \frac{j-j'}{f}$$

and hence

$$-2 < \lfloor \frac{i-i'}{e} \rfloor + \lfloor \frac{j-j'}{f} \rfloor =: n.$$

Now $\mathcal{H}om_A(P(i'p+j'q), P(ip+jq)) = \mathcal{O}(n)$ so taking cohomology shows that $\mathrm{Ext}^1_A(P(i'p+j'q), P(ip+jq)) = H^1(C, \mathcal{O}(n)) = 0$ as desired.

To finish the proof of the theorem, note that the previous lemma implies $\mathrm{Ext}^1_A(Q, P(ip+jq)) = 0$ so the sequence (4) splits.

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