# PRE-BALANCED DUALIZING COMPLEXES

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Abstract. We study pre-balanced dualizing complexes over noncommutative complete semilocal algebras and prove an analogue of Van den Bergh's theorem [VdB, 6.3]. The relationship between pre-balanced dualizing complexes and Morita dualities is studied. Some immediate applications to classical ring theory are also given.

## 0. Introduction

The noncommutative dualizing complex in the sense of Yekutieli [Ye1] is a very useful tool in studying the homological properties of noncommutative rings. For example, noncommutative versions of the Auslander-Buchsbaum formula, Bass theorem, the no-holes theorem can be proved by using dualizing complexes (see [Jo1, Jo2, Jo3] in the graded case, and [WZ1, WZ2, WZ3] in the ungraded case). Other applications of dualizing complexes can be found in the work of Yekutieli [Ye1, Ye2, Ye3, YZ1, YZ2].

The dualizing complex is equivalent to the cotilting bimodule complex defined by Miyachi in [Mi], where he studied Morita duality theory for derived categories.

The main existence theorem for dualizing complexes is due to Van den Bergh [VdB, 6.3]. Van den Bergh's result was generalized from the graded case to the complete local case in [WZ2] and [Ch].

The dualizing complexes constructed by Van den Bergh's method have good properties such as bifiniteness and Cdim-symmetry [Theorem 1.5], which are important in studying some other ring-theoretic properties. Pre-balanced dualizing complexes [Definition 1.6] appear naturally in various ways and the existence of these is proved for several classes of algebras. The main purpose of this paper is to show that every pre-balanced dualizing complex over a complete semilocal ring is equivalent to one constructed by Van den Bergh's method. Consequently, such dualizing complexes have lots of good properties.

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**Theorem 0.1.** *Let $A$ be a left noetherian algebra and $B$ be a right noetherian algebra. Suppose $A$ and $B$ are semilocal and complete with respect to their Jacobson radicals. Let $R$ be a pre-balanced dualizing complex over $(A, B)$. Then the following assertions hold.*

(1) *There is a Morita duality between $A$ and $B$ induced by $R$.*
(2) *$A$ and $B^{\circ}$ have finite cohomological dimension and satisfy the left $\chi$ condition.*
(3) *$R$ is* Cdim *-symmetric.*
(4) *If $A$ and $B$ are noetherian and satisfy the $\chi$ condition, and if $A/\mathfrak{m}$ is weakly symmetric, then $R$ is bifinite.*

A basic ingredient in Van den Bergh's construction is Morita duality, which is a noncommutative version of Matlis duality. We prove part (1) by using *truncated Morita dualities* between the artinian algebras $A/\mathfrak{m}^n$ and $B/\mathfrak{n}^n$. Here $\mathfrak{m}$ and $\mathfrak{n}$ are the Jacobson radicals of $A$ and $B$ respectively. This idea was first used by Jategaonkar in [Ja]. Other parts follows from the local duality formula [Proposition 3.4] and results in [WZ2] and [Ch].

Jategaonkar also showed that every noetherian complete semilocal algebra $A$ with $A/\mathfrak{m}$ finite dimensional is Morita self-dual [Ja, 2.7]. This allows us to show an analogue of Van den Bergh's result [VdB, 6.3] and Yekutieli's result [Ye1, 4.10].

**Theorem 0.2 [Ch].** *Let $A$ be a complete noetherian algebra with Jacobson radical $\mathfrak{m}$ such that $A/\mathfrak{m}$ is finite dimensional over the base field. Then the following are equivalent:*

(1) *$A$ has a pre-balanced dualizing complex.*
(2) *$A$ has a balanced dualizing complex.*
(3) *$A$ satisfies the $\chi$ condition and has finite left and right cohomological dimension.*

The definition of a balanced dualizing complex is given in Definition 3.7 for the algebras which appear in Theorem 0.2. We have attempted to define the balanced condition for the algebras which appear in Theorem 0.1, but it seems that there is no way to define the balanced condition in general. Theorems 0.1 and 0.2 suggest that the pre-balanced condition is a good replacement for the balanced condition.

Theorems 0.1 and 0.2 have some immediate consequences. Note that not every noetherian semilocal complete ring has a Morita duality. Also, it is unknown if every noetherian semilocal complete PI ring is Morita self-dual. The next corollary gives a criterion for the existence of Morita self-duality.

**Corollary 0.3.** *Every AS-Gorenstein noetherian complete semilocal algebra is Morita self-dual.*

The property of Cdim -symmetry is crucial in the proof of the following corollary, which is an analogue of a result of Yekutieli's [YZ2, 6.23]. It is still an open question whether an AS-Gorenstein noetherian local ring has an artinian fraction ring.

**Corollary 0.4.** *If $A$ is an Auslander Gorenstein complete local noetherian ring, then $A$ has a QF artinian fraction ring.*

The following corollary is useful for studying the structure of the dualizing complex. See [YZ4] for some details.

**Corollary 0.5.** *Let $A$ be a noetherian complete algebra with Jacobson radical $\mathfrak{m}$ such that $A/\mathfrak{m}$ is finite dimensional over the base field. Suppose that $A$ has a balanced dualizing complex $R_A$. If $B$ is a factor ring of $A$, then $B$ has a balanced dualizing complex $R_B$ and*

$$R_B \cong \mathrm{RHom}_A(B, R_B) \cong \mathrm{RHom}_{A^\circ}(B, R_A)$$

*in $D(A \otimes A^\circ)$.*

In Section 1, we review some basic definitions concerning dualizing complexes and Morita duality. Section 2 contains some results about Morita duality. We prove Theorems 0.1 and 0.2 in Section 3 and prove Corollaries 0.3 and 0.4 in Section 4. In Section 5, we study the behavior of Morita duality and dualizing complexes under finite extensions. The proof of Corollary 0.5 is given in Section 5.

## 1. Definitions and Previous Results

In this section we will review some definitions about dualizing complexes and other definitions related to Van den Bergh's construction in the complete semilocal case. We refer to [Ha] for basic notions about complexes and derived categories.

Throughout the paper, we fix a base field $k$ and all objects will be assumed to be defined over $k$. Let $A$ be an algebra. The opposite ring of $A$ is denoted by $A^\circ$. Unless otherwise stated, we will work with left modules. We say an $A$-module is *finite* if it is finitely generated over $A$.

Let $D(A)$ ($D^b(A)$, $D^+(A)$ and $D^-(A)$ respectively) denote the derived category of (bounded, bounded below, bounded above, respectively) complexes of $A$-modules. Let $D_f(A)$ denote the derived category of complexes of $A$-modules with finite cohomology. The noncommutative version of a dualizing complex was introduced by Yekutieli.

**Definition 1.1 [Ye1] [YZ2].** Let $A$ be a left noetherian algebra and $B$ be a right noetherian algebra. An object $R \in D^b(A \otimes B^\circ)$ is called a *dualizing complex over* $(A, B)$ if it satisfies the following three conditions:

(1) $R$ has finite injective dimension over $A$ and $B^\circ$.
(2) $R$ has finite cohomology over $A$ and $B^\circ$.
(3) The canonical morphisms $B \longrightarrow \mathrm{R\,Hom}_A(R, R)$ and $A \longrightarrow \mathrm{R\,Hom}_{B^\circ}(R, R)$ are isomorphisms in $D(B \otimes B^\circ)$ and $D(A \otimes A^\circ)$ respectively.

When $A = B$, we say that $R$ is a *dualizing complex over* $A$.

When we say that $R$ is a dualizing complex over $(A, B)$, we will assume implicitly that $A$ is left noetherian and $B$ is right noetherian.

Van den Bergh proved the following remarkable theorem about the existence of dualizing complexes in the graded case.

**Theorem 1.2 [VdB, 6.3].** *Let $A$ be a noetherian connected graded algebra. Then $A$ has a balanced dualizing complex if and only if $A$ satisfies the $\chi$ condition and has finite left and right cohomological dimension.*

The terminology used above is defined in [Ye1] and [VdB]. We will review the ungraded versions of this terminology only, since we are primarily interested in complete semilocal rings.

The Jacobson radical of $A$ is denoted by $\mathfrak{m}$. We say that $A$ is *semilocal* if $A/\mathfrak{m}$ is a semisimple artinian ring. Left (or right) artinian rings are semilocal.

Let $(A, \mathfrak{m})$ be a left noetherian semilocal ring and let $A_0 = A/\mathfrak{m}$. We say that $A$ satisfies the *left* $\chi$ *condition* if $\operatorname{Ext}^i_A(A_0, M)$ is of finite length as an $A_0$-module for every $i$ and every finite $A$-module $M$. The right $\chi$ condition is defined similarly. If moreover $A$ is noetherian, we say that $A$ satisfies the $\chi$ *condition* when $A$ satisfies the left and the right $\chi$ condition. Stafford showed that noetherian semilocal PI algebras satisfy the $\chi$ condition [SZ1, 3.5]. However, not every noetherian local algebra satisfies $\chi$ [SZ2, 2.3] [WZ1, 9.4].

For any $A$-module $M$, the $\mathfrak{m}$-*torsion functor* $\Gamma_{\mathfrak{m}}$ is defined to be

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid \mathfrak{m}^n x = 0, \text{ for } n \gg 0\}.$$

The derived functor $\mathrm{R}\,\Gamma_{\mathfrak{m}}$ is defined on the derived category $D^+(A)$. We define the *ith local cohomology* of $X \in D^+(A)$ to be

$$\mathrm{H}^i_{\mathfrak{m}}(X) = \mathrm{R}^i\,\Gamma_{\mathfrak{m}}(X).$$

The *local cohomological dimension* of an $A$-module $M$ is defined to be

$$lcd(M) = \sup\{i \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\}.$$

The *cohomological dimension* of $A$ (or of $\Gamma_{\mathfrak{m}}$) is defined to be

$$cd(A) = \sup\{lcd(M) \mid \text{for all } A\text{-modules } M\}.$$

Obviously, $\Gamma_{\mathfrak{m}}(M) = \varinjlim \operatorname{Hom}_A(A/\mathfrak{m}^n, M)$, which implies that

$$\mathrm{H}^i_{\mathfrak{m}}(X) = \varinjlim \operatorname{Ext}^i_A(A/\mathfrak{m}^n, X)$$

for all $X \in D^+(A)$. Since $\mathrm{H}^i_{\mathfrak{m}}$ commutes with direct limits,, we have

$$cd(A) = \sup\{lcd(M) \mid \text{for all finite } A\text{-modules } M\}.$$

If $cd(A)$ is finite, then $cd(A) = lcd({}_AA)$.

There are two basic ingredients in Van den Bergh's construction [VdB]. One is local cohomology defined above. The other is graded Matlis duality. The ungraded version of Matlis duality is Morita duality, which we now review.

Let $B$ be another algebra and let ${}_AE_B$ be an $(A, B)$-bimodule. We say that $E$ induces a *Morita duality* between $A$ and $B$ if

(1) ${}_AE$ and $E_B$ are injective cogenerators in the categories of left $A$-modules and right $B$-modules, respectively;
(2) the canonical ring homomorphisms $A \to \operatorname{End} E_B$ and $B^\circ \to \operatorname{End}{}_AE$ are isomorphisms.

In this case we say that $A$ is *left Morita* and $B$ is *right Morita*, and that $A$ is *Morita dual to* $B$ (or $A$ and $B$ are in *Morita duality*). If $A = B$, then $A$ is Morita self-dual, or has a Morita self-duality. We refer to [AF, Xu] for some basic properties of a Morita duality.

Since a Morita duality is a duality between categories of modules and a dualizing complex induces a duality between derived categories, a dualizing complex can be viewed as a generalization of a Morita duality [Mi]. In fact if $A$ and $B$ are local and artinian, every dualizing complex is given by a Morita duality and a complex shift [WZ2, 3.7].

Graded Matlis duality (i.e. graded vector space duality) exists trivially. However, not every two-sided artinian algebra is left Morita [Xu, 2.9]. Some criteria for the existence of Morita dualities for artinian rings were worked out by Azumaya, Fuller, Jategaonkar, Morita, Xue and others. See [AF, Xu] for their results.

With an extra condition on $A/\mathfrak{m}$, a left artinian ring $A$ is left Morita if and only if $A$ is artinian [Proposition 1.3]. A left artinian algebra $A$ is *left weakly symmetric* if

[LWS] for every $B$ and every two-sided artinian bimodule ${}_AM_B$ and every left artinian $A$-module ${}_AN$, $\mathrm{Hom}_A(M, N)$ is a left $B$-module of finite length.

Right weak symmetry for right artinian algebras is defined similarly. If $A$ is artinian, we say that $A$ is *weakly symmetric* if $A$ is left and right weakly symmetric. It is easy to check that this definition of weak symmetry is equivalent to the definition given in [WZ1, WZ2]. By [WZ1, 7.3 and 7.4] artinian PI algebras and stratiform simple artinian algebras are weakly symmetric. The stratiform simple artinian algebra was introduced by Schofield in [Sc]. The Weyl skew fields and division algebras of skew polynomial rings are stratiform.

**Proposition 1.3.** *Suppose that $A$ is left artinian and that $A/\mathfrak{m}$ is weakly symmetric. Then the following are equivalent.*

(1) *$A$ is left Morita.*
(2) *$A$ is right Morita.*
(3) *$A$ is artinian.*

*Proof.* (1) $\Rightarrow$ (3) This follows from [WZ1, 7.5].

(3) $\Rightarrow$ (1) In this case $\mathfrak{m}/\mathfrak{m}^2$ is artinian on both sides. The weak symmetry implies that $\mathrm{Hom}_A({}_A\mathfrak{m}/\mathfrak{m}^2, {}_A A/\mathfrak{m})$ is of finite length. By [Xu, 11.3], $A$ is left Morita. In this implication we only use left weak symmetry of $A$.

Similarly (2) is equivalent to (3). □

The following partial converse is easy to prove: Let $A_0$ be a semisimple artinian algebra. If every artinian algebra $A$ with $A/\mathfrak{m} = A_0$ is left Morita, then $A_0$ is left weakly symmetric.

The next example shows that Proposition 1.3 does not hold when $A/\mathfrak{m}$ is not weakly symmetric.

**Example 1.4 [AF, Exercise 24.9].** Let $C \subset D$ be division rings such that $D_C$ is finite dimensional and ${}_CD$ is not (see [Co]). Let $A = \begin{pmatrix} D & D \\ 0 & C \end{pmatrix}$. Then $A$ is left and right artinian but not left Morita. Both $A/\mathfrak{m}$ and $D$ are not left weakly symmetric because $\operatorname{Hom}_D({}_DD_C, {}_DD)$ is not a finite right $C$-module [SZ1, 3.1].

The next result was proved in [WZ2, 0.1]. A similar result appeared in [Ch].

**Theorem 1.5.** *Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be complete noetherian semilocal algebras and let ${}_AE_B$ be a bimodule which induces a Morita duality between $A$ and $B$. Suppose that*

*(i) $A$ and $B^\circ$ have finite cohomological dimension,*
*(ii) $A$ and $B$ satisfy the (left and right) $\chi$ condition, and*
*(iii) $A_0 = A/\mathfrak{m}$ is weakly symmetric.*

*Then*

(1) $R := \operatorname{Hom}_A(\mathrm{R}\Gamma_{\mathfrak{m}}(A), E)$ *is isomorphic to* $\operatorname{Hom}_{B^\circ}(\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(B), E)$ *in* $D(A \otimes B^\circ)$*;*
(2) *$R$ is a dualizing complex over $(A, B)$;*
(3) *$R$ is bifinite,* Cdim *-symmetric and pre-balanced.*

The proof of this was similar to the proof of [VdB, 3.6]. Some terms in part (3) need to be defined. Let $R$ be a dualizing complex over $(A, B)$ and let $M$ be an $A$-module. The *grade* (or *j-number*) of $M$ with respect to $R$ is

$$j(M) = \inf\{q \mid \operatorname{Ext}^q_A(M, R) \neq 0\}.$$

The grade of a $B^\circ$-module is defined similarly. The *canonical dimension* with respect to a dualizing complex $R$ is defined to be

$$\operatorname{Cdim} M = -j(M)$$

for all finite $A$-(or $B^\circ$)-modules $M$.

A dualizing complex $R$ over $(A, B)$ is called Cdim *-symmetric* if for every $(A, B)$-bimodule $M$ finite on both sides, one has $\operatorname{Cdim} {}_AM = \operatorname{Cdim} M_B$. A dualizing complex $R$ over $(A, B)$ is called *bifinite* if the following conditions hold:

(1) for every $A$-bimodule $M$ finite on both sides, $\operatorname{Ext}^q_A(M, R)$ is finite on both sides;
(2) the same holds after $A$ and $B^\circ$ are exchanged.

The next notion is a central object of this paper.

**Definition 1.6 [Ye1] [WZ2].** A dualizing complex $R$ over $(A, B)$ is *pre-balanced* if

(1) for every simple $A$-module $S$, $\operatorname{Ext}^i_A(S, R) = 0$ for all $i \neq 0$ and $\operatorname{Ext}^0_A(S, R)$ is a simple $B^\circ$-module.
(2) the same statement holds after $A$ and $B^\circ$ are exchanged.

To end this section we show that the pre-balanced condition is automatic in certain cases, which suggests that this condition is natural. Recall from [SZ1, 3.5] that every noetherian PI semilocal algebra satisfies the $\chi$ condition so the next proposition applies in particular to such algebras.

**Proposition 1.7.** *Let $A$ and $B^\circ$ be noetherian local algebras satisfying the left $\chi$ condition and let $R$ be a dualizing complex over $(A, B)$. Then a complex shift of $R$ is pre-balanced.*

*Proof.* First, we show that $\mathrm{Ext}^i_A(A/\mathfrak{m}, R)$ is a finite bimodule. It follows from properties of the dualizing complex that $\mathrm{Ext}^i_A(A/\mathfrak{m}, R)$ is a finite $B^\circ$-module. It remains only to show that it is finite as an $A$-module, so we may as well forget the $B^\circ$-structure on $R$. We will show that if $X \in D^b_f(A)$, then $\mathrm{Ext}^i_A(A/\mathfrak{m}, X)$ is a finite $A$-module. By the long exact sequence and induction on the length of $X$, one may reduce to the case that $X$ is a finite $A$-module in which case the assertion is precisely the left $\chi$ condition.

Let $S$ be the simple $A$-module and $T$ be the simple $B^\circ$-module. Let

$$a = \max\{i \mid \mathrm{Ext}^i_A(S, R) \neq 0\}, \qquad b = \min\{i \mid \mathrm{Ext}^i_A(S, R) \neq 0\}$$

and

$$c = \max\{i \mid \mathrm{Ext}^i_{B^\circ}(T, R) \neq 0\}, \qquad d = \min\{i \mid \mathrm{Ext}^i_{B^\circ}(T, R) \neq 0\}.$$

It follows from the induction that, for any nonzero $A$-module $M$ of finite length and any nonzero $B^\circ$-module $N$ of finite length, we have

$$a = \max\{i \mid \mathrm{Ext}^i_A(M, R) \neq 0\}, \qquad b = \min\{i \mid \mathrm{Ext}^i_A(M, R) \neq 0\}$$

and

$$c = \max\{i \mid \mathrm{Ext}^i_{B^\circ}(N, R) \neq 0\}, \qquad d = \min\{i \mid \mathrm{Ext}^i_{B^\circ}(N, R) \neq 0\}.$$

We saw in the first paragraph that $\mathrm{Ext}^i_A(A/\mathfrak{m}, R)$ is finite. Since $\mathrm{Ext}^i(A/\mathfrak{m}, R)$ is an $A/\mathfrak{m}$-module, it is an $A$-module of finite length. By Lenagan's lemma [GW, 7.10], it is also of finite length as a $B^\circ$-module. Its summand $\mathrm{Ext}^i_A(S, R)$ is thus also a $B^\circ$-module of finite length. By the definitions of $a, b, c, d$ we see that

$$\mathrm{Ext}^j_{B^\circ}(\mathrm{Ext}^i_A(S, R), R) \neq 0$$

for $(i, j) = (a, c)$, $(a, d)$, $(b, c)$ and $(b, d)$. Consider the convergent spectral sequence

$$E^2_{p,q} := \mathrm{Ext}^p(\mathrm{Ext}^q(S, R), R) \Longrightarrow S$$

[YZ2, 1.7], we see that all possible nonzero terms in the $E^2$-pages are in the rectangle bounded by the four corner vertices $(a, c)$, $(a, d)$, $(b, c)$ and $(b, d)$. Since the boundary maps at vertices $(a, d)$ and $(b, c)$ are zero, these the $E^2$-terms at these two vertices will survive in the the $E^\infty$-page. Since the spectral sequence converges, the only possibility for this to happen is when $a = b = c = d$. After shifting $R$, we may assume $a = b = c = d = 0$. Hence $\mathrm{Ext}^i_A(S, R) = 0$ for all $i \neq 0$. This implies that $\mathrm{Ext}^0(-, R)$ is exact on modules of finite length. Finally the spectral sequence

$$\mathrm{Ext}^p(\mathrm{Ext}^q(M, R), R) \Longrightarrow M$$

shows that $\operatorname{Ext}^0(-,R)$ induces a duality between the category of finite length $A$-modules and the category of finite length $B^\circ$-modules. Therefore $\operatorname{Ext}^0(S,R) \cong T$. By symmetry, the same statement holds for $T$. Thus $R$ is pre-balanced. □

## 2. Truncated Morita Dualities

In this section we recall some results due to Müller and Jategaonkar, which play a key role in constructing Morita dualities for algebras with pre-balanced dualizing complexes.

Let $A$ and $B$ be two algebras with ideals $\mathfrak{m} \subset A$ and $\mathfrak{n} \subset B$. Assume that $A/\mathfrak{m}$ and $B/\mathfrak{n}$ are semisimple artinian. Suppose that for every $n$,

(1) $A/\mathfrak{m}^n$ is left artinian and $B/\mathfrak{n}^n$ is right artinian;
(2) there is a Morita duality between $A/\mathfrak{m}^n$ and $B/\mathfrak{n}^n$;
(3) the Morita duality between $A/\mathfrak{m}^{n-1}$ and $B/\mathfrak{n}^{n-1}$ is the restriction of the Morita duality between $A/\mathfrak{m}^n$ and $B/\mathfrak{n}^n$.

Let $E^n$ be an $(A/\mathfrak{m}^n, B/\mathfrak{n}^n)$-bimodule which induces the Morita duality. Hence $E^n$ is artinian on both sides. We call $\{(A/\mathfrak{m}^n, B/\mathfrak{n}^n, E^n) | n \in \mathbb{N}\}$ *a system of truncated Morita dualities.*

For any $n$, there are two contravariant functors

$$F^n := \operatorname{Hom}_{A/\mathfrak{m}^n}(-, E^n) \quad \text{and} \quad G^n := \operatorname{Hom}_{(B/\mathfrak{n}^n)^\circ}(-, E^n),$$

which give rise to a duality between the category of artinian left $A/\mathfrak{m}^n$-modules and the category of artinian right $B/\mathfrak{n}^n$-modules.

Define the essential length of an artinian module as follows:
$\operatorname{el}(M) = 0$ if $M = 0$,
$\operatorname{el}(M) = 1$ if $M = \operatorname{soc}(M)$ where $\operatorname{soc}(M)$ is the sum of all simple submodules of $M$, and
$\operatorname{el}(M) = \operatorname{el}(M/\operatorname{soc}(M)) + 1$.

Note that an $\mathfrak{m}$-torsion $A$-module $M$ is an $A/\mathfrak{m}^t$-module if and only if $\operatorname{el}(M) \leq t$.

**Lemma 2.1.** *There is a bimodule embedding $E^n \to E^{n+1}$ such that the image is*

$$\operatorname{Hom}_{A/\mathfrak{m}^{n+1}}(A/\mathfrak{m}^n, E^{n+1}) = \operatorname{Hom}_{(B/\mathfrak{n}^{n+1})^\circ}(B/\mathfrak{n}^n, E^{n+1}).$$

*Proof.* If ${}_AE_B$ defines a Morita duality between $A$ and $B$, then the lattices of the ideals of $A$ and $B$ are isomorphic [AF, 24.6(1)]. By the proof of [AF, 24.6(1)] (see Lemma 5.7), if $I$ is an ideal of $A$ and $J$ is the corresponding ideal of $B$, then $A/I$ and $B/J$ are Morita dual via the bimodule

$$\operatorname{Hom}_A(A/I, E) = \operatorname{Hom}_{B^\circ}(B/J, E).$$

Applying this statement to $(A/\mathfrak{m}^{n+1}, B/\mathfrak{n}^{n+1}, E^{n+1})$, we see that the ideal $\mathfrak{m}^t/\mathfrak{m}^{n+1}$ corresponds to $\mathfrak{n}^t/\mathfrak{n}^{n+1}$ for $t \leq n$ because duality preserves essential length. Furthermore, the bimodule inducing the Morita duality between $A/\mathfrak{m}^t$ and $B/\mathfrak{n}^t$ is

$$N := \operatorname{Hom}_{A/\mathfrak{m}^{n+1}}(A/\mathfrak{m}^t, E^{n+1}) = \operatorname{Hom}_{(B/\mathfrak{n}^{n+1})^\circ}(B/\mathfrak{n}^t, E^{n+1}).$$

Since $E^n$ is induced from $E^{n+1}$, there is a right $B/\mathfrak{n}^n$-module isomorphism

$$\tau_M : \operatorname{Hom}_A(M, E^n) \cong \operatorname{Hom}_A(M, E^{n+1})$$

for all $A/\mathfrak{m}^n$-modules $M$. If $M = A/\mathfrak{m}^n$, then $\tau_M$ induces an isomorphism from $E^n_B \to N_B$. Since $\tau$ is natural, this is also $A/\mathfrak{m}^n$-linear. □

By Lemma 2.1, we have a direct system of bimodules $\{E^n \mid n \geq 0\}$. Define $E = \varinjlim E^n$. For simplicity we identify $E^n$ with its image in $E$, so $E^n$ is a subbimodule of $E$. If $E^n \subset E$ is as in Lemma 2.1, we call $\{E^n \mid n > 0\}$ *a system of truncated injective modules*. The completions of $A$ and $B$ are defined to be $\hat{A} = \varprojlim A/\mathfrak{m}^n$ and $\hat{B} = \varprojlim B/\mathfrak{n}^n$ respectively. Note that $E$ is a bimodule over $(A, B)$ and over $(\hat{A}, \hat{B})$.

**Lemma 2.2.** *Let $E = \varinjlim E^n$. Then $\operatorname{Hom}_A(E, E) = \operatorname{Hom}_{\hat{A}}(E, E) = \hat{B}$ and the equality also holds when $A$ and $B^\circ$ are exchanged.*

*Proof.* Since $E^n$ induces a Morita duality between $A/\mathfrak{m}^n$ and $B/\mathfrak{n}^n$,

$$B/\mathfrak{n}^n = \operatorname{Hom}_A(E^n, E^n) = \operatorname{Hom}_A(E^n, E).$$

Now the identity follows from the formula $\operatorname{Hom}(\varinjlim E^n, -) = \varprojlim \operatorname{Hom}(E^n, -)$. □

Note that the lattice of $A$-submodules of $E$ is identical with the lattice of $\hat{A}$-submodules of $E$. The following is a simplified version of the main result in [Ja].

**Theorem 2.3 [Ja, 2.2].** *Let $A$ be a left noetherian complete semilocal algebra and $B$ a complete right noetherian semilocal algebra. Suppose that $\{(A/\mathfrak{m}^n, B/\mathfrak{n}^n, E^n)\}_n$ is a system of truncated Morita dualities, then $(A, B, E)$ is a Morita duality. In particular, E is artinian on both sides.*

Jategaonkar also strengthens a result of Müller [Mu, 8] as follows.

**Theorem 2.4 [Ja, 2.4] [Mu, 8].** *Let $A$ be a left noetherian complete semilocal algebra, ${}_AE$ be an injective cogenerator with finite essential socle, and $B = \operatorname{End}({}_AE)$. Then the following conditions are equivalent:*

*(1) $A$ is left Morita.*
*(2) ${}_AE$ is artinian.*
*(3) $B$ is right noetherian.*

**Corollary 2.5.** *Let $A$ be a left noetherian complete semilocal algebra with $A/\mathfrak{m}$ finite dimensional. Let $E^n = \operatorname{Hom}_k(A/\mathfrak{m}^n, k)$ and $E = \varinjlim E^n$.*

(1) *${}_AE$ is artinian if and only if $A$ is right noetherian.*
(2) *If $A$ is right noetherian, then $E$ is an artinian injective cogenerator for $A$-modules and it induces a Morita self-duality of $A$.*

*Proof.* (1) follows from Theorem 2.4. (2) follows from [Ja, 2.2 and 2.7]. □

## 3. Pre-balanced Dualizing Complexes

The aim of this section is to prove Theorems 0.1 and 0.2. As usual, $A$ and $B$ denote semilocal rings with Jacobson radicals $\mathfrak{m}$ and $\mathfrak{n}$ respectively. We will use the truncated Morita dualities given in the previous section. The following is a key proposition.

**Proposition 3.1.** *Let $A$ and $B$ be semilocal algebras. If $R$ is a pre-balanced dualizing complex over $(A, B)$, then the functors*

$$\{(\mathrm{Ext}^0_A(-,R)|_{A/\mathfrak{m}^n}, \mathrm{Ext}^0_{B^\circ}(-,R)|_{B/\mathfrak{n}^n}) \mid n > 0\}$$

*induces a system of truncated Morita dualities.*

*If moreover $A$ and $B$ are complete, then*

$$\mathrm{R}\,\Gamma_{\mathfrak{m}}(R) \cong \mathrm{H}^0_{\mathfrak{m}}(R) \cong \mathrm{H}^0_{\mathfrak{n}^\circ}(R) \cong \mathrm{R}\,\Gamma_{\mathfrak{n}^\circ}(R).$$

*Further, the Morita duality between $A$ and $B$ induced by the limit of the above system is given by the bimodule $\mathrm{H}^0_{\mathfrak{m}}(R) \cong \mathrm{H}^0_{\mathfrak{n}^\circ}(R)$.*

*Proof.* Let el be the essential length defined before Lemma 2.1. Note that an artinian $A$-module $M$ is an $A/\mathfrak{m}^n$-module if and only if $\mathrm{el}(M) \le n$.

By definition, if ${}_AM$ is simple, then $\mathrm{Ext}^0_A(M,R)$ is $B^\circ$-simple. So if $\mathrm{el}(M) = 1$, then $\mathrm{el}(\mathrm{Ext}^0_A(M,R)) = 1$. By induction and the fact $\mathrm{Ext}^0(M,R)$ is exact on modules of finite length, $\mathrm{Ext}^0(-,R)$ induces a duality between finite $A/\mathfrak{m}^n$-modules and finite $(B/\mathfrak{n}^n)^\circ$-modules. Therefore the first statement follows.

We now assume $A$ and $B$ are complete. Observe first that $\mathrm{R}\,\Gamma_{\mathfrak{m}}(R)$ and $\mathrm{R}\,\Gamma_{\mathfrak{n}^\circ}(R)$ have nonzero cohomology in cohomological degree 0 only, since $R$ is pre-balanced. The pre-balanced condition also shows that $\mathrm{H}^i_{\mathfrak{m}}(R)$ is right $\mathfrak{n}$-torsion and $\mathrm{H}^i_{\mathfrak{n}^\circ}(R)$ is $\mathfrak{m}$-torsion for every $i$. By [WZ2, 2.6 and 2.7], $\mathrm{R}\,\Gamma_{\mathfrak{m}}(R) \cong \mathrm{R}\,\Gamma_{\mathfrak{n}^\circ}(R)$ as $(A,B)$-bimodule complexes. Taking 0-th cohomology, we find that

$$\mathrm{H}^0_{\mathfrak{m}}(R) \cong \mathrm{H}^0_{\mathfrak{n}^\circ}(R).$$

For the final statement note that the system of truncated Morita dualities is induced by the bimodules

$$E^n = \mathrm{Ext}^0_A(A/\mathfrak{m}^n, R) \cong \mathrm{Ext}^0_{B^\circ}(B/\mathfrak{n}^n, R).$$

By Jategaonkar's theorem (see Theorem 2.3),

$$\varinjlim E^n = \mathrm{H}^0_{\mathfrak{m}}(R) \cong \mathrm{H}^0_{\mathfrak{n}^\circ}(R)$$

induces the desired Morita duality. □

**Remark 3.2.** In [Ye1, 4.1], Yekutieli introduced the notion of a balanced dualizing complex for a noncommutative graded algebra. There is an obvious generalization of his definition to our setting. Suppose $A$ and $B$ are Morita dual via a bimodule $E$. A dualizing complex is said to be *balanced with respect to* $E$ if

$$\mathrm{R}\,\Gamma_{\mathfrak{m}}(R) \cong E \cong \mathrm{R}\,\Gamma_{\mathfrak{n}^{\circ}}(R)$$

in $D(A \otimes B^{\circ})$. Proposition 3.1 shows that a pre-balanced dualizing complex is automatically balanced with respect to its intrinsic Morita duality.

Later in this section, we will introduce a notion of balanced dualizing complexes which depends only on the algebra [Definition 3.7]. The next lemma shows that a balanced dualizing complex is pre-balanced.

**Lemma 3.3.** *Let $A$ and $B$ be complete semilocal algebras and let $E$ be a bimodule which induces a Morita duality between $A$ and $B$. If $R$ is a dualizing complex over $(A, B)$ such that*

$$\mathrm{R}\,\Gamma_{\mathfrak{m}}(R) \cong E \cong \mathrm{R}\,\Gamma_{\mathfrak{n}^{\circ}}(R)$$

*in $D(A \otimes B^{\circ})$, then $R$ is pre-balanced.*

*Proof.* Let $I$ be a minimal injective resolution of ${}_AR$. Then $\mathrm{R}\,\Gamma_{\mathfrak{m}}(R) \cong E$ implies that $\Gamma_{\mathfrak{m}}(I)^i = 0$ for all $i \neq 0$ and $\Gamma_{\mathfrak{m}}(I)^0 \cong {}_AE$. Hence if $S$ is a simple $A$-module, $\mathrm{Ext}^i(S, R) = \mathrm{Ext}^i(S, I) = 0$ for all $i \neq 0$ and $\mathrm{Ext}^0(S, R) = \mathrm{Hom}(S, \mathrm{soc}({}_AE))$. By induction $\mathrm{Ext}^0_A(-, R)$ is exact on $A$-modules of finite length.

Since $A$ and $B$ are Morita dual via $E$, $A/\mathfrak{m}$ and $B/\mathfrak{n}$ are Morita dual via $\mathrm{soc}({}_AE) = \mathrm{soc}(E_B)$. Hence $\mathrm{Ext}^0(S, R)$ is a finite right $B/\mathfrak{n}$-module.

Similar statements hold for $\mathrm{Ext}^0_{B^{\circ}}(-, R)$ on $B^{\circ}$-modules of finite length. This shows that $\mathrm{Ext}^0(-, R)$ induces a duality between the category of $A$-modules of finite length and that of $B^{\circ}$-modules. Therefore $\mathrm{Ext}^0_A(S, R)$ is a simple $B^{\circ}$-module. If $T$ is a simple $B^{\circ}$-module, similar statements hold $\mathrm{Ext}^i_{B^{\circ}}(T, R)$ by symmetry. Thus $R$ is pre-balanced. $\square$

The next proposition is a version of local duality for rings with pre-balanced dualizing complexes. It was proved in the graded case by Yekutieli in [Ye1, 4.18]. We will give a new proof here which is comparatively easy. A more general version appears in [YZ3, 5.2].

**Proposition 3.4.** *Let $A$ and $B$ be complete semilocal algebras. If $R$ is a pre-balanced dualizing complex over $(A, B)$ and $E = \mathrm{R}\,\Gamma_{\mathfrak{m}}(R) = \mathrm{R}\,\Gamma_{\mathfrak{n}^{\circ}}(R)$ as above, then we have the following isomorphism*

$$\mathrm{Hom}_A(\mathrm{R}\,\Gamma_{\mathfrak{m}}(M), E) \cong \mathrm{RHom}_{\mathrm{A}}(M, R)$$

*which is natural in $M \in D^b_f(A)$.*

*If $C$ is another algebra, then the above isomorphism holds for bounded complexes $M$ of $A \otimes C^\circ$-modules whose cohomology is finite over $A$.*

*Proof.* Note first that there is a natural transformation

$$\Phi : \mathrm{RHom_A}\,(-, R) \longrightarrow \mathrm{Hom}_A(\mathrm{R}\,\Gamma_{\mathfrak{m}}(-), \mathrm{R}\,\Gamma_{\mathfrak{m}}(R)) =: F$$

given by applying the functor $\Gamma_{\mathfrak{m}}$ to an homomorphism. Let $D$ denote the functor $\mathrm{RHom_A}\,(-, R)$ and $D^\circ$ denote $\mathrm{RHom_{B^\circ}}\,(-, R)$. Now $D^\circ$ is a duality so in fact, we need only show $DD^\circ = id \longrightarrow FD^\circ$ is an isomorphism of functors on bounded complexes of $C \otimes B^\circ$-modules whose cohomology is finite over $B^\circ$. Since the restriction from $C \otimes B^\circ$-modules to $B^\circ$-modules commutes with natural transformations, it suffices to show that $DD^\circ = id \longrightarrow FD^\circ$ is an isomorphism of functors on $D^b_f(B^\circ)$. Recall that the dualizing complex $R$ has finite injective dimension, so both $D^\circ$ and $F$ are way-out left functors in the sense of [Ha, p.68]. Hence $FD^\circ$ is way-out left. By [Ha, I.7.1(i)] and the dual version of [Ha, I.7.1(iv)], it suffices to show that this is an isomorphism when evaluated on $B$. Now

$$FD^\circ(B) = \mathrm{RHom_A}\,(\mathrm{R}\,\Gamma_{\mathfrak{m}}(\mathrm{RHom_{B^\circ}}\,(B, R)), \mathrm{R}\,\Gamma_{\mathfrak{m}}(R))$$

$$= \mathrm{RHom_A}\,(\mathrm{R}\,\Gamma_{\mathfrak{m}}(R), \mathrm{R}\,\Gamma_{\mathfrak{m}}(R)) = \mathrm{RHom_A}\,(E, E) = B$$

as was to be shown. □

Here is a partial converse of Theorem 1.5.

**Corollary 3.5.** *Let $A$ and $B$ be complete semilocal algebras. Let $R$ be a pre-balanced dualizing complex over $(A, B)$ and let $E = \mathrm{R}\,\Gamma_{\mathfrak{m}}(R) \cong \mathrm{R}\,\Gamma_{\mathfrak{n}^\circ}(R)$ as in Proposition 3.1. Then*

(1) $R \cong \mathrm{Hom}_A(\mathrm{R}\,\Gamma_{\mathfrak{m}}(A), E) \cong \mathrm{Hom}_{B^\circ}(\mathrm{R}\,\Gamma_{\mathfrak{n}^\circ}(B), E)$,
(2) $\mathrm{cd}(A)$ *and* $\mathrm{cd}(B^\circ)$ *are finite,*
(3) *$A$ and $B^\circ$ satisfy the left $\chi$ condition.*

*Proof.* We prove only the left-handed statements, the right-handed ones being symmetric.

(1) This follows by taking $M$ to be the $A$-bimodule $A$ in Proposition 3.4.

(2) By Proposition 3.4,

$$\mathrm{Hom}_A(\mathrm{R}\,\Gamma_{\mathfrak{m}}(M), E) \cong \mathrm{RHom_A}\,(M, R)$$

for finite $A$-modules $M$. Since $\mathrm{Hom}_A(-, E)$ is exact, we have

$$\mathrm{Hom}_A(\mathrm{R}^{-i}\,\Gamma_{\mathfrak{m}}(M), E) \cong \mathrm{Ext}^i_A(M, R)$$

for all $i$. Since $\mathrm{Ext}^i_A(M, R)$ is a finite $B^\circ$-module, $\mathrm{R}^{-i}\,\Gamma_{\mathfrak{m}}(M)$ is an artinian $A$-module. Therefore [WZ2, 3.3(1)] implies that

$$\mathrm{R}\,\Gamma_{\mathfrak{m}}(M) \cong \mathrm{Hom}_{B^\circ}(\mathrm{RHom_A}\,(M, R), E).$$

Let $\inf R = \min\{i | \mathrm{H}^i(R) \neq 0\}$. For every $A$-module $M$ and every $i < \inf R$, $\mathrm{Ext}^i_A(M, R) = 0$, and hence

$$\mathrm{R}^{-i}\,\Gamma_{\mathfrak{m}}(M) = \mathrm{Hom}_{B^\circ}(\mathrm{Ext}^i_A(M, R), E) = 0. \tag{3.5.1}$$

Thus $cd(A)$ is bounded by $-\inf R$.

(3) Let $M$ be a finite $A$-module. Since $\mathrm{Ext}^{-i}_A(M, R)$ is finite $B^\circ$-module, the Morita dual $\mathrm{Hom}_{B^\circ}(\mathrm{Ext}^{-i}_A(M, R), E)$ is artinian. Then the local duality formula (3.5.1) implies that $\mathrm{R}^i\,\Gamma_{\mathfrak{m}}(M)$ is an artinian module for every $i$. Now $A$ and $B$ are Morita dual so injective hulls of simple $A$-modules are artinian. Hence we may apply [WZ2, 2.3] to yield the left $\chi$ condition. □

**Remark 3.6.** It follows from Proposition 3.1 and Corollary 3.5(1) that there is a one-to-one correspondence between the isomorphism classes of the Morita dualities between $A$ and $B$ and the isomorphism classes of the pre-balanced dualizing complexes over $(A, B)$ (see Proposition 4.7). If a pre-balanced dualizing complex $R$ satisfies Corollary 3.5(1), we say that $R$ is *associated to* $E$. It is clear that dualizing complexes associated to $E$ are unique up to isomorphism.

*Proof of Theorem 0.1.* (1) was proved in Proposition 3.1 and (2) was proved in Corollary 3.5. It remains to show (3) and (4).

(3) By (3.5.1),

$$\mathrm{Ext}^i_A(M, R) \cong \mathrm{Hom}_A(\mathrm{R}^{-i}\,\Gamma_{\mathfrak{m}}(M), E)$$

for all finite $A$-module $M$. Thus

$$\mathrm{Cdim}\, M := -\inf\{i | \mathrm{Ext}^i_A(M, R) \neq 0\} = \sup\{i | \mathrm{R}^i\,\Gamma_{\mathfrak{m}}(M)\} =: lcd(M).$$

If $M$ is an $(A, B)$-bimodule finite on both sides, then $lcd$ is symmetric, i.e., $lcd({}_AM) = lcd(M_B)$ [WZ2, 2.9]. Thus $R$ is Cdim-symmetric, i.e., $\mathrm{Cdim}\,({}_AM) = \mathrm{Cdim}\,(M_B)$.

(4) This follows from Corollary 3.5 and Theorem 1.5. □

Our definition of a balanced dualizing complex in Remark 3.2 depended on the Morita duality. We wish now to give a definition which depends only on $A$. To do so we restrict ourselves to the following situation.

Suppose now that $A$ is a noetherian complete semilocal ring with $\dim_k A/\mathfrak{m}$ finite. Let $E^n$ be the module $\mathrm{Hom}_k(A/\mathfrak{m}^n, k)$ and let $E = \varinjlim E^n$. It is clear that each $E^n$ is finite dimensional and that $\{E^n\}$ is a system of truncated injective modules. By Corollary 2.5(2), ${}_AE_A$ induces a Morita self-duality. We consider $E$ to be the natural choice of Morita duality for $A$. We can now copy the graded version of balanced dualizing complexes [Ye1, 4.1] to the complete semilocal case.

**Definition 3.7.** Let $A$ be a noetherian complete semilocal algebra. Suppose $A/\mathfrak{m}$ is finite dimensional over $k$. Let $E_A = \varinjlim \mathrm{Hom}_k(A/\mathfrak{m}^n, k)$. A dualizing complex $R$ over $A$ is called *balanced* if

$$\mathrm{R}\,\Gamma_{\mathfrak{m}}(R) \cong E_A \cong \mathrm{R}\,\Gamma_{\mathfrak{m}^\circ}(R)$$

in $D(A \otimes A^\circ)$.

It follows from Lemma 3.3 that the balanced condition is stronger than the pre-balanced condition. As a consequence of Proposition 3.1 and Corollary 3.5(1), a balanced dualizing complex over $A$ (if it exists) is unique up to isomorphism. From now on the balanced dualizing complex over $A$ is denoted by $R_A$.

If $A$ is local, then any two Morita self-dualities differ by an automorphism of $A$ as the next lemma shows.

**Lemma 3.8.** *(1) Let $C, C'$ and $B$ be algebras. Suppose ${}_CE_B$ and ${}_{C'}F_B$ are bimodules such that the canonical maps $C \to \operatorname{End}(E_B)$ and $C' \to \operatorname{End}(F_B)$ are isomorphisms. If $f : E_B \to F_B$ is an isomorphism of right $B$-modules. Then $f$ induces an algebra isomorphism $\phi : C' \to C$ such that $f$ is a bimodule isomorphism from ${}^{\phi}{}_CE_B \to_{C'} F_B$.*

*(2) Suppose $A$ is a noetherian complete local algebra with $\dim_k A/\mathfrak{m} < \infty$. If an $A$-bimodule $E$ induces a Morita self-duality, then there is an automorphism $\phi$ of $A$ such that ${}^{\phi}E \cong E_A$ as $A$-bimodules.*

*Proof.* (1) By the canonical homomorphism we may identify $C$ with $\operatorname{End}(E_B)$ and $C'$ with $\operatorname{End}(F_B)$. We define $\phi : \operatorname{End}(F_B) \to \operatorname{End}(E_B)$ by $\phi(a) = f^{-1}af$ for all $a \in \operatorname{End}(F_B)$. Hence $\phi$ is an isomorphism of algebras. We now define a left $C'$-module structure on $E$ by

$$a * x = \phi(a)x (= f^{-1}af(x)).$$

By this we see that $f$ is $C'$-linear. Therefore $f$ is a bimodule isomorphism ${}^{\phi}{}_CE_B \to_{C'} F_B$.

(2) Let $E_A$ be as in Definition 3.7 and let $E$ be another $A$-bimodule which induces a Morita self-dual of $A$. Since $A$ is local, both $E$ and $E_A$ are the injective hull of the $A^\circ$-module $A/\mathfrak{m}$. Hence $E \cong E_A$ as $A^\circ$-modules. The assertion now follows from (1). □

We now prove Theorem 0.2. The graded version of this was proved by Van den Bergh [VdB] and partly by Yekutieli [Ye1].

**Corollary 3.9.** *Let $A$ be a noetherian complete semilocal algebra with $A/\mathfrak{m}$ finite dimensional. Then the following are equivalent:*

1. *$A$ has a balanced dualizing complex*
2. *$A$ has a pre-balanced dualizing complex*
3. *$A$ satisfies $\chi$ and $\operatorname{cd}(A)$ and $\operatorname{cd}(A^\circ)$ are finite.*

*If moreover $A$ is local, then every pre-balanced dualizing complex is isomorphic to ${}^{\phi}R_A$ where $R_A$ is the balanced dualizing complex over $A$.*

*Proof.* (1) $\Longrightarrow$ (2) is Lemma 3.3.

(2) $\Longrightarrow$ (3) is Corollary 3.5.

(3) $\Longrightarrow$ (1) Since $A/\mathfrak{m}$ is finite dimensional over $k$, it is weakly symmetric. By Theorem 1.5, $R := \operatorname{Hom}_A(\mathrm{R}\,\Gamma_{\mathfrak{m}}(A), E_A)$ is a dualizing complex over $A$.

It follows from the local duality theorem [WZ2, 3.6(2)] that

$$\operatorname{Hom}_A(\mathrm{R}\,\Gamma_{\mathfrak{m}}(R), E_A) \cong \mathrm{R}\operatorname{Hom}_A(R, R) \cong A$$

in $D(A \otimes A^\circ)$. Taking Morita duals, we find that

$$\mathrm{R}\,\Gamma_{\mathfrak{m}}(R) \cong E_A.$$

Similarly, $\mathrm{R}\,\Gamma_{\mathfrak{m}^\circ}(R) \cong E_A$. Therefore $R$ is balanced.

For the last statement, we assume $A$ is local and $R_A$ is a balanced dualizing complex over $A$. If $R$ is a pre-balanced dualizing complex over $A$. Then by Corollary 3.5, there is an $A$-bimodule $E$, which induces a Morita self-duality of $A$ such that

$$R \cong \mathrm{Hom}_{A^\circ}(\mathrm{R}\,\Gamma_{\mathfrak{m}^\circ}(A), E).$$

By Lemma 3.8(2), there is an automorphism $\phi$ of $A$ such that $E \cong^{\phi} E_A$. Thus

$$R \cong \mathrm{Hom}_{A^\circ}(\mathrm{R}\,\Gamma_{\mathfrak{m}^\circ}(A), {}^{\phi} E_A) \cong {}^{\phi}(\mathrm{Hom}_{A^\circ}(\mathrm{R}\,\Gamma_{\mathfrak{m}^\circ}(A), E_A)) \cong {}^{\phi}R_A.$$

□

## 4. Applications

In this section, we present some applications of the relation between dualizing complexes and Morita dualities established in the last section. First we prove Corollaries 0.3 and 0.4.

An algebra is called *Quasi-Frobenius* (or *QF*) if it is artinian and has injective dimension 0. If $A$ is QF, then the bimodule ${}_AA_A$ induces a Morita self-duality. A generalization of QF algebras to higher injective dimension is the so called Artin-Schelter Gorenstein (or AS-Gorenstein) ring. Recall that a noetherian algebra $A$ is *AS-Gorenstein* if

(1) $A$ has finite left and right injective dimension $d$,
(2) For every simple (left) $A$-module $S$, $\mathrm{Ext}^i(S, A) = 0$ for all $i \neq d$ and $\mathrm{Ext}^d_A(S, A)$ is a simple right $A$-module, and
(3) Part (2) holds when 'left' and 'right' are exchanged.

An artinian (or noetherian) algebra $A$ is QF if and only if it is AS-Gorenstein of injective dimension 0 [AF, Chapter 30]. We now generalize the fact that every QF algebra has a Morita self-duality to the higher dimensional case.

*Proof of Corollary 0.3.* Since $A$ has finite left and right injective dimension, $A$ is a dualizing complex over $A$. The AS-Gorenstein condition (2,3) shows that the complex shift $A[d]$ is pre-balanced dualizing complex over $A$. Now the assertion follows from Theorem 0.1(1). □

Since every noetherian local PI algebra with finite injective dimension is AS-Gorenstein [SZ1, 3.10], we have the following: every complete local noetherian PI algebra of finite injective dimension has a Morita self-duality.

**Definition 4.1.** [Ye2] [YZ2] A dualizing complex $R$ over $(A, B)$ is *Auslander* if

(1) for every finite $A$-module $M$, every $q$, and every $B^\circ$-submodule $N \subset \mathrm{Ext}^q_A(M, R)$ one has $j(N) \geq q$.
(2) the same holds after $A$ and $B^\circ$ are exchanged.

By [Le, 4.5] and [YZ2, 2.10], if $R$ is Auslander then Cdim defined in Section 1 is an exact, finitely partitive dimension function (for the definition of dimension function, see [MR, 6.8.4]). An algebra is called *Auslander Gorenstein* if $A$ has finite left and right injective dimension and $A$ is an Auslander dualizing complex over $A$. If, moreover, $A$ has finite global dimension, then $A$ is *Auslander regular.*

Levasseur [Le, 4.8] showed that every connected graded or local Auslander regular algebra is a domain. Yekutieli [YZ2, 6.23] showed that every connected graded (or filtered) Auslander Gorenstein algebra has a QF ring of fractions. We now generalize [YZ2, 6.23] to the complete local case.

**Proposition 4.2.** *Let $A$ be a noetherian complete semilocal ring. If $A$ is AS-Gorenstein and Auslander-Gorenstein of injective dimension $d$, then the following hold:*

(1) *If $\mathfrak{p}$ is a minimal prime ideal of $A$, then* $\operatorname{Cdim} A/\mathfrak{p} = d$.
(2) *$A$ has a QF artinian ring of fractions.*

*Proof.* Since $A$ is AS-Gorenstein, $R := A[d]$ is a pre-balanced dualizing complex over $A$. By Theorem 0.1(3), $R$ is $\operatorname{Cdim}_R$-symmetric. Shifting $R = A[d]$ back to $A$, the Cdim changes by $+d$. Hence $A$ is $\operatorname{Cdim}_A$-symmetric. If we let the dimension function $\delta$ in [ASZ, Theorem 6.1] be $\operatorname{Cdim}_A$, then the Auslander Gorenstein condition ensures that all the hypotheses of that theorem hold. The assertions now follow from [ASZ, 6.1]. □

If the algebra is local, then Auslander-Gorenstein implies AS-Gorenstein. The proof of the graded case [Le, 6.3] can be copied to the local case without any trouble.

**Lemma 4.3 [Le, 6.3].** *Suppose $A$ is noetherian and local. If $A$ is Auslander-Gorenstein, then it is AS-Gorenstein.*

Corollary 0.4 follows from Proposition 4.2 and Lemma 4.3.

Next is an application of the idea of truncated Morita dualities to the non-semilocal case. It follows from [WZ2, 5.6] that if $A$ is local and $R$ is a pre-balanced dualizing complex over $(A, B)$ then $\operatorname{id}_A R = \operatorname{id} R_B = 0$. This condition was assumed when we proved the homological identities for non-local noncommutative rings [WZ3]. We now show that this is true for any pre-balanced dualizing complex. Since we do not assume that $A$ is semilocal, we need a more general local duality result proved in [YZ3, 5.2].

**Proposition 4.4.** *Let $R$ be a pre-balanced dualizing complex over $(A, B)$. Then $R$ is normal, in the sense that*

$$\operatorname{id}_A R = \operatorname{id} R_B = 0.$$

*Proof.* We say that an $A$-module $M$ is torsion if every finite submodule of $M$ has finite length. When $A$ is semilocal, this agrees with $\mathfrak{m}$-torsion. Let $\mathcal{M}$ be the full subcategory of torsion $A$-modules. The torsion $A$-modules form a *hereditary torsion class*, i.e., it is closed under subquotients, extensions and infinite direct sums. Hence $\mathcal{M}$ is a *localizing subcategory* of $A - \operatorname{Mod}$. These notions for $B^\circ$-modules are defined similarly and the full subcategory of torsion $B^\circ$-module is denoted by $\mathcal{N}$.

The torsion functor $\Gamma_{\mathcal{M}}$ is defined similarly to $\Gamma_{\mathfrak{m}}$. A more general version is given in [YZ3, Section 1].

By [YZ3, (1.1)],

$$\Gamma_{\mathcal{M}}(-) = \varinjlim \operatorname{Hom}_A(A/\mathfrak{a}, -)$$

where $\mathfrak{a}$ runs over all left ideals of $A$ such that $A/\mathfrak{a}$ is artinian. Hence

$$E := \mathrm{R}^0\,\Gamma_{\mathcal{M}}(R) = \varinjlim \operatorname{Ext}^0_A(A/\mathfrak{a}, R).$$

The pre-balanced condition ensures that $E_B$ is torsion.

Let $I$ be the minimal injective resolution of ${}_AR$. Since ${}_AR$ is pre-balanced, only $I^0$ contains nonzero torsion $A$-modules. Hence $\mathrm{R}\,\Gamma_{\mathcal{M}}R \cong \mathrm{R}^0\,\Gamma_{\mathcal{M}}R = E$.

Let $T$ be any simple $B^\circ$-module. It follows from the pre-balanced condition and the spectral sequence

$$\operatorname{Ext}^p_A(\operatorname{Ext}^q_{B^\circ}(T, R), R) \Longrightarrow T$$

that there is a simple $A$-module $S = A/\mathfrak{b}$ such that $\operatorname{Ext}^0_A(S, R) \cong T$. Since $\operatorname{Ext}^0_A(-, R)$ is exact on artinian modules, if $\mathfrak{a}$ is a left ideal of $A$ contained in $\mathfrak{b}$ such that $A/\mathfrak{a}$ is artinian then the canonical map $\operatorname{Ext}^0_A(A/\mathfrak{b}, R)) \to \operatorname{Ext}^0_A(A/\mathfrak{a}, R)$ is injective. Since $\varinjlim$ is exact, we obtain an injective map

$$\varinjlim{}_{\mathfrak{a}\subset\mathfrak{b}} \operatorname{Ext}^0_A(A/\mathfrak{b}, R) \to \varinjlim{}_{\mathfrak{a}\subset\mathfrak{b}} \operatorname{Ext}^0_A(A/\mathfrak{a}, R),$$

where $\mathfrak{a}$ runs over all left ideals of $A$ contained in $\mathfrak{b}$ such that $A/\mathfrak{a}$ is artinian. The left term is isomorphic to $\operatorname{Ext}^0_A(A/\mathfrak{b}, R)$, which is isomorphic to $T$, and the right term is isomorphic to $E$. Thus $T$ is a submodule of $E_B$.

The pre-balanced condition ensures that $\operatorname{id} R \geq 0$. If $\operatorname{id}{}_A R = i > 0$, then there is a finite $A$-module $M$ such that $\operatorname{Ext}^i_A(M, R) \neq 0$. Since $\operatorname{Ext}^i_A(M, R)$ is a finite $B^\circ$, it has a simple factor, say $T$. Thus there is a nonzero map

$$\operatorname{Ext}^i_A(M, R) \to T \to E$$

which gives a nonzero element in $\operatorname{Ext}^{-i}_{B^\circ}(\mathrm{R}\operatorname{Hom}_A(M, R), E)$. By Yekutieli's local duality [YZ3, 5.2], for any finite $A$-module $M$, we have

$$\mathrm{R}\,\Gamma_{\mathcal{M}}M \cong \mathrm{R}\operatorname{Hom}_{B^\circ}(\mathrm{R}\operatorname{Hom}_A(M, R), \mathrm{R}\,\Gamma_{\mathcal{M}}R) \cong \mathrm{R}\operatorname{Hom}_{B^\circ}(\mathrm{R}\operatorname{Hom}_A(M, R), E).$$

Thus

$$\mathrm{R}^{-i}\,\Gamma_{\mathcal{M}}M \cong \operatorname{Ext}^{-i}_{B^\circ}(\mathrm{R}\operatorname{Hom}_A(M, R), E) \neq 0.$$

This is a contradiction. Therefore $\operatorname{id}{}_A R = 0$. □

In the rest of this section we show the statement made in Remark 3.6.

**Lemma 4.5 [Xu,4.6].** *Suppose $A$ and $B$ are Morita dual via a bimodule ${}_AE_B$.*

(1) *If $A'$ and $A$ are Morita equivalent via an invertible bimodule ${}_{A'}P_A$, then $A'$ and $B$ are Morita dual via $E' := P \otimes_A E \cong \operatorname{Hom}_A(P', E)$ where $P' = \operatorname{Hom}_{A^\circ}(P, A)$.*

(2) *If $A'$ and $B$ are Morita dual via $E'$, then $E' = P \otimes_A E$ where $P = \operatorname{Hom}_{B^\circ}(E, E')$. And $B$ and $B'$ are Morita equivalent via $P$.*

A similar statement holds for dualizing complexes.

**Lemma 4.6.** *Suppose $R$ is a dualizing complex over $(A, B)$. In part (2) we also assume that $A$ and $B$ are complete semilocal and are Morita dual via a bimodule $E$.*

(1) *If $A'$ and $A$ are Morita equivalent via $P$, then*

$$R' := P \otimes_A R = \operatorname{Hom}_A(P', R)$$

*is a dualizing complex over $(A', B)$. If $R$ is pre-balanced, then so is $R'$.*

(2) *If $R$ is associated to $E$, then $R'$ is associated to $E' := P \otimes_A E$.*

*Proof.* (1) Recall that $P' = \operatorname{Hom}_{A^\circ}(P, A)$. First note that $\operatorname{Hom}_A(P', -)$ is naturally isomorphic to $P \otimes_A -$ because $A$ and $A'$ are Morita equivalent. If $M$ is in $D^b_f(A')$, then we have

$$\mathrm{R}\operatorname{Hom}_{A'}(M, R') = \mathrm{R}\operatorname{Hom}_{A'}(M, \operatorname{Hom}_A(P', R)) = \mathrm{R}\operatorname{Hom}_A(P' \otimes_{A'} M, R). \tag{4.6.1}$$

This shows that $\operatorname{id}_{A'} R'$ is finite. Other axioms in Definition 1.1 can also be shown easily for $R'$ and therefore $R'$ is a dualizing complex over $(A', B)$.

Suppose $R$ is pre-balanced. Let $S'$ be a simple $A'$-module. Then $S := P' \otimes_{A'} S'$ is a simple $A$-module. Then (4.6.1) shows that $\operatorname{Ext}^i_{A'}(S', R') \cong \operatorname{Ext}^i_A(S, R)$. Therefore $\operatorname{Ext}^i(S', R') = 0$ for all $i \neq 0$ and $\operatorname{Ext}^0(S', R')$ is a simple $B^\circ$-module. Given any finite (or simple) $B^\circ$-module $T$, we have

$$\mathrm{R}\operatorname{Hom}_{B^\circ}(T, R') = \mathrm{R}\operatorname{Hom}_{B^\circ}(T, P \otimes_A R) = P \otimes_A \mathrm{R}\operatorname{Hom}_{B^\circ}(T, R). \tag{4.6.2}$$

This shows the other half of the pre-balanced condition.

(2) Now suppose $A$ and $B$ are complete semilocal. Then so is $A'$. Suppose $R$ is associated to $E$. Let $T = B/\mathfrak{n}^n$ in (4.6.2), we have

$$\operatorname{Ext}^0_{B^\circ}(B/\mathfrak{n}^n, R') = P \otimes_A \operatorname{Ext}^0_{B^\circ}(B/\mathfrak{n}^n, R).$$

After taking $\varinjlim$, we see that $E' = P \otimes_A E$. By Proposition 3.1, $A'$ and $B$ are Morita dual via $E'$ and by Corollary 3.5, $R'$ is associated to $E'$. □

**Proposition 4.7.** *Suppose $R$ is a dualizing complex over $(A, B)$ where $A$ and $B$ are complete semilocal algebras. Then there is a one-to-one correspondence between the following two classes.*

*(a) Isomorphism classes of pre-balanced dualizing complexes over $(A, B)$.*

*(b) Isomorphism classes of bimodules which induce Morita dualities between $A$ and $B$.*

*Proof.* For any pre-balanced dualizing complex $R$ in class (a), we define $E = \mathrm{H}^0_{\mathfrak{m}}(R)$. By Proposition 3.1, $E$ is in class (b). We know from Corollary 3.5(1) that this map is one-to-one. To show that the map is onto, we pick any $E'$ in class (b). By Lemma 4.5(1), $E' = P \otimes_A E$ where $P$ is some invertible $A$-bimodule. Also, by Lemma 4.6(2) there is an $R'$ in class (a) which is associated to $E'$. Thus the pre-image of $E'$ is $R'$. □

**Corollary 4.8.** *Assume the hypotheses of Proposition 4.7. Suppose $A'$ and $B'$ are Morita equivalent to $A$ and $B$ respectively. Then there is a one-to-one correspondence between the following two classes*

*(a) Isomorphism classes of pre-balanced dualizing complexes over $(A', B')$.*

*(b) Isomorphism classes of bimodules which induce Morita dualities between $A'$ and $B'$.*

*Proof.* The assertion follows from Lemma 4.6 and Proposition 4.7. □

## 5. Finite Extensions

In this section we prove that the pre-balanced dualizing complex extends for finite ring extensions. As before we assume that $A$ and $B$ are Morita dual via an artinian bimodule $E$. As a consequence, $A$ and $B^\circ$ are left noetherian complete semilocal algebras.

In this section we also assume that $A$ and $B$ are two-sided noetherian. For the next construction we further assume that $A/\mathfrak{m}$ and hence $B/\mathfrak{n}$ is weakly symmetric.

We consider a ring extension $f : A \longrightarrow A'$ which is *finite* in the sense that $A'$ is a finite $A$-module on both sides, and is *local* in the sense that $f(\mathfrak{m}) \subseteq \mathfrak{m}'$ where $\mathfrak{m}'$ denotes the Jacobson radical of $A'$. These two hypotheses ensure that $A'$ is a complete noetherian semilocal ring and that $A'/\mathfrak{m}'$ is weakly symmetric. Also, the local hypothesis shows that the filtrations $\{A'\mathfrak{m}^r\}_{r\in\mathbb{N}}, \{\mathfrak{m}^r A'\}_{r\in\mathbb{N}}, \{\mathfrak{m}'^r\}_{r\in\mathbb{N}}$ are all cofinal. We wish to study Morita duality for $A'$.

In general, Morita duality does not behave well even with respect to finite ring extensions [Xu, 7.5]. This situation is remedied by our assumption that $A/\mathfrak{m}$ is weakly symmetric. We will in fact show that

$$E' := \mathrm{Hom}_A(A', E)$$

induces a Morita duality for $A'$. First note that by the adjunction formula, the $A'$-module $E'$ is an injective cogenerator. Note that the functor $\mathrm{Hom}_A(-, E)$ restricts to $\mathrm{Hom}_{A'}(-, E')$ on $\mathrm{Mod}(A')$.

**Lemma 5.1.** *Assume the notation given above. Suppose that $A/\mathfrak{m}$ is weakly symmetric. Then ${}_AE'_B$ is an artinian bimodule.*

*Proof.* Morita duality shows that $E'$ is artinian over $B$. We seek to show that it is artinian over $A$ by showing that it is $\mathfrak{m}$-torsion and has finite length socle. To this end, let $f \in E' = \operatorname{Hom}_A(A', E)$ and note that since $A'$ is finite over $A$, im $f \subseteq E^r := \operatorname{Hom}_A(A/\mathfrak{m}^r, E)$ for some $r$. We can choose $s \in \mathbb{N}$ such that $A'\mathfrak{m}^s \subseteq \mathfrak{m}^r A'$. Hence $\mathfrak{m}^s f = 0$ showing that $E'$ is indeed $\mathfrak{m}$-torsion. To bound the socle we note that for $s$ large enough we have,

$$\operatorname{soc} E' = \operatorname{Hom}_A(A/\mathfrak{m}, \operatorname{Hom}_A(A', E)) = \operatorname{Hom}_A(A'/A'\mathfrak{m}, E) \subseteq \operatorname{Hom}_A(A'/\mathfrak{m}^s A', E)$$

Weak symmetry now ensures that this is finite length. Hence ${}_AE'$ is indeed artinian. □

As a consequence, ${}_{A'}E'$ is also artinian. This artinian module $E'$ does induce a Morita duality for $A'$.

**Lemma 5.2.** *Assume the notation given above. Suppose that $A/\mathfrak{m}$ is weakly symmetric. Then $E'$ induces a Morita duality for $A'$.*

*Proof.* By [Xu, 7.3], we need to show that ${}_AA'$ and ${}_AE'$ are linearly compact (see [Xu, Section 3] for a definition). Now $A'$ is linearly compact since it is finite over $A$ (see [Xu, 3.3]) while $E'$ is linearly compact since, by [Xu, 3.1] artinian modules are. □

We let $B' = \operatorname{End}_{A'} E'$ denote the Morita dual of $A'$ induced by $E'$ and let $\mathfrak{n}'$ be its Jacobson radical. We wish to show that it plays a symmetric role to that of $A$. Now $B'$ is a semilocal ring since $B'/\mathfrak{n}'$ is Morita dual to $A'/\mathfrak{m}'$ and so also is semisimple artinian. Note that there is a natural action of $B$ on $E'$ which commutes with the action of $A'$. This induces a natural map $g : B \longrightarrow B'$. To see that $B'$ is a noetherian bimodule over $B$ we consider the following isomorphism of $B$-bimodules,

$$B' \simeq \operatorname{Hom}_{A'}(E', \operatorname{Hom}_A(A', E)) \simeq \operatorname{Hom}_A(E', E)$$

Since $E'$ is an artinian $(A, B)$-bimodule, [WZ1, 7.6] ensures this last term is noetherian on both sides. We can compute $B'/\mathfrak{n}'$ from [WZ1, 7.2(4)] which gives

$$B'/\mathfrak{n}' = \operatorname{Hom}_{A'}(\operatorname{Hom}_{A'}(A'/\mathfrak{m}', E'), E')$$

If we let $E^1$ denote $\operatorname{Hom}_A(A/\mathfrak{m}, E)$ then this last term can be rewritten as

$$\operatorname{Hom}_{A'}(\operatorname{Hom}_A(A'/\mathfrak{m}', E^1), E') = \operatorname{Hom}_A(\operatorname{Hom}_A(A'/\mathfrak{m}', E^1), E^1)$$

This last term is annihilated by $\mathfrak{n}$ (on both sides) which shows that $g(\mathfrak{n}) \subseteq \mathfrak{n}'$. We have thus proved part 1 of the following.

**Proposition 5.3.** *Assume the notation above. Suppose that $A/\mathfrak{m}$ is weakly symmetric.*

(1) *The Morita dual $B'$ is a finite local ring extension of $B$.*
(2) *As an $(A, B')$-bimodule, $E'$ is isomorphic to $\operatorname{Hom}_{B^\circ}(B', E)$.*

*Proof.* Consider $E'$ as an $(A, B')$-bimodule. Then

$$E' = \operatorname{Hom}_{B^\circ}(\operatorname{Hom}_A(E', E), E) = \operatorname{Hom}_{B^\circ}(\operatorname{Hom}_{A'}(E', E'), E) = \operatorname{Hom}_{B^\circ}(B', E)$$

where the first equality holds since $E'$ is an artinian bimodule and hence is $E$-reflexive. □

**Corollary 5.4.** *Let $A$ be an AS-Gorenstein noetherian complete semilocal ring. If $A/\mathfrak{m}$ is weakly symmetric, then every finite local ring extension of $A$ is left and right Morita.*

*Proof.* The assertion follows from Corollary 0.3 and Lemma 5.2. □

We say that $(A', B', E')$ is a *finite extension* of $(A, B, E)$ if there are algebra homomorphisms $A \to A'$ and $B \to B'$ such that

(1) $A$ and $B$ are noetherian,
(2) the maps $A \to A'$ and $B \to B'$ are finite and local,
(3) $A/\mathfrak{m}$ is weakly symmetric,
(4) $E' \cong \operatorname{Hom}_A(A', E)$ as $(A', B)$-bimodule, and
(5) $E' \cong \operatorname{Hom}_{B^\circ}(B', E)$ as $(A, B')$-bimodule.

By Proposition 5.3, when we have a finite local map $A \to A'$ and when $A/\mathfrak{m}$ is weakly symmetric, then there are $B'$ and $E'$ such that $(A', B', E')$ is a finite extension of $(A, B, E)$.

Next we show that when $(A, B, E)$ has a finite extension, the pre-balanced dualizing complex extends too. The following lemma was proved in [WZ1, 6.5].

**Lemma 5.5.** *Suppose $A$ is noetherian complete and semilocal and $A \to A'$ is finite and local. Then $\mathrm{R}\,\Gamma_{\mathfrak{m}'}$ is the restriction of $\mathrm{R}\,\Gamma_{\mathfrak{m}}$ to $D^+(A')$.*

**Theorem 5.6.** *Suppose $(A', B', E')$ is a finite extension of $(A, B, E)$. If $R$ is a pre-balanced dualizing complex associated to $E$, then there is a pre-balanced dualizing complex $R'$ over $(A', B')$ associated to $E'$. Further*

$$R' \cong \mathrm{R}\operatorname{Hom}_A(A', R)$$

*holds in $D(A' \otimes B^\circ)$ and*

$$R' \cong \mathrm{R}\operatorname{Hom}_{B^\circ}(B', R)$$

*holds in $D(A \otimes (B')^\circ)$.*

*Proof.* By Corollary 3.5(2), $\mathrm{R}\,\Gamma_{\mathfrak{m}}$ and $\mathrm{R}\,\Gamma_{\mathfrak{n}^\circ}$ have finite cohomological dimension so Lemma 5.5 shows that $\mathrm{R}\,\Gamma_{\mathfrak{m}'}$ and $\mathrm{R}\,\Gamma_{\mathfrak{n}'^\circ}$ also have finite cohomological dimension. Also, Corollary 3.5(3) shows that $A$ and $B^\circ$ satisfy the $\chi$-condition, so by Lemma 5.5 and [WZ2, 2.3], so do $A'$ and $(B')^\circ$.

Let $F'$ be the functor $\mathrm{Hom}_{A'}(\mathrm{R}\Gamma_{\mathfrak{m}'}(-), E')$ and $G'$ the functor $\mathrm{Hom}_{B'}(\mathrm{R}\Gamma_{\mathfrak{n}'^\circ}(-), E')$. Then $(F', G')$ are contravariant functors between $D(A')$ and $D(B'^\circ)$. From Lemma 5.5 and the fact that $\mathrm{Hom}_A(-, E)$ restricts to $\mathrm{Hom}_{A'}(-, E')$, we see that $F := \mathrm{Hom}_A(\mathrm{R}\Gamma_{\mathfrak{m}}(-), E)$ restricts to $F'$ when applied to $D^+(A')$. Similarly $G := \mathrm{Hom}_{B^\circ}(\mathrm{R}\Gamma_{\mathfrak{n}^\circ}(-), E)$ restricts to $G'$. By local duality (see Proposition 3.4), $F \cong \mathrm{Hom}_A(-, R)$ and $G \cong \mathrm{Hom}_{B^\circ}(-, R)$. Hence $(F, G)$ defines a duality between $D^b_f(A)$ and $D^b_f(B^\circ)$. This restricts to a duality $(F', G')$ between $D^b_f(A')$ and $D^b_f(B'^\circ)$, thus verifying condition (D1) in [Mi, p.156]. Since $\mathrm{R}\Gamma_{\mathfrak{m}'}$ and $\mathrm{R}\Gamma_{\mathfrak{n}'^\circ}$ have finite cohomological dimension, conditions (D2r) and (D2l) of [Mi, p.156] hold. Therefore $A'$ and $B'$ are Morita derived dual in the sense of [Mi, p.156]. By [Mi, 3.3], there is a dualizing complex ${}_{A'}U_{B'}$ (or cotilting bimodule complex in Miyachi's terminology) between $A'$ and $B'$.

We now prove that $U$ is pre-balanced. By [Mi, 3.5], the complex $U$ constructed in [Mi, 3.3] has the following property: for any $X \in D^b_f(A')$,

$$F'(X) = \mathrm{R}\mathrm{Hom}_{A'}(X, U)$$

in $D(B'^\circ)$. Let $X$ be a simple $A'$-module $S$. Then $\mathrm{R}\mathrm{Hom}_{A'}(S, U) = \mathrm{Hom}_{A'}(S, E')$, which is a simple right $B'$-module. This shows half of the pre-balanced condition. The other half follows from the double Ext spectral sequence [YZ2, 1.7]. Therefore $U$ is pre-balanced. By Proposition 4.7, there is a pre-balanced dualizing complex $R'$ associated to $E'$.

Finally, by local duality Corollary 3.5(1) and Proposition 3.4,

$$R' \cong \mathrm{R}\mathrm{Hom}_{A'}(\mathrm{R}\Gamma_{\mathfrak{m}'}(A'), E') \cong \mathrm{R}\mathrm{Hom}_A(\mathrm{R}\Gamma_{\mathfrak{m}}(A'), E) \cong \mathrm{R}\mathrm{Hom}_A(A', R)$$

in $D(A' \otimes B^\circ)$. The other formula holds similarly. □

Let us mention two special cases. The first case is when $A'$ is a factor ring of $A$.

**Lemma 5.7 [AF, 24.6].** *Suppose $A$ and $B$ are Morita dual via a bimodule $E$. Then there is an isomorphism $\phi$ from the lattice of ideals of $A$ to the lattice of ideals of $B$. Further, if $I$ is an ideal of $A$ and $J = \phi(I)$, then $A/I$ and $B/J$ are Morita dual via the bimodule*

$$E' := \mathrm{Hom}_A(A/I, E) = \mathrm{Hom}_{B^\circ}(B/J, E).$$

*In particular, if $A$ and $B$ are noetherian and $A/\mathfrak{m}$ is weakly symmetric, then $(A/I, B/J, E')$ is a finite extension of $(A, B, E)$.*

The next result is an immediate consequence of Theorem 5.6.

**Corollary 5.8.** *Let $(A/I, B/J, E')$ be as in Lemma 5.7. Suppose $R$ is a pre-balanced dualizing complex over $(A, B)$ associated to $E$. Then there is a pre-balanced dualizing complex over $(A/I, B/J)$ associated to $E'$. Further,*

$$R' \cong \mathrm{R}\mathrm{Hom}_A(A/I, R) \cong \mathrm{R}\mathrm{Hom}_{B^\circ}(B/J, R)$$

*in $D(A \otimes B^\circ)$.*

The second case is when $A/\mathfrak{m}$ is finite dimensional over $k$. The following is not hard to check and the proof is omitted.

**Lemma 5.9.** *Let $A$ be a noetherian complete semilocal algebra such that $A/\mathfrak{m}$ is finite dimensional. Let $E_A = \varinjlim \operatorname{Hom}_k(A/\mathfrak{m}^n, k)$. Let $(A, A, E_A)$ be the Morita duality.*

(1) *If $M$ is a finite $A$-module, then*

$$\operatorname{Hom}_A(M, E_A) = \varinjlim \operatorname{Hom}_k(A/\mathfrak{m}^n \otimes_A M, k).$$

(2) *If $M$ is an artinian $A$-module, then*

$$\operatorname{Hom}_A(M, E_A) = \varprojlim \operatorname{Hom}_k(\operatorname{Hom}_A(A/\mathfrak{m}^n, M), k).$$

(3) *If $A \to C$ is a finite local map, then $(C, C, E_C)$ is a finite extension of $(A, A, E_A)$.*

(4) *If $C = A/I$ for some ideal $I$, then $\phi(I) = I$ (where $\phi$ is the map described in Lemma 5.7). And*

$$E_C = \operatorname{Hom}_A(A/I, E_A) = \operatorname{Hom}_{A^\circ}(A/I, E_A).$$

The following corollary follows from Theorem 5.6 and Lemma 5.9.

**Corollary 5.10.** *Let $A$ be a noetherian complete semilocal algebra such that $A/\mathfrak{m}$ is finite dimensional. Let $R_A$ be a balanced dualizing complex over $A$. If $A \to C$ is a finite local map, then $C$ has a balanced dualizing complex $R_C$ and*

$$R_C \cong \operatorname{R}\operatorname{Hom}_A(C, R_A) \cong \operatorname{R}\operatorname{Hom}_{A^\circ}(C, R_A)$$

*in $D(A \otimes A^\circ)$.*

Corollary 0.5 is a special case of Corollary 5.10.

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