Noncommutative Cyclic Covers and Maximal Orders on Surfaces

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Abstract

In this paper, we construct various examples of maximal orders on surfaces, including some del Pezzo orders, some ruled orders and some numerically Calabi-Yau orders. The method of construction is a noncommutative version of the cyclic covering trick. These noncommutative cyclic covers are very computable and we give a formula for their ramification data. This often allows us to determine if a maximal order, described via ramification data can be constructed as a noncommutative cyclic cover. The construction also has applications to Brauer-Severi varieties and, in the quaternion case, we show how to obtain some Brauer-Severi varieties from G-Hilbert schemes of $\mathbb{P}^1$-bundles.

Dedicated to Mike Artin on the occasion of his 70th birthday.

Throughout, all objects and maps are assumed to be defined over some algebraically closed base field k of characteristic zero and all schemes are noetherian.

1 Introduction

Over the last few years, there has been a program to classify maximal orders on surfaces as a prelude to the general classification of noncommutative surfaces [AdJ], [CK03],[CI]. So far, the classification has focussed on determining possible ramification data. It is based on a sequence of Artin and Mumford [AM, theorem 1] which gives necessary and sufficient conditions for maximal orders with given ramification data to exist. Unfortunately, though we know maximal orders with various ramification data exist, there are very few maximal orders which have been written down explicitly. The purpose of this paper is to give a natural simple construction of a large class of maximal orders on surfaces. More precisely, given ramification data on a projective surface, we seek to construct maximal orders with that ramification. This can be considered the second phase of the classification program for orders on surfaces.

The construction we use is called the noncommutative cyclic cover and can be viewed as a noncommutative analogue of the cyclic covering trick. It involves replacing the line bundle in the cyclic covering trick with its noncommutative incarnation, the invertible bimodule. There are some subtleties which appear in that an overlap condition must be satisfied before the commutative definition can be made to work in this setting. Noncommutative cyclic covers have also been considered in [LVV] in the affine case under the name roll-up algebras, and our construction can also be viewed as a globalisation of theirs. The noncommutative cyclic cover can also be described as invariant rings of certain trivial Azumaya algebras. In this guise, they can be viewed as globalisations of Artin's local construction in [A86].

In section 2, we recall some preliminaries primarily concerning invertible bimodules. In section 3, we define noncommutative cyclic covers, compute their ramification indices and give a sufficient criterion for when Brauer classes of function fields can be represented by noncommutative cyclic covers. Section 4 is devoted primarily to studying when the overlap condition holds. The next few sections are concerned with constructing some of the maximal orders on surfaces which appear in the (as yet still incomplete) classification of possible ramification data. The ones which do arise in this way are special but, fortunately, wide in variety. We construct some del Pezzo orders, some ruled orders and some numerically Calabi-Yau orders.

It is hoped that the explicit nature of the noncommutative cyclic cover will enable us to study these maximal orders better. We illustrate the computatibility of this construction by our formula for ramification and our demonstration (in section 8) that certain ruled orders which can be constructed as noncommutative cyclic covers are noncommutative $\mathbb{P}^1$-bundles in the sense of [VdB01]. The latter is done explicitly, with an actual formula for the relative tautological bundle. It is also hoped that the explicit computations will be helpful in studying Brauer-Severi varieties. Indeed, unramified noncommutative cyclic covers have the form $A = (\mathcal{E}nd\, V)^G$ for V some vector bundle and G a cyclic group. Hence the corresponding Brauer-Severi variety is birationally $\mathbb{P}(V)/G$. We give explicitly both V and the action of G on $\mathbb{P}(V)$ in this case. Furthermore, we show that for quaternion orders arising this way, the Brauer-Severi variety can be obtained nicely from the G-Hilbert scheme of $\mathbb{P}(V)$.

Acknowledgements: Firstly, I would like to thank my teacher, Mike Artin for his mathematical inspiration. He once asked me as a graduate student to compile a compendium of noncommutative rings. I never got far with this task, but this paper is a compendium of maximal orders on surfaces. I would also like to thank Paul Hacking for filling in some gaps in my algebraic geometry.

2 Background

We need a noncommutative generalisation of the notion of invertible sheaves as studied by Artin and Van den Bergh [AV]. In this section, we briefly review this concept and more generally the notion of bimodules over a scheme as well as the notion of $\mathbb{Z}$-algebras which are used to define noncommutative ruled surfaces. The reader is referred to [AV], [Vdb96, §2, §3] and [Vdb01, §3] for details.

Let Y, Y' be schemes of finite type over a base scheme S. A coherent sheaf $\mathcal{B}$ on $Y \times_S Y'$ is a *coherent* $\mathcal{O}_S$*-central* (Y, Y')*-bimodule* if the projection maps Supp $\mathcal{B} \longrightarrow Y$, Supp $\mathcal{B} \longrightarrow Y'$ are finite. A *quasi-coherent* $\mathcal{O}_S$*-central* (Y, Y')*-bimodule* is a filtered direct limit of such coherent bimodules. Pushing forward to Y or Y' give the left and right module structures ${}_{\mathcal{O}_Y}\mathcal{B}$ and $\mathcal{B}_{\mathcal{O}_{Y'}}$ respectively. If $\mathcal{B}'$ is a (Y', Y'')-bimodule for some scheme Y'', then there is a naturally defined (Y, Y'')-bimodule $\mathcal{B} \otimes_{\mathcal{O}_{Y'}} \mathcal{B}'$. For the cases we need, a formula for computing this tensor product is given below. We will primarily be interested in (Y, Y)-bimodules which are also called $\mathcal{O}_Y$*-bimodules*. Quasi-coherent $\mathcal{O}_Y$-bimodules form a monoidal category [VdB01] so there is a notion of algebra objects in this category, called $\mathcal{O}_Y$*-bimodule algebras* as well as a notion of modules over an $\mathcal{O}_Y$-bimodule algebra. The definitions are the natural ones and can be found in [VdB96, definition 3.1, definition 3.3].

The invertible objects in the category of coherent $\mathcal{O}_Y$-bimodules are called *invertible bimodules*. These were classified in [AV, proposition 2.15] as follows. Let $L \in \operatorname{Pic} Y, \sigma \in \operatorname{Aut} Y$ and $\Gamma \subset Y \times Y$ be the graph of σ. If $\pi : \Gamma \longrightarrow Y$ is the projection then we define $L_\sigma = \pi^* L$. Then L_σ is an invertible bimodule and every invertible bimodule arises in this fasion. Tensoring of invertible bimodules is given by the simple formula

$$L_\sigma \otimes_{\mathcal{O}_Y} M_\tau \simeq (L \otimes_{\mathcal{O}_Y} \sigma^* M)_{\tau\sigma}.$$

This formula for tensor products of bimodules will suffice for our computational purposes and the reader is welcome to take it as a definition.

For the cases we are interested in, the base scheme is k, but the bimodules $\mathcal{B}$ will have support in $Y \times_Z Y \subset Y \times Y$ where Z is scheme with $Y \longrightarrow Z$ finite. In this case, the bimodule can be regarded as being $\mathcal{O}_Z$-central and $\mathcal{B}$ is a bimodule over the sheaf of algebras $\mathcal{O}_Y$ on Z. Furthermore, if $\mathcal{B}$ is also an $\mathcal{O}_Y$-bimodule algebra then it can be viewed as a noncommutative sheaf of algebras on Z. We will construct orders on Z in this fashion.

For section 8, we need a generalisation of the concept of graded algebras, called in the literature, $\mathbb{Z}$-algebras. They perhaps should more accurately be referred to as $\mathbb{Z}$-indexed algebras but we will follow current conventions. We will briefly sketch the definition here and leave it to the reader to examine [BP] and [VdB01, §3.2] for details.

Let $T_n, n \in \mathbb{Z}$ be schemes of finite type. A *sheaf $\mathbb{Z}$-algebra* on Z is a collection of quasi-coherent (T_m, T_n)-bimodules $\mathcal{A}_{mn}, m \leq n$ together with multiplication maps $\mathcal{A}_{lm} \otimes_{\mathcal{O}_{T_m}} \mathcal{A}_{mn} \longrightarrow \mathcal{A}_{ln}$ and identity maps $\mathcal{O}_{T_N} \longrightarrow \mathcal{A}_{nn}$ satisfying the usual unit and associativity axioms. We will usually write $\mathcal{A} = \oplus \mathcal{A}_{mn}$ to denote the sheaf $\mathbb{Z}$-algebra. A *(graded) right module over* $\mathcal{A}$ is a collection $M = \oplus_{n\in} M_n$ of quasi-coherent $\mathcal{O}_{T_n}$-modules M_n with a scalar multiplication $M_m \otimes_{\mathcal{O}_{T_m}} \mathcal{A}_{mn} \longrightarrow M_n$ satisfying the usual unit and associativity axioms for modules. We let $\mathrm{Gr}\,\mathcal{A}$ denote the category of such graded modules. A module $M \in \mathrm{Gr}\,\mathcal{A}$ is said to be *right bounded* if $M_n = 0$ for $n >> 0$. Let tors denote the Serre subcategory of direct limits of right bounded modules and $\mathrm{Proj}\,\mathcal{A} := \mathrm{Gr}\,\mathcal{A}/\mathrm{tors}$.

Van den Bergh uses sheaf $\mathbb{Z}$-algebras to define noncommutative ruled surfaces (see [VdB01, §4] for details). A *noncommutative* $\mathbb{P}^1$*-bundle* is a category of the form $\mathrm{Proj}\,\mathcal{A}$ where $\mathcal{A}$ is a sheaf $\mathbb{Z}$-algebra constructed as follows. The schemes $T_n = T$ if n is even and T' if n is odd, are both smooth. The data involved to construct $\mathcal{A}$ is a set of (T_n, T_{n+1})-bimodules $\mathcal{B}_n, n \in N$ which are locally free of rank two on the left and on the right and a set of invertible bimodules $Q_n \subset \mathcal{B}_n \otimes_{\mathcal{O}_{T_{n+1}}} \mathcal{B}_{n+1}$. We consider the $\mathcal{B}_n$ as generators for the $\mathbb{Z}$-algebra in the sense that we set $\mathcal{A}_{n,n+1} = \mathcal{B}_n$ and view the Q_n as quadratic relations. We assume that the relations Q_n are non-degenerate in the sense that the natural map $\mathcal{B}_n^* \otimes_{\mathcal{O}_{T_n}} Q_n \longrightarrow \mathcal{B}_{n+1}$ is an isomorphism. We define $\mathcal{A}$ to be the sheaf $\mathbb{Z}$-algebra generated by $\mathcal{B}_n$ with defining relations Q_n. In particular, we will have $\mathcal{A}_{nn} \simeq \mathcal{O}_{T_n}$ and $\mathcal{A}_{n,n+2} = (\mathcal{B}_n \otimes_{\mathcal{O}_{T_{n+1}}} \mathcal{B}_{n+1})/Q_n$. A *noncommutative ruled surface* is a noncommutative $\mathbb{P}^1$-bundle constructed from smooth curves T, T'.

3 Noncommutative Cyclic Covers

Let Y be an integral scheme. Let σ be an automorphism of Y and $L \in \mathrm{Pic}\,Y$. Let $D \geq 0$ be an effective Cartier divisor and suppose there is an integer e and an isomorphism of invertible bimodules $\phi : L_\sigma^e \xrightarrow{\sim} \mathcal{O}(-D)$. We consider the composite morphism $\mathrm{L}_\sigma^e \longrightarrow \mathcal{O}_\mathrm{Y}(-D) \hookrightarrow \mathcal{O}_\mathrm{Y}$, also denoted ϕ as a relation on the tensor algebra

$$T(Y; L_\sigma) := \bigoplus_{n\geq 0} L_\sigma^n.$$

Note that for the relation to exist we must have $\sigma^e = \mathrm{id}$. We will assume that e is the order of σ. Also, we will usually assume that ϕ satisfies the *overlap condition*, that is, that the following diagram is commutative.

$$\begin{array}{ccc} L_\sigma \otimes_Y L_\sigma^{e-1} \otimes_Y L_\sigma & \xrightarrow{1\otimes\phi} & L_\sigma \otimes_Y \mathcal{O}_\mathrm{Y} \\ \downarrow{\scriptstyle \phi\otimes 1} & & \downarrow{\scriptstyle \phi} \\ \mathcal{O}_\mathrm{Y} \otimes_Y L_\sigma & \xrightarrow{\phi} & L_\sigma \end{array} \tag{1}$$

Definition 3.1 *Suppose the relation ϕ satisfies the overlap condition (1). In this case, let $A(Y; L_\sigma, \phi)$ denote the algebra with relations $T(L_\sigma)/(\phi)$.The* noncommutative cyclic cover *of Y with respect to L_σ and ϕ is $\mathcal{S}pec_Y\, A(Y; L_\sigma, \phi)$ and $A(Y; L_\sigma, \phi)$ is called the* cyclic algebra.

An immediate consequence of the generalised Bergman's diamond lemma (see [C, theorem 2.2]) is

Proposition 3.2 *If the relation $\phi : L_\sigma^e \longrightarrow \mathcal{O}_\mathrm{Y}$ satisfies the overlap condition then*

$$A(Y; L_\sigma, \phi) = \bigoplus_{i=0}^{e-1} L_\sigma^i.$$

Example 3.3 *Cyclic division algebras.*

Let $Y = \mathrm{Spec}\, K$ where K is a field and $\sigma \in \mathrm{Aut}\, K$. Let G be the cyclic group generated by σ and $F = K^G$ so that K/F is a cyclic field extension. We shall write the invertible bimodule K_σ as Kz so that scalars skew commute through z via $z\alpha = \sigma(\alpha)z$. The tensor powers are of course $(Kz)^{\otimes n} \simeq Kz^n$ where scalars skew commute through z^n in the expected way, $z^n\alpha = \sigma^n(\alpha)z^n$. Suppose we are given a relation of the form $\phi : Kz^e \xrightarrow{\sim} K$. Note, ϕ is given by multiplication by some $\alpha \in K$ so that $z^e = \alpha$. The overlap condition is equivalent to $\alpha z = z\alpha$, in other words, $\alpha \in K^G = F$. The resulting cyclic algebra $A(Y; Kz, \phi)$ is none other than the classical cyclic central simple algebra $K[z;\sigma]/(z^e - \alpha)$. Note that the above computation for the cyclic algebra works so long as Y is an affine scheme and $L = \mathcal{O}_Y\, z$ for some two-sided generator $z \in L$.

This gives the following

Corollary 3.4 *Let Y be a normal integral Cohen-Macaulay scheme, $\sigma \in AutY$ be an automorphism satisfying $\sigma^e = id$ and let $G = \langle\sigma\rangle$. Suppose that $Z := Y/G$ is a scheme. Let $\phi : L_\sigma^e \longrightarrow \mathcal{O}_Y$ be a relation as above. Then ϕ satisfies the overlap condition if and only if generically ϕ is multiplication by some $\alpha \in K(Z)$ and in that case the cyclic algebra $A(Y; L_\sigma)$ is a reflexive order in the cyclic algebra $K(Y)[z;\sigma]/(z^e - \alpha)$. Furthermore, the centre of A is Z.*

Consider a relation $\phi : L_\sigma^e \longrightarrow \mathcal{O}_Y$. Given an isomorphism of invertible bimodules $\beta : M_\tau \longrightarrow L_\sigma$, there is an induced relation $\phi' : M_\tau^e \longrightarrow \mathcal{O}_Y$. We will say ϕ, ϕ' are *isomorphic* relations in this case. Of course, the cyclic algebras formed from isomorphic relations are isomorphic.

We now fix $\sigma \in \mathrm{Aut}\, Y$. The set of (isomorphism classes of) relations Rel forms a monoid in the following fashion. Consider two relations $\phi : L_\sigma^e \longrightarrow \mathcal{O}_Y, \psi : M_\sigma^e \longrightarrow \mathcal{O}_Y$. Then we can define their product to be the relation

$$\phi \otimes \psi : (L \otimes_Y M)_\sigma^e \xrightarrow{\sim} L_\sigma^e \otimes_Y M_\sigma^e \xrightarrow{\phi\otimes\psi} \mathcal{O}_Y \otimes_Y \mathcal{O}_Y = \mathcal{O}_Y\,.$$

The identity is the canonical morphism $\mathcal{O}_\sigma^e \xrightarrow{\sim} \mathcal{O}_Y$. Note that the subset Rel_o of relations satisfying the overlap condition is a submonoid and the subset Rel_i of relations $\phi : L_\sigma^e \xrightarrow{\sim} \mathcal{O}_Y$ which are isomorphisms is the subgroup of invertible elements. Finally, the intersection $\mathrm{Rel}_{io} = \mathrm{Rel}_i \cap \mathrm{Rel}_o$ is a subgroup of Rel.

Automorphisms of L induce isomorphic relations. We need to examine these. Consider a relation $\phi : \mathrm{L}_\sigma^e \longrightarrow \mathcal{O}_Y$. For any $\beta \in \mathcal{O}(Y)^*$ there are induced isomorphisms $L_\sigma^i \xrightarrow{\sim} L_\sigma^i$ obtained by multiplication by $\beta\,\sigma\,\beta \ldots \sigma^{i-1}(\beta)$. These sum together to give an isomorphism of algebras $A(Y; L_\sigma, \phi) \xrightarrow{\sim} A(Y; L_\sigma, \mathrm{Nr}\,(\beta)\phi)$ where Nr denotes the norm with respect to the action of G on Y. More generally, if $\beta \in K(Y)^*$ is such that $\mathrm{Nr}\,\beta \in \mathcal{O}(Y)^*$ then changing ϕ to $\mathrm{Nr}\,(\beta)\phi$ yields a generically isomorphic algebra. Consequently, we shall say two relations are *0-equivalent* if they differ by a norm in this fashion. Let E_0 be the subgroup of Rel of relations which are 0-equivalent to the trivial relation. Then two relations are 0-equivalent if and only if they differ in Rel by an element of E_0. Note that $E_0 < \mathrm{Rel}_{io}$.

There is also an easy way to change the relation to pass to a Morita equivalent algebra which is generically isomorphic to the original. Let $M \in \mathrm{Pic}\, Y$ and $\phi : L_\sigma^e \longrightarrow \mathcal{O}_Y$ be a relation satisfying the overlap condition. We may alter L_σ to

$$L'_\sigma := M \otimes_Y L_\sigma \otimes_Y M^{-1} = (M \otimes_Y L \otimes_Y \sigma^*(M^{-1}))_\sigma$$

and the relation to

$$\phi' : (M \otimes_Y L_\sigma \otimes_Y M^{-1})^e = M \otimes_Y L_\sigma^e \otimes_Y M^{-1} \xrightarrow{1\otimes\phi\otimes 1} M \otimes_Y \mathcal{O}_Y \otimes_Y M^{-1} \xrightarrow{\sim} \mathcal{O}_Y\,.$$

With this relation, it is easy to see that

$$A(Y; L'_\sigma, \phi') \simeq M \otimes_Y A(Y; L_\sigma, \phi) \otimes_Y M^{-1}.$$

Consequently, $A(Y; L', \phi')$ is generically isomorphic to $A(Y; L_\sigma, \phi)$ and is Morita equivalent with

$$- \otimes_{\mathcal{O}_Y} M : \quad \text{mod} - A(Y; L'_\sigma) \longrightarrow \quad \text{mod} - A(Y; L_\sigma)$$

giving the module category equivalence. We shall say that the above two relations are *1-equivalent*. Let E_1 be the subgroup of relations 1-equivalent to the trivial relation. Then two relations are 1-equivalent if and only if they differ by an element of E_1. Note $E_1 < \text{Rel}_{io}$. We let E be the subgroup of Rel generated by E_0, E_1.

In the unramified case, relations can be classified neatly using cohomology. Let $G = \langle\sigma\rangle$ and consider Pic Y as a G-set. Recall that group cohomology of a G-set M can be computed as the cohomology of the periodic sequence

$$\cdots \xrightarrow{N} M \xrightarrow{D} M \xrightarrow{N} M \xrightarrow{D} \cdots$$

where the differentials are multiplication by $N = 1 + \sigma + \ldots + \sigma^{e-1}$ and $D = 1 - \sigma$. From this, we see that the 1-cocycles L of Pic Y are precisely the invertible bimodules of the form L_σ such that $L_\sigma^e \simeq \mathcal{O}_Y$.

Let $\lambda \in H^1(G, \text{Pic } Y)$ correspond to L. We assume that $\pi : Y \longrightarrow Z$ is an étale Galois cover and, as in [A], we consider the Hochschild-Serre spectral sequence

$$H^p(G, H^q(Y, \mathbb{G}_m)) \Rightarrow H^{p+q}(Z, \mathbb{G}_m).$$

Writing E_2^{pq} terms out explicitly we have

$$\begin{array}{llll} B(Y)^G & & & \\ \text{Pic}\,(Y)^G & H^1(G, \text{Pic } Y) & & \\ \mathcal{O}(Y)^{*G} & H^1(G, \mathcal{O}(Y)^*) & H^2(G, \mathcal{O}(Y)^*) & H^3(G, \mathcal{O}(Y)^*) \end{array} \tag{2}$$

where $B(-)$ denotes the second cohomology group $H^2(-, \mathbb{G}_m)$.

Lemma 3.5 *Suppose that Z is smooth and that $\pi : Y \longrightarrow Z$ is étale. There is an exact sequence of the form*

$$Pic\,(Y)^G \xrightarrow{d_2} H^2(G, \mathcal{O}(Y)^*) \longrightarrow Rel_{io}/E \xrightarrow{f} H^1(G, Pic\ Y) \xrightarrow{d_2} H^3(G, \mathcal{O}(Y)^*)$$

where d_2 is the second differential in the Hochschild-Serre spectral sequence (2) above and f is the forgetful map sending $\phi : L_\sigma^e \xrightarrow{\sim} \mathcal{O}_Y$ to the cohomology class λ representing L.

Proof. Let $K = K(Y)$ denote the constant sheaf of rational functions on Y and $L \in \text{Rel}_i$. We will embed L in K so that all $\sigma^{*i} L$ are also subsheaves of K. Now $L_\sigma^e \simeq \mathcal{O}_Y$ if and only if there exists $\alpha \in K^*$ with $L(\sigma^* L) \ldots (\sigma^{*(e-1)} L) = \alpha\, \mathcal{O}_Y$. Note that α is a 2-cocycle of the G-set $K^*/\,\mathcal{O}(Y)^*$. The corresponding cohomology class $[\alpha] \in H^2(G, K^*/\,\mathcal{O}(Y)^*)$ can be described using the exact sequence

$$0 \longrightarrow K^*/\,\mathcal{O}(Y)^* \longrightarrow \text{Div } Y \longrightarrow \text{Pic } Y \longrightarrow 0$$

where Div Y denotes the group of divisors. If $\delta : H^1(G, \text{Pic } Y) \longrightarrow H^2(G, K^*/\,\mathcal{O}(Y)^*)$ denotes the connecting homomorphism arising from the long exact sequence in cohomology then $[\alpha] = \delta(\lambda)$. From the exact sequence

$$0 \longrightarrow \mathcal{O}(Y)^* \longrightarrow K^* \longrightarrow K^*/\,\mathcal{O}(Y)^* \longrightarrow 0$$

we obtain another connecting homomorphism $\partial : H^2(G, K^*/\,\mathcal{O}(Y)^*) \longrightarrow H^3(G, \mathcal{O}(Y)^*)$. Note that $d_2 = \partial\delta : H^1(G, \text{Pic } Y) \longrightarrow H^3(G, \mathcal{O}(Y)^*)$.

A relation $\phi : L_\sigma^e \xrightarrow{\sim} \mathcal{O}_Y$ is given by multiplication by $\alpha^{-1}\beta : L(\sigma^* L) \ldots (\sigma^{*(e-1)} L) \longrightarrow \mathcal{O}_Y$ where $\beta \in \mathcal{O}(Y)^*$. As in example 3.3, this satisfies the overlap condition if and only if $\alpha^{-1}\beta \in K^G$ or in other words $\alpha\,\sigma(\alpha)^{-1} = \beta\,\sigma(\beta)^{-1}$. Now the set of $\beta\,\sigma(\beta)^{-1}$ are the 3-coboundaries of the G-set $\mathcal{O}(Y)^*$ so β satisfying the overlap condition exists if and only if

$$d_2(\lambda) = \partial([\alpha]) = [\alpha\,\sigma(\alpha)^{-1}]$$

is trivial in $H^3(G, \mathcal{O}(Y)^*)$. Hence f surjects onto $\ker d_2 \subseteq H^1(G, \text{Pic}\, Y)$.

We now check exactness at Rel_{io}/E. By example 3.3, a relation of the form $\mathcal{O}_Y \simeq \mathcal{O}_Y$ satisfies the overlap condition if and only if it is multiplication by some element of $\mathcal{O}(Y)^{*G}$. Now $H^2(G, \mathcal{O}(Y)^*) = \mathcal{O}(Y)^{*G}/\,\text{Nr}\;\mathcal{O}(Y)^*$ and we saw that changing relations by a norm yields an isomorphic relation. Hence, there is a surjective map $H^2(G, \mathcal{O}(Y)^*) \longrightarrow \ker f$.

Finally, we check exactness at the $H^2(G, \mathcal{O}(Y)^*)$ term. As above, we factor d_2 as

$$\text{Pic}\,(Y)^G \longrightarrow H^1(G, K^*/\,\mathcal{O}(Y)^*) \longrightarrow H^2(G, \mathcal{O}(Y)^*).$$

Now Div Y is free abelian so the first map is surjective while the second map has image precisely the norms Nr β of those elements $\beta \in K^*$ with Nr $\beta \in \mathcal{O}(Y)^*$. This gives the desired exactness.

Let A be an order on a normal integral Cohen-Macaulay scheme Z. We define the dualising module of A to be $\omega_A := \mathcal{H}om_{\mathcal{O}_Z}(A, \omega_Z)$ in accordance with noncommutative duality theory. Recall also that A is *normal* ([CI, definition 2.3]) if it is reflexive and is hereditary in codimension one and in fact at every codimension one point we have $\omega_A \simeq A$ as a left A-module and a right A-module. This latter condition is also called standard by Artin [A, definition 2.13] and principal by Hijikata-Nishida [HN]. It is easy to check if a reflexive order A is normal at some codimension one point C of its centre Z. If R is the strict henselisation of the local ring $\mathcal{O}_{Z,C}$ and t is a uniformising parameter, then A is normal at C if and only if it has the form

$$A = \begin{pmatrix} R & \ldots\ldots\ldots & R \\ tR & \ddots & \vdots \\ \vdots & \ddots \quad \ddots & \vdots \\ tR & \ldots \quad tR & R \end{pmatrix}^{n\times n} .$$

Theorem 3.6 *Let $G = \langle \sigma \,|\, \sigma^e = 1\rangle$ act generically faithfully on a normal, integral, Cohen-Macaulay scheme Y. Suppose that the quotient $Z := Y/G$ is actually a scheme. Let L_σ be an invertible bimodule, D an effective reduced Cartier divisor and $\phi : L_\sigma^e \xrightarrow{\sim} \mathcal{O}_Y(-D)$ a relation satisfying the overlap condition. Suppose that the quotient map $\pi : Y \longrightarrow Z$ is unramified generically at the components of D. Then D is G-invariant and $A(Y; L_\sigma, \phi)$ is a normal order. Furthermore, if C is a codimension one prime of Z not contained in $\pi(D)$ then the ramification index of A at C is the ramification index of π above C. If C is a component of $\pi(D)$ then the ramification index of A above C is e.*

Proof. Reflexivity has already been observed so we need only verify the theorem in the case where $Z =$ Spec R is a discrete valuation ring say with uniformising parameter t. Furthermore, since the property of normality and the ramification indices are stable under étale extensions, we may assume that $Y =$ Spec S where

$$S = \prod_{i=1}^{d} S_i \quad \text{and} \quad S_i \simeq R[t^{1/c}] \quad \text{for} \quad c = \frac{e}{d}.$$

Let $s = t^{1/c}$ and ζ be a primitive c-th root of unity. We may assume that σ cycles through the S_i and $\sigma^d : s \mapsto \zeta\, s$. More specifically, $\sigma : S \longrightarrow S : (s_1, \ldots, s_d) \mapsto (s_2, s_3, \ldots, s_d, \sigma^d(s_1))$. We will write S' for $R[t^{1/c}]$ which we can view as a subalgebra of S via the diagonal morphism.

As computed in example 3.3, the overlap condition implies that $A = S[z; \sigma]/(z^e - \alpha t^a)$ where $\alpha \in R^*$. Note that D corresponds to the zeros of αt^a so $a = 0, 1$ and D is invariant under the action of G. We may assume that $\alpha = 1$ by taking étale extensions if necessary and scaling z. We consider the subalgebra $A' = S'\langle z^d\rangle$ of A. We will need the following

Lemma 3.7 *There is an injective algebra morphism $A \longrightarrow A'^{(d\times d)}$ defined by*

$$z \mapsto \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ z^d & 0 & \dots & \dots & 0 \end{pmatrix} \quad , \quad (s_1, \dots, s_d) \mapsto \begin{pmatrix} s_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & s_d \end{pmatrix}$$

where $(s_1, \dots, s_d) \in S$. The image of the morphism is

$$\begin{pmatrix} A' & \dots\dots\dots & A' \\ z^d A' & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ z^d A' & \dots & z^d A' & A' \end{pmatrix}.$$

We omit the proof which is a simple computation.

If D is not zero then by assumption, S/R is unramified so $d = e, a = 1, A' = R$ so $z^d A' = tR$. The lemma shows that A is normal with ramification index e.

Suppose now that $D = 0$ so that $z^e = 1$. This implies that z^d is invertible in A' so by the lemma, A is just a full matrix algebra in A'. We may thus assume that S/R is totally ramified. The theorem now follows from

Lemma 3.8 *There is an isomorphism of R-algebras*

$$\phi : A \longrightarrow \begin{pmatrix} R & \dots\dots\dots & R \\ tR & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ tR & \dots & tR & R \end{pmatrix}$$

defined by

$$s \mapsto \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ t & 0 & \dots & \dots & 0 \end{pmatrix} \quad , \quad z \mapsto \begin{pmatrix} \zeta^{e-1} & 0 & \dots & 0 \\ 0 & \zeta^{e-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

The theorem shows that if $Y \longrightarrow Z$ is Galois étale then, for a relation $\phi : L_\sigma^e \xrightarrow{\sim} \mathcal{O}_Y$ in Rel_{io}, $A(Y; L_\sigma, \phi)$ is Azumaya in codimension one. Moreover, if Z is smooth (so Y is too), then the Brauer class of $A(Y; L_\sigma, \phi)$ lives in $B(Y/Z) := \ker(B(Z) \longrightarrow B(Y)^G)$ since this is true generically and $B(Y)$ embeds in $B(K(Y))$. Hence we obtain a map

$$\Phi : \mathrm{Rel}_{io} \longrightarrow B(Y/Z).$$

Theorem 3.9 *Let Y be a smooth scheme and $G = \langle\sigma\rangle \subseteq Aut Y$. Suppose that the quotient map $\pi : Y \longrightarrow Z := Y/G$ is étale. Then the map Φ factors through a morphism $Br : Rel_{io}/E \longrightarrow B(Y/Z)$ which is a group isomorphism.*

Proof. Since Z is smooth, $B(Z)$ embeds in $B(K(Z))$. Hence to show that Φ is a group homomorphism we need only show the composite $\mathrm{Rel}_{io} \xrightarrow{\Phi} B(Z) \longrightarrow B(K(Z))$ is a group homomorphism. Consider two relations $\phi : L_\sigma^e \xrightarrow{\sim} \mathcal{O}_Y, \psi : M_\sigma^e \xrightarrow{\sim} \mathcal{O}_Y$. If we embed L, M and their twisted tensor powers in $K(Y)$ then we can consider

$$\phi : L(\sigma^* L) \dots (\sigma^{(e-1)} L) \xrightarrow{\sim} \mathcal{O}_Y, \psi : M(\sigma^* M) \dots (\sigma^{(e-1)} M) \xrightarrow{\sim} \mathcal{O}_Y$$

as multiplication by $\alpha, \beta \in K(Y)^G$. The product $\phi \otimes \psi$ is then multiplication by $\alpha\beta$. Since generically, $A(Y; L_\sigma, \phi), A(Y; M_\sigma, \psi)$ are the cyclic algebras $K(Z)[z;\sigma]/(z^e - \alpha), K(Z)[z;\sigma]/(z^e - \beta)$, Φ is a group homomorphism. We have already seen that equivalent relations yield generically isomorphic algebras so Br is well-defined.

It suffices now to show that Br makes the following diagram commute

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(G, \mathcal{O}(Y)^*)/\operatorname{im} d_2 & \longrightarrow & \mathrm{Rel}_{io}/E & \longrightarrow & \ker d_2 & \longrightarrow & 0 \\
 & & \downarrow \mathrm{id} & & \downarrow \mathrm{Br} & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & H^2(G, \mathcal{O}(Y)^*)/\operatorname{im} d_2 & \longrightarrow & B(Y/Z) & \longrightarrow & \ker d_2 & \longrightarrow & 0
\end{array}$$

where the top row is a short exact sequence obtained by truncating the exact sequence in lemma 3.5 and the bottom short exact sequence is derived from the Hochschild-Serre spectral sequence.

Let $\delta : \ker d_2 \longrightarrow H^2(G, K(Y)^*/\mathcal{O}(Y)^*)$ be the connecting homomorphism of lemma 3.5. Then since

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(G, \mathcal{O}(Y)^*)/\operatorname{im} d_2 & \longrightarrow & B(Y/Z) & \longrightarrow & \ker d_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \gamma & & \downarrow \delta & & \\
0 & \longrightarrow & H^2(G, \mathcal{O}(Y)^*)/\operatorname{im} d_2 & \longrightarrow & B(K(Y)/K(Z)) & \longrightarrow & H^2(G, K(Y)^*/\mathcal{O}(Y)^*) & &
\end{array}$$

commutes and γ, δ are injective, it suffices to show

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(G, \mathcal{O}(Y)^*)/\operatorname{im} d_2 & \longrightarrow & \mathrm{Rel}_{io}/E & \longrightarrow & \ker d_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(G, \mathcal{O}(Y)^*)/\operatorname{im} d_2 & \longrightarrow & B(K(Y)/K(Z)) & \longrightarrow & H^2(G, K(Y)^*/\mathcal{O}(Y)^*) & &
\end{array}$$

commutes. This follows from calculations in the proof of lemma 3.5.

4 The Overlap Condition in the Ramified Case

Suppose from now on that Y is a smooth quasi-projective variety and $G = \langle \sigma \rangle \subseteq \operatorname{Aut} Y$ is a cyclic subgroup. This will, henceforth, be the usual hypotheses and can be assumed to hold unless specifically mentioned otherwise.

Proposition 4.1 *Suppose that $\mathcal{O}(Y)^* = k^*$. Let the ramification indices of the quotient map $\pi : Y \longrightarrow Z := Y/G$ be $\{e_i\}$. If the lowest common multiple of the e_i's is e then the natural map $d_2 : H^1(G, Pic\, Y) \longrightarrow H^3(G, \mathcal{O}(Y)^*)$ is zero. In particular, in this case, cyclic algebras constructed from any element of $H^1(G, Pic\, Y)$ automatically satisfy the overlap condition.*

Proof. Let $L \in H^1(G, \mathrm{Pic}\, Y)$ so that

$$L(\sigma^* L) \dots (\sigma^{*(e-1)} L) = \alpha\, \mathcal{O}_Y .$$

From the proof of lemma 3.5

$$d_2 L = \alpha\,\sigma(\alpha)^{-1} \in H^3(G, \mathcal{O}(Y)^*) \simeq \mu_e.$$

Hence $\alpha \in K(Y)$ is an eigenvector for σ and we need to show that the eigenvalue is one in this case.

We need to study the eigenspace decomposition of $K(Y)$. We can write $K(Y) = K(Z)[y]/(y^e - f)$ where $f \in K(Z)$ has divisor $(f) = \sum n_i C_i$. The hypothesis on the e_i ensures that the n_i generate $\mathbb{Z}/e$. The eigenspaces for $K(Y)$ are $K(Z)y^i$ for $i = 0, 1, \ldots, e-1$.

Let D_i be an irreducible divisor above C_i and ν_i the valuation along D_i. Since Y is quasi-projective, there is a G-invariant open set U which contains the generic point of each D_i. By multiplying L by a rational function and shrinking U if necessary, we may assume that $L = \mathcal{O}_{\mathrm{Y}}$ on U. Hence $\nu_i(\alpha) = 0$ for every i. We wish to show that this implies that α lies in the eigenspace $K(Z)$.

Suppose to the contrary that $\alpha \in K(Z)y^j$ for $1 \le j < e$. The hypothesis on the ramification indices implies there is some i for which j is not a multiple of e_i. Now for any $\beta \in K(Z)$, $\nu_i(\beta)$ is a multiple of e_i. On the other hand $\nu_i(y)$ generates $\mathbb{Z}/e_i$ so $\nu_i(y^j)$ is not a multiple of e_i. This contradiction finishes the proof of the proposition.

We have a partial converse.

Proposition 4.2 *Let Y be a smooth projective variety and suppose the quotient map $\pi : Y \longrightarrow Z := Y/G$ is totally ramified at an irreducible divisor $D \subset Y$ (and unramified away from D). If $Y' := Y - D$ then we have an exact sequence*

$$H^1(G, Pic\ Y) \longrightarrow H^1(G, Pic\ Y') \xrightarrow{d_2} H^3(G, \mathcal{O}(Y')^*)$$

where the first map is the functorially defined one.

Proof. We apply the argument of the proof of the previous proposition. The sequence is a complex by the previous proposition. We let $L \in \ker d_2 \subseteq H^1(G, \mathrm{Pic}\ Y')$ so that

$$L(\sigma^* L)\ldots(\sigma^{*(e-1)} L) = \alpha\, \mathcal{O}_{\mathrm{Y}}(jD)$$

for some $j \in \mathbb{Z}$. If locally near the generic point of D we have $L = \mathcal{O}_{\mathrm{Y}}(iD)$ then since π is totally ramified along D, α has a pole of order $ei - j$ there. Now $d_2 L = \alpha\,\sigma(\alpha)^{-1} = 1$ implies that j is a multiple of e. Consequently, adjusting L by some tensor power of $\mathcal{O}_{\mathrm{Y}}(D)$ we may assume $j = 0$, that is, L lifts to $H^1(G, \mathrm{Pic}\ Y)$.

We will give an example in section 6 where $H^1(G, \mathrm{Pic}\ Y)$ maps to a proper subgroup of $H^1(G, \mathrm{Pic}\ Y')$ (see equation 4). This gives an example where the overlap condition fails to hold. In that case, Y' is affine. There are also projective examples such as

Example 4.3 *Failure of the overlap condition in projective case.*

Let $Z = E$ be an elliptic curve with zero p_0 and let p be an e-torsion point. Let Y be the Galois cover corresponding to p. More precisely, if $f \in K(E)$ has divisor $(f) = ep - ep_0$ then $K(Y) = K(E)[y]/(y^e - f)$. As usual, write $G = \langle\sigma\rangle$ for the Galois group of Y/Z. If $q \in \pi^{-1}(p), q_0 \in \pi^{-1}(p_0)$ then

$$(\pi^* f) = (eq + e\,\sigma(q) + \ldots + e\,\sigma^{e-1}(q)) - (eq_0 + e\,\sigma(q_0) + \ldots + e\,\sigma^{e-1}(q_0)).$$

Hence

$$(y) = (q + \sigma(q) + \ldots + \sigma^{e-1}(q)) - (q_0 + \sigma(q_0) + \ldots + \sigma^{e-1}(q_0)) \tag{3}$$

Note that $L := \mathcal{O}_{\mathrm{Y}}(q - q_0) \in H^1(G, \mathrm{Pic}\ Y)$ by equation (3). However,

$$L(\sigma^* L)\ldots(\sigma^{*(e-1)} L) = y\, \mathcal{O}_{\mathrm{Y}}$$

and $y \notin K(E)$. Hence no relation satisfies the overlap condition for this L. The computation above shows that, in this case, the differential $d_2 : H^1(G, \text{Pic } Y) \longrightarrow H^3(G, \mathcal{O}(Y)^*) \simeq \mu_e$ is in fact surjective.

Putting together lemma 3.5, theorem 3.9 and propositions 4.1 and 4.2, we obtain the following corollary which exhibits many Brauer classes as noncommutative cyclic covers.

Corollary 4.4 *Let Y be a smooth projective variety and supoose that the quotient map $\pi : Y \longrightarrow Z := Y/G$ is totally ramified at an irreducible divisor $D \subset Y$ (and unramified away from D). Suppose further that D is not a torsion divisor in Pic Y and that Z is smooth. Then there is a group monomorphism $\Psi : H^1(G, \text{Pic } Y) \longrightarrow B(K(Y)/K(Z))$ given explicitly as follows. If $L \in$ Pic Y represents a 1-cocycle in $H^1(G, \text{Pic } Y)$ as in section 3, then any isomorphism $\phi : L_\sigma^e \xrightarrow{\sim} \mathcal{O}_Y$ satisfies the overlap condition and $\Psi(L)$ is the Brauer class of $K(Z) \otimes_Z A(Y; L_\sigma, \phi)$. The image of Ψ consists precisely of those Brauer classes ramified only on D.*

Proof. Let $Y' = Y - D$ and note that the exact sequence

$$0 \longrightarrow \mathbb{Z}\, D \longrightarrow \text{Pic } Y \longrightarrow \text{Pic } Y' \longrightarrow 0$$

yields the following exact sequence

$$0 = H^1(G, \mathbb{Z}\, D) \longrightarrow H^1(G, \text{Pic } Y) \longrightarrow H^1(G, \text{Pic } Y').$$

Hence by proposition 4.2,

$$H^1(G, \text{Pic } Y) \simeq \ker(H^1(G, \text{Pic } Y') \xrightarrow{d_2} H^3(G, \mathcal{O}(Y')^*).$$

Also, $H^2(G, \mathcal{O}(Y')^*) = 0$ since Y is projective so lemma 3.5 and theorem 3.9 show that in fact $H^1(G, \text{Pic } Y) \simeq B(Y'/\pi(Y'))$ which gives the corollary.

Finally, we show that not only ramification indices, but also ramification data is extremely easy to compute in the totally ramified case.

Proposition 4.5 *Suppose that $Y, Z := Y/G$ are smooth quasi-projective surfaces and that the quotient map $\pi : Y \longrightarrow Z$ is totally ramified at $D \subset Y$. Consider the cyclic algebra $A = A(Y; L_\sigma, \phi)$ arising from a relation of the form $\phi : L_\sigma^e \simeq \mathcal{O}_Y$. Then the ramification of A along $\pi(D)$ is the cyclic cover of D defined by the e-torsion line bundle $L|_D$.*

Proof. Note first that as G fixes D, ϕ restricts to a bimodule isomorphism $(L|_D)^e \simeq \mathcal{O}_D$. Let $I_D := \mathcal{O}_Y(-D) \otimes_{\mathcal{O}_Y} A = A \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-D)$ which is a two-sided ideal of A. Now $I^e = A \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(-\pi(D))$. By [AM, theorem 1], the ramification of A at $\pi(D)$ can be computed as the centre of the residue ring at $\pi(D)$ modulo the radical. Now

$$A/I_D \simeq \mathcal{O}_D \oplus L|_D \oplus \ldots \oplus (L|_D)^e$$

which proves the proposition.

5 Classification of Orders on Surfaces

In the past few years, there has been a program to classify maximal orders on projective surfaces along the same lines as Enriques' classification of commutative surfaces. A picture has emerged which gives an almost complete classification of the ramification data that can occur. There is a sequence of Artin-Mumford which guarantees the existence of maximal orders with given ramification data, but it is non-constructive. The goal of the next few sections is to construct using noncommutative cyclic algebras some of the orders which appear in the classification. Consequently, we briefly review the relevant classification theory in this section. It is not logically necessary for the rest of the paper, but has been included because it i) motivates the choice of maximal orders that we wish to construct, ii) introduces the requisite language we use concerning orders and iii) emphasises the relationship between the maximal order and the "maximal commutative subalgebra" Y used to construct it.

Let A be a maximal order on a smooth projective surface Z with ramification curves D_i and corresponding ramification indices e_i. Then we define the *canonical divisor of* A to be $K_A := K_Z + \Delta$ where

$$\Delta := \sum \left(1 - \frac{1}{e_i}\right) D_i \in \text{Div } Z.$$

This formula of Artin's is suggested by the dualising module (see [CK03; proposition 5]). It is possible to classify maximal orders on surfaces via the Kodaira dimension of the log surface (Z, Δ). The usual approach to this classification is via a noncommutative adaptation of Mori's minimal model program (see [CI]).

The analogue of smooth surfaces is played by terminal orders, the definition of which can be found in [CI]. We will only mention here that any maximal order with smooth discriminant curve $\cup D_i$ is terminal. Most of the maxinal orders we wish to construct as noncommutative cyclic algebras will be of this form. There is an analogue of Castelnuovo's contraction theorem [CI, theorem 5.2] which allows us to speak of minimal models of orders on surfaces. A terminal order on Z is *minimal* if Z does not contain a K_A-negative curve E with $E^2 = -1$.

Theorem 5.1 *Let A be a maximal order on a smooth projective surface ramified on a smooth curve D with ramification index e. If A is minimal and K_A is not nef then either*

i. (Del Pezzo) $-K_A$ is ample. In this case, $Z = \mathbb{P}^2$ and $\deg D = 3, 4$ or 5. If $\deg D = 4$ then $e = 2$ or 3 and if $\deg D = 5$ then $e = 2$.

ii. (Ruled) There is a $\mathbb{P}^1$-fibration $\rho : Z \longrightarrow C$ to a smooth curve C and $-K_A$ is ρ-ample. In this case, D is either a bisection of ρ or, if $e = 2$, it may also be a trisection.

If K_A is nef then the Kodaira dimension is 0,1 or 2. The classification here is incomplete. Terminal orders with K_A numerically trivial are called *numerically Calabi-Yau*. They are minimal orders of Kodaira dimension zero and have been classified in [CK]. We will give part of the classification here.

Theorem 5.2 *Let A be a numerically Calabi-Yau order on a smooth projective surface Z with a smooth ramification curve D with ramification index e. Then $-K_Z$ is nef. In particular, Z is either rational, ruled over an elliptic curve or has Kodaira dimension 0.*

i. If $Z = \mathbb{P}^2$ then $\deg D = 6$ or 4 and $e = 2$ or 3 respectively.

ii. If Z is ruled over an elliptic curve C then $D \longrightarrow C$ is an étale cover of degree 4 or 3 and $e = 2$ or 3 respectively.

iii. If Z has Kodaira dimension 0 then A is unramified.

To construct a maximal order as described in either of the two theorems above, we need to construct a degree e cover of Z ramified on D. By the Riemann-Hurwitz formula and the formula for K_A we see that Y is del Pezzo if A is, Y is birationally ruled if A is ruled and Y is minimal Kodaira dimension 0 if A is numerically Calabi-Yau.

In the next 4 sections we shall construct certain maximal orders as noncommutative cyclic covers. In particular, we will describe the cover Y and give explicit lines bundles L representing elements of $H^1(G, \text{Pic } Y)$.

6 Del Pezzo Orders

From theorem 5.1, we see that the only del Pezzo orders on $\mathbb{P}^2$ with smooth discriminant that can be constructed as noncommutative cyclic covers are those of degree two ramified on a quartic or those of degree 3 ramified on a cubic. We construct all of these explicitly as well as some whose discriminant curve is not smooth.

Let $Z = \mathbb{P}^2$ and $C \subset Z$ be a smooth quartic. Let Y be the double cover of Z ramified on C and $\pi : Y \longrightarrow Z$ be the quotient map. Let $D = \pi^{-1}C$ and let G be the group $\{1, \sigma\}$ so that $Z = Y/G$. Note that Y is the blowing up of $\mathbb{P}^2$ at 7 points in general position. Indeed, the Riemann-Hurwitz formula shows that

$$K_Y \cong \pi^* K_Z + D \cong -\pi^* H$$

where H is a line. Being anti-ample, Y must be del Pezzo and computing

$$K_Y^2 = \pi^* H.\pi^* H = \pi_* \pi^* H.H = 2 = K_{\mathbb{P}^2}^2 - 7$$

we see that Y is $\mathbb{P}^2$ with 7 points in general position blown up.

We wish to compute $H^1(G, \mathrm{Pic}\, Y)$. Let $f : Y \longrightarrow \mathbb{P}^2$ be the blowing down which contracts the exceptional curves $E_1, \ldots, E_7$ to $p_1, \ldots, p_7$. Let $\tilde{\mathrm{H}} \in \mathrm{Pic}\, Y$ be the pull-back of the line on $\mathbb{P}^2$ so that $\mathrm{Pic}\, Y = \mathbb{Z}\, \tilde{\mathrm{H}} \oplus (\oplus_i \mathbb{Z}\, E_i)$.

We need to identify the action of G on Pic Y. There are 28 bitangents of the quartic $C \subset \mathbb{P}^2$ (see [Hart; chapter IV, exercise 2.3h)]) and the inverse image of each is the union of two (-1)-curves. From [Dem, page 34, table 1], we know there are 56 (-1)-curves and they can be described in a different fashion as follows.

For each $i \in \{1, \ldots, 7\}$ there is the exceptional curve E_i. Also, consider the cubic passing through p_i with multiplicity 2 and all the other p_j. Its strict transform E_i' is also a (-1)-curve.

Let $i, j \in \{1, \ldots, 7\}$ be distinct and consider the line through p_i, p_j. Its strict transform E_{ij} in Y is a (-1)-curve. Similarly, the conic passing through the other 5 points p_l has strict transform E_{ij}' which is also a (-1)-curve.

The automorphism σ swaps E_i with E_i' and E_{ij} with E_{ij}'. This allows us to compute the matrix of σ acting on Pic Y with respect to the basis $\tilde{\mathrm{H}}, E_1, \ldots, E_7$ to be

$$\mathrm{Pic}\, \sigma = \begin{pmatrix} 8 & 3 & \ldots & \ldots & \ldots & 3 \\ -3 & -2 & -1 & \ldots & \ldots & -1 \\ \vdots & -1 & \ddots & & & -1 \\ \vdots & \vdots & & \ddots & & \vdots \\ -3 & -1 & & & -2 & -1 \\ -3 & -1 & -1 & \ldots & -1 & -2 \end{pmatrix}$$

To compute $H^1(G, \mathrm{Pic}\, Y)$, note that $\ker(1+\sigma)$ is generated by $h := \tilde{\mathrm{H}} - 3E_1, e_i := E_i - E_{i+1}, i = 1, \ldots, 6$. It is also easy to see that $\mathrm{im}\,(1-\sigma)$ is generated by $2\ker(1+\sigma)$ and $h + e_2 + e_4 + e_6$ so $H^1(G, \mathrm{Pic}\, Y) \simeq (\mathbb{Z}/2\,\mathbb{Z})^6$. From this description, we see that $H^1(G, \mathrm{Pic}\, Y)$ is generated by differences between exceptional curves. Conversely, given any two exceptional curves $E, E' \in Y$, we have $L := \mathcal{O}_Y(E - E') \in H^1(G, \mathrm{Pic}\, Y)$ since $(1+\sigma)(E - E') = \pi^*(\pi(E) - \pi(E'))$ and the difference of the two bitangents $\pi(E), \pi(E')$ is linearly equivalent to zero.

Corollary 4.4 shows that $H^1(G, \mathrm{Pic}\, Y)$ classifies the Brauer classes ramified on C with ramification index two. Furthermore, a representative order for each class can be constructed as noncommutative cyclic covers with L described as above. On the other hand, the Artin-Mumford sequence [AM, theorem 1] shows that such Brauer classes are classified by the étale double covers of C. Note that the 64 elements of $H^1(G, \mathrm{Pic}\, Y)$ correspond to the 64 étale double covers of C.

Let $Y' = Y - D$. As promised after proposition 4.2, we will show that

$$d_2 : H^1(G, \text{Pic } Y') \longrightarrow H^3(G, \mathcal{O}(Y')^*) = \mu_2 \quad (4)$$

is surjective so that $H^1(G, \text{Pic } Y) \neq H^1(G, \text{Pic } Y')$. We use the notation in the proof of lemma 3.5. First we compute D in Pic Y. From our description of exceptional curves via bitangents, we see that $2 = D.E_i = D.E_{ij}$. We deduce that $D \sim 6\,\tilde{\text{H}} - 2\sum E_i$. Let $A = \tilde{\text{H}} - E_1$ so that $A + \sigma^* A \sim D$. Choose a rational function h with divisor $A + \sigma^* A - D$ so that h represents the image of A under the map $\delta : H^1(G, \text{Pic } Y') \longrightarrow H^2(G, K^*/\,\mathcal{O}(Y')^*)$. Now h has a zero of order 1 along D so $d_2 A = h^{-1}\,\sigma^*(h) = -1$.

Of course, the procedure above works fairly generally as the following argument shows. We consider here del Pezzo orders A on $Z = \mathbb{P}^2$ of index 3 ramified on the union of a cubic C and a transverse line C'. Let $\pi : Y \longrightarrow Z$ be the triple cover of Z ramified on C. As before, we can compute that Y is del Pezzo and in fact, is the blowup of $\mathbb{P}^2$ at 6 points.

We recall the following theorem of Artin's [A, theorem 2.15] which, though stated only in the local case is true in the generality given below. His argument extends globally because the construction is canonical.

Theorem 6.1 *Let A be a normal order on Z with say ramification indices e_C over any divisor C. Let $\pi : Y \longrightarrow Z$ be a ramified Galois cover with Galois group G and ramification indices r_C over C. Suppose that r_C divides e_C for every C. Then there exists a canonically defined normal order B containing $\pi^* A$ such that*

i. $B = \pi^* A$ over the étale locus of π.

ii. The action of G on $\pi^* A$ extends to B in such a way that $B^G = A$.

iii. The ramification indices of B above C are e_C / r_C.

We shall call the order B in the theorem the Artin cover of A with respect to π. In our case, the Artin cover B is unramified except possibly on $D' := \pi^{-1} C'$. Since the residue field of D' is isomorphic to the cyclic extension corresponding to the ramification of A over Z, we see in fact that at the generic point of C', $B = \pi^* A$ is also unramified.

Now Br $Y = 0$ since Y has negative Kodaira dimension (see [DF, theorem 1.1]). Hence, by theorem 3.9, there is some line bundle $\tilde{L}$ on $Y' := Y - \pi^{-1}(C \cup C')$ and a relation $\tilde{\phi} : \tilde{L}^3 \longrightarrow \mathcal{O}_{Y'}$ satisfying the overlap condition whose corresponding cyclic algebra $A(Y'; \tilde{L}_\sigma, \tilde{\phi})$ has the same Brauer class as A.

Extend $\tilde{L}$ arbitrarily to a line bundle L on Y. The relation extends to a relation of the form $\phi : L^3 \xrightarrow{\sim} \mathcal{O}(E)$ where E is a divisor supported on $D \cup D'$. Tensoring L by an appropriate multiple of $\mathcal{O}(D)$ and $\mathcal{O}(D')$ we may assume in fact that $E = -iD - jD'$ where $i, j \in \{0, 1, 2\}$. From propositions 4.1, 4.2, we see that a relation of the form $\phi : L^3 \xrightarrow{\sim} \mathcal{O}(-iD - jD')$ satisfies the overlap condition if and only if $i = 0$. Also, theorem 3.6 shows that $j = 1, 2$ otherwise we obtain a maximal order ramified on the cubic C only.

We examine now the $j = 0$ case. The noncommutative cyclic covers ramified on the cubic C can all be described explicitly. Recall that the 27 lines of Y occur in triples which are the inverse images of the 9 inflexion lines of C. Let E, E' be two exceptional curves lying over distinct inflexion lines. As in the quartic ramification case, $L := \mathcal{O}_{\text{Y}}(E - E') \in H^1(G, \text{Pic } Y)$. The cyclic algebra $A(Y; L_\sigma)$ is non-trivial in the Brauer group. The easiest way to see this is to compute its ramification using proposition 4.5. This tells us that the ramification of A over C is the triple cover defined by the 3-torsion divisor which is the difference of the inflexion points $E \cap C$ and $E' \cap C$. This computation also shows that the 9 Brauer classes can all be represented by noncommutative cyclic covers.

7 Ruled Orders

We consider now a ruled order A with centre a $\mathbb{P}^1$-bundle $\rho : Z \longrightarrow \overline{\mathrm{T}}$ as defined in theorem 5.1(ii). If A arises as a noncommutative cyclic cover with smooth ramification divisor D then we see we must have the ramification index $e = 2$ and the divisor $D \sim 2C_0 + 2\rho^* P$ where $P \in \mathrm{Pic}\, \overline{\mathrm{T}}$ and C_0 is a section of ρ.

This ensures there is a double cover $\pi : Y \longrightarrow Z$ of Z ramified along D which can be described as follows. The Riemann-Hurwitz formula shows that the double cover $D \longrightarrow \overline{\mathrm{T}}$ has $2r$ ramification points where $r = g(D) - 2g + 1$. Let $p_1, \dots, p_{2r}$ be the ramification points and $\tilde{\mathrm{F}}_i := (\rho\pi)^{-1}(p_i)$ be the corresponding fibres. Note that $\tilde{\mathrm{F}}_i$ is the union of two exceptional lines say F_i, F_i'. The general fibre of $\rho\pi$ is a smooth rational curve with self-intersection 0. Hence contracting $F_1, \dots, F_{2r}$ yields a surface geometrically ruled over $\overline{\mathrm{T}}$ which we denote by $\overline{\mathrm{Y}}$.

Note that $\mathrm{Br}\, Y = 0$ so by theorem 3.9, proposition 4.1 and Artin's theorem 6.1, we see that up to Morita equivalence, all ruled orders ramified on D with ramification index 2 can be constructed as noncommutative cyclic covers and in fact these are classified by $H^1(G, \mathrm{Pic}\, Y)$ where $G = \mathbb{Z}/2$. We shall construct explicit generators for the group $H^1(G, \mathrm{Pic}\, Y)$. The Brauer classes representing orders with the given ramification data are determined by the Artin-Mumford sequence [AM, theorem 1]. The permissible double covers of D defining ramification data are classified by the kernel of $\gamma : H^1(D, \mathbb{Z}/2) \longrightarrow H^1(\overline{\mathrm{T}}, \mathbb{Z}/2)$ which is Poincaré dual to the usual pull-back map $H^1(\overline{\mathrm{T}}, \mathbb{Z}/2) \longrightarrow H^1(D, \mathbb{Z}/2)$. The latter has kernel $\mathbb{Z}/2$ when $D \longrightarrow \overline{\mathrm{T}}$ is étale and is zero otherwise. Hence $H^1(G, \mathrm{Pic}\, Y) \simeq \ker\gamma$ is isomorphic to $(\mathbb{Z}/2)^{2g(D)-2g+1} = (\mathbb{Z}/2)^{2g-1}$ in the étale case and $(\mathbb{Z}/2)^{2g(D)-2g} = (\mathbb{Z}/2)^{2r+2g-2}$ otherwise.

We construct generators as follows. Firstly, pick a basis $d_1, \dots, d_{2g}$ for the 2-torsion subgroup $(\mathrm{Pic}\, \overline{\mathrm{T}})_2$ of $\mathrm{Pic}\, \overline{\mathrm{T}}$. Also, for each $i \in [2, 2r]$, pick $q_i \in \mathrm{Div}\, \overline{\mathrm{T}}$ so that $p_1 - p_i \sim 2q_i$. Then $Z_i := F_1 - F_i - \pi^*\rho^* q_i$ is a 1-cocycle representing an element in $H^1(G, \mathrm{Pic}\, Y)$. We consider the 1-cocycles $\pi^*\rho^* d_1, \dots, \pi^*\rho^* d_{2g}, Z_2, \dots, Z_{2r}$ of which there are

- $2g$ in the étale case $r = 0$.
- $2r + 2g - 1$ if $r \neq 0$.

We wish to show that, modulo 2, there is only one relation among these cocycles so comparing with the order of $H^1(G, \mathrm{Pic}\, Y)$ computed above, we see that they generate the cohomology group. Let V be the subgroup of $\mathrm{Pic}\, Y$ generated by these elements. We compute coboundaries. Since $\overline{\mathrm{Y}}$ is a ruled surface,

$$\mathrm{Pic}\, Y = \bigoplus \mathbb{Z}\, F_i \oplus \pi^*\rho^*\, \mathrm{Pic}\, \overline{\mathrm{T}} \oplus \mathbb{Z}\, C_0$$

where C_0 is the pull-back of a section of $\overline{\mathrm{Y}} \longrightarrow \overline{\mathrm{T}}$. If σ is the non-trivial element in G then note that $1 - \sigma$ annihilates $\pi^*\rho^*\, \mathrm{Pic}\, \overline{\mathrm{T}}$ and

$$(1 - \sigma)F_i = 2F_i - \pi^*\rho^* p_i.$$

Let W be the group generated by these elements. Computing explicitly, we see that $W \cap V \subseteq 2V$. Hence in $H^1(G, \mathrm{Pic}\, Y)$, there can only be a single relation (modulo 2) in the above generators coming from $(1 - \sigma)(C_0 + F)$ where F is an appropriate linear combination of the F_i. We summarise in

Theorem 7.1 *The ruled orders ramified on D with ramification index 2 can all be constructed by noncommutative cyclic covers. Up to Morita equivalence, they are classified by $H^1(G, \mathrm{Pic}\, Y)$ which is isomorphic to $(\mathbb{Z}/2)^{2g-1}$ in the étale case and $(\mathbb{Z}/2)^{2r+2g-2}$ otherwise. This group is generated by $\pi^*\rho^* d_1, \dots, \pi^*\rho^* d_{2r}, F_1 - F_2 - \pi^*\rho^* q_2, \dots, F_1 - F_{2r} - \pi^*\rho^* q_{2r}$.*

8 Quantum Quadrics

We freely use the notation of the previous section. Let $Z = \mathbb{P}^1 \times \mathbb{P}^1, \rho : Z \longrightarrow \mathbb{P}^1 =: \overline{\mathrm{T}}$ be the projection onto the first factor and $D \subset Z$ be a smooth anti-canonical divisor. There is a smooth double cover

Y of Z ramified on D. By the previous section, the maximal orders of degree two ramified on D can be constructed, up to Morita equivalence, as cyclic algebras $A = A(Y; L_\sigma)$ where $L = \mathcal{O}(F_1 - F_2)$ and $\pi(F_1), \pi(F_2)$ are two fibres of ρ which are tangential to D. In this case, $r = 2$ so there are 4 such tangential fibres. We wish to show explicitly how A is a noncommutative ruled surface in the sense of Van den Bergh [VdB01]. By [AdJ, §4.3], we are forced to use the $\mathbb{Z}$-algebra formulation.

Let E be an exceptional curve in Y such that $\pi(E)$ is a $(0,1)$-divisor and let $E' := \sigma(E)$. Note that $\pi(E)$ is a tangent of D which intersects $\pi(F_i)$ transversely. We will work simultaneously with the Morita equivalent cyclic algebra $A' := \mathcal{O}(-E) \otimes_Y A \otimes_Y \mathcal{O}(E) = A(Y; L(E'-E)_\sigma)$. The role of the relative tautological bundle is played by a pair of bimodules, ${}_A N_{A'} := A \otimes_Y \mathcal{O}(E)$ and ${}_{A'} N'_A := A' \otimes_Y \mathcal{O}(-E)$.

Considering the configuration of exceptional curves on Y, we may choose E to intersect F'_1, F_2, F'_3, F_4 so that E' intersects F_1, F'_2, F_3, F'_4. This choice is made so that

$$L(E'-E) = \mathcal{O}_{\mathrm{Y}}(F_1 - F_2 + E' - E) \simeq \mathcal{O}(F_4 - F_3).$$

To see this we need only compute intersection products with the basis $F_1, F'_1, F_2, F_3, F_4, C_0$ of Pic Y. This emphasises the symmetry between A and A'.

We consider $X :=$ Spec $A, X' :=$ Spec A' as ringed spaces on Z and $T :=$ Spec $\rho_* A, T' :=$ Spec $\rho_* A'$ as ringed spaces on $\overline{\mathrm{T}}$. To simplify notation, we will use the convention that $\mathrm{A}^\cdot$ can signify either A or A' depending on the case at hand and define $\mathrm{T}^\cdot, \mathrm{N}^\cdot$ similarly. The choice will always be uniquely determined by the context. We also define

$$N_n := N \otimes_{A'} N' \otimes_A N \otimes \ldots \otimes \mathrm{N}^\cdot$$

where there are n tensor factors on the right hand side.

To study these, we will need to examine N along the fibres of $\rho\pi : Y \longrightarrow \overline{\mathrm{T}}$. The generic fibre $\tilde{\mathrm{F}}$ is $\mathbb{P}^1$. There are 4 special fibres $\tilde{\mathrm{F}}_i = F_i \cup F'_i$, $i = 1,2,3,4$. By [Lip, proposition 11.1], line bundles on $\tilde{\mathrm{F}}_i$ are determined by their degree on F_i, F'_i. We denote by $\mathcal{O}_{\tilde{\mathrm{F}}_i}(m,n)$ the line bundle on $\tilde{\mathrm{F}}_i$ with degree m on F_i and degree n on F'_i. Now

$$N_{2j} = A \otimes_{\mathcal{O}_{\mathrm{Y}}} \mathcal{O}_{\mathrm{Y}}(jE + jE') \ , \ N_{2j+1} = A \otimes_{\mathcal{O}_{\mathrm{Y}}} \mathcal{O}_{\mathrm{Y}}((j+1)E + jE')$$

so restricting to fibres yields

$$\begin{array}{lll}
\text{On } \tilde{\mathrm{F}} : N_n \simeq \mathcal{O}_{\tilde{\mathrm{F}}}(n) \oplus \mathcal{O}_{\tilde{\mathrm{F}}}(n)_\sigma & & \\
\text{On } \tilde{\mathrm{F}}_1 : N_{2n} \simeq \mathcal{O}_{\tilde{\mathrm{F}}_1}(n,n) \oplus \mathcal{O}_{\tilde{\mathrm{F}}_1}(n-1,n+1)_\sigma, & N_{2n+1} \simeq \mathcal{O}_{\tilde{\mathrm{F}}_1}(n,n+1) \oplus \mathcal{O}_{\tilde{\mathrm{F}}_1}(n,n+1)_\sigma & \\
\text{On } \tilde{\mathrm{F}}_2 : N_{2n} \simeq \mathcal{O}_{\tilde{\mathrm{F}}_2}(n,n) \oplus \mathcal{O}_{\tilde{\mathrm{F}}_2}(n+1,n-1)_\sigma, & N_{2n+1} \simeq \mathcal{O}_{\tilde{\mathrm{F}}_2}(n+1,n) \oplus \mathcal{O}_{\tilde{\mathrm{F}}_2}(n+1,n)_\sigma & (5) \\
\text{On } \tilde{\mathrm{F}}_3 : N_{2n} \simeq \mathcal{O}_{\tilde{\mathrm{F}}_3}(n,n) \oplus \mathcal{O}_{\tilde{\mathrm{F}}_3}(n,n)_\sigma, & N_{2n+1} \simeq \mathcal{O}_{\tilde{\mathrm{F}}_3}(n,n+1) \oplus \mathcal{O}_{\tilde{\mathrm{F}}_3}(n+1,n)_\sigma & \\
\text{On } \tilde{\mathrm{F}}_4 : N_{2n} \simeq \mathcal{O}_{\tilde{\mathrm{F}}_4}(n,n) \oplus \mathcal{O}_{\tilde{\mathrm{F}}_4}(n,n)_\sigma, & N_{2n+1} \simeq \mathcal{O}_{\tilde{\mathrm{F}}_4}(n+1,n) \oplus \mathcal{O}_{\tilde{\mathrm{F}}_4}(n,n+1)_\sigma &
\end{array}$$

The higher cohomology of these sheaves vanish so $R\rho_* N_n = \rho_* N_n$ and the fibres of $\rho_* N_n$ can be computed as the global sections of the fibres listed above. The next proposition computes T, T'.

Proposition 8.1 *The spaces T, T' are commutative double covers of $\overline{\mathrm{T}}$. The map $T \longrightarrow \overline{\mathrm{T}}$ is ramified over p_1, p_2 while $T' \longrightarrow \overline{\mathrm{T}}$ is ramified over p_3, p_4. Also, $\rho_*(N_m^{-1} \otimes_A N_n)$ is a locally free $(\mathrm{T}^\cdot, \mathrm{T}^\cdot)$-bimodule of rank $n - m + 1$ where $\mathrm{T}^\cdot = T, T'$ as the case may be. The support of $\rho_* N, \rho_* N'$ (in $\mathrm{T}^\cdot \times \mathrm{T}^\cdot$) are smooth elliptic curves.*

Proof. We study $\rho_*(N_m^{-1} \otimes_A N_n)$ in the case $m = 0$ since the general case is similar. Note that $\rho_* N_n$ is torsion-free and hence a locally free sheaf on $\overline{\mathrm{T}}$. We determine the rank of $\rho_* N_n$ by computing on a fibre $\tilde{\mathrm{F}} \simeq \mathbb{P}^1$ of $\rho\pi : Y \longrightarrow \overline{\mathrm{T}}$ above a generic point $q \in \overline{\mathrm{T}}$. Now $N_n|_{\tilde{\mathrm{F}}} \simeq \mathcal{O}_{\tilde{\mathrm{F}}}(n) \oplus \mathcal{O}_{\tilde{\mathrm{F}}}(n)$ so computing global sections we see that $\rho_* N_n$ has rank $2(n+1)$ over $\overline{\mathrm{T}}$. The $n = 0$ case shows that $\mathcal{O}_T$ is a rank 2

$\overline{\mathrm{T}}$-bimodule. Now $\rho_* \mathcal{O}_{\mathrm{Y}} = \mathcal{O}_{\overline{\mathrm{T}}}$ which is central so $\mathcal{O}_T$ is commutative. The other $\mathcal{O}_{\overline{\mathrm{T}}}$-summand of $\mathcal{O}_T$ is $\rho_* \mathcal{O}_{\mathrm{Y}}(F_1 - F_2) \simeq \mathcal{O}_{\overline{\mathrm{T}}}(-1)$ since

$$h^0(\overline{\mathrm{T}}, \rho_* \mathcal{O}_{\mathrm{Y}}(F_1 - F_2)) = 0$$
$$h^0(\overline{\mathrm{T}}, \rho_* \mathcal{O}_{\mathrm{Y}}(F_1 - F_2) \otimes \mathcal{O}_{\overline{\mathrm{T}}}(1)) = h^0(Y, \mathcal{O}_{\mathrm{Y}}(F_1 + \sigma(F_2))) = 1$$

We compute the ramification locus of $T \longrightarrow \overline{\mathrm{T}}$. In the following, we view L as a subsheaf of the constant sheaf $K = K(Y)$. Suppose f is a section of L which generates L on the complement of $\tilde{\mathrm{F}}$. We may assume that $f \in H^0(Y, \mathcal{O}_{\mathrm{Y}}(F_1 - F_2 + \tilde{\mathrm{F}}))$. Since $F_1 + F_2'$ is the unique effective divisor linearly equivalent to $F_1 - F_2 + \tilde{\mathrm{F}}$ we see in fact that $(f) = \tilde{\mathrm{F}} - F_2 - F_2'$. We need to compute the square of f in $\mathcal{O}_T$ or equivalently, in A. It is given by $\alpha f\, \sigma^* f$ where multiplication by α induces an isomorphism $L\, \sigma^* L \simeq \mathcal{O}_{\mathrm{Y}}$, that is, $(\alpha) = F_2 + F_2' - F_1 - F_1'$. Now

$$(\alpha f\, \sigma^* f) = 2\,\tilde{\mathrm{F}} - F_2 - F_2' - F_1 - F_1' = \rho\pi^{-1}(2q - p_1 - p_2)$$

so the ramification locus is the two points p_1, p_2.

The support of N is $T \times_{\overline{\mathrm{T}}} T'$ which is an elliptic curve as can be seen by applying the Riemann-Hurwitz formula to $T \times_{\overline{\mathrm{T}}} T' \longrightarrow T$. This completes the proof of the proposition.

We construct a sheaf $\mathbb{Z}$-algebra à la Van den Bergh as follows. Consider the functors $s_{n*} : X \longrightarrow \mathrm{T}^{\cdot}, s_n^* : \mathrm{T}^{\cdot} \longrightarrow X$ defined by

$$s_{n*} = \rho_*(- \otimes_A N_n)\ ,\ \ s_n^* = \rho^*(-) \otimes_{\mathrm{A}^{\cdot}} N_n^{-1}.$$

Note that $\mathcal{O}_{T^{\cdot}}$ is commutative so the projection formula for $X \longrightarrow T^{\cdot}$ holds with the usual proof as for example is found in [Har66, Chapter 2, §5]. The projection formula shows that s_{n*}, s_n^* are adjoint. The $\mathbb{Z}$-algebra $\oplus_{n \geq m} s_{n*} s_m^*$ is induced by the sheaf $\mathbb{Z}$-algebra $\mathcal{A} := \oplus_{n \geq m} \mathcal{A}_{mn}$ where

$$\mathcal{A}_{mn} = \rho_*(N_m^{-1} \otimes_A N_n)$$

is considered a $(\mathrm{T}^{\cdot}, \mathrm{T}^{\cdot})$-bimodule.

Theorem 8.2 *The category Proj $\mathcal{A}$ is a noncommutative ruled surface (see §2) and Proj $\mathcal{A} \simeq$ Proj A.*

Proof. The previous proposition shows that $\mathcal{A}$ is a graded sum of locally free bimodules of the correct rank so for the first statement it suffices to show that i) it is generated in degree one ii) the kernels Q, Q' of the multiplication maps $\rho_* N \otimes \rho_* N' \longrightarrow \rho_*(N \otimes_A N'), \rho_* N' \otimes \rho_* N \longrightarrow \rho_*(N' \otimes_A N)$ are non-degenerate. The category equivalence will be proved by appealing to the relative version of the Artin-Zhang theorem given by Van den Bergh (see theorem 8.5 below). These form the next three lemmas.

Lemma 8.3 *The algebra $\mathcal{A}$ is generated in degree one.*

Proof. We shall prove

$$\rho_* N_n \otimes_{\mathrm{T}^{\cdot}} \rho_* \mathrm{N}^{\cdot} \longrightarrow \rho_* N_{n+1}$$

is surjective, the general case being similar. It suffices to prove this on fibres of closed points in $\overline{\mathrm{T}}$. We will do this for the fibre $\tilde{\mathrm{F}}_1$, the other fibre computations being similar or easier. For the rest of the proof, all objects will be on the fibre $\tilde{\mathrm{F}}_1$ and we will leave off the notation denoting the restriction to the fibre so that for example, Y denotes $\tilde{\mathrm{F}}_1$. We show that

$$H^0(N_m) \otimes_{T'} H^0(\mathrm{N}^{\cdot}) \longrightarrow H^0(N_{m+1})$$

is surjective when $m = 2n$ is even, the odd case being similar. By the fibre computations in equation (5), it suffices to note that

$$H^0(\mathcal{O}(n,n) \oplus \mathcal{O}(n-1,n+1)_\sigma) \otimes_k H^0(\mathcal{O}(0,1)) \longrightarrow H^0(\mathcal{O}(n,n+1) \oplus \mathcal{O}(n,n+1)_\sigma)$$

is already surjective.

Lemma 8.4 *The relations Q, Q' are non-degenerate.*

Proof. Let $\rho_* N^*$ denote the dual (T', T)-bimodule to $\rho_* N$. By definition of non-degeneracy, we need to show that the composite map

$$\phi : \rho_* N^* \otimes_T Q \longrightarrow \rho_* N^* \otimes_T \rho_* N \otimes_T \rho_* N' \longrightarrow \rho_* N'$$

is an isomorphism and similarly for Q'. Now both $\rho_* N^* \otimes_T Q$ and $\rho_* N'$ are line bundles supported on the smooth elliptic curve $C := T' \times_{\overline{\mathrm{T}}} T$ so it suffices to show that these two line bundles have the same degree and the map ϕ is non-zero.

We compute degrees of line bundles first. Note that since $R\rho_* N_n = \rho_* N_n$ we have by the Leray-Serre spectral sequence $\chi_Y(N_n) = \chi_{\overline{\mathrm{T}}}(\rho_* N_n)$ so in future we will denote both simply by $\chi(N_n)$. We compute χ using Riemann-Roch on Y and the fact that E, E', F_i are (-1)-curves and so have intersection -1 with K. We find

$$\begin{aligned}\chi(N) &= \chi(\mathcal{O}_{\mathrm{Y}}(E)) + \chi(\mathcal{O}_{\mathrm{Y}}(F_1 - F_2 + E')) \\ &= 2\chi(\mathcal{O}_{\mathrm{Y}}) + \frac{1}{2}E.(E - K) + \frac{1}{2}(F_1 - F_2 + E').(F_1 - F_2 + E' - K) \\ &= 2\end{aligned}$$

and similarly $\chi(N') = 2$. Hence $\rho_* N, \rho_* N'$ have degree 2 on C. Also, the left module structure on $\rho_* N^*$ is $\mathrm{Hom}_{\mathcal{O}_{\mathrm{T}^{\cdot}}}(\rho_* N, \mathcal{O}_{\mathrm{T}^{\cdot}})$ where the Hom uses the right module structure on $\rho_* N$. The right module structure on $\rho_* N^*$ is described similarly. Now $\rho_* N$ has degree 0 on $\mathrm{T}^{\cdot}$ and hence, so does $\rho_* N^*$. Consequently, $\rho_* N^*$ and $\rho_* N'$ have the same degree on C.

By lemma 8.3 we have the following exact sequence

$$0 \longrightarrow Q \longrightarrow \rho_* N \otimes_{T'} \rho_* N' \longrightarrow \rho_* N_2 \longrightarrow 0$$

and a similar one for Q'. Proposition 8.1 shows that the last two terms are locally free (T, T)-bimodules so comparing ranks we see that Q is an invertible bimodule. We seek to show it has degree 0 over T. Note first that the bimodule $\rho_* N \otimes_{T'} \rho_* N'$ considered as an invertible sheaf on $T \times_{\overline{\mathrm{T}}} T' \times_{\overline{\mathrm{T}}} T$ has degree 8. Also, from the description of $\mathrm{T}^{\cdot} \longrightarrow \overline{\mathrm{T}}$ in proposition 8.1, we see that $T \times_{\overline{\mathrm{T}}} T' \times_{\overline{\mathrm{T}}} T$ must be the union of two smooth elliptic curves intersecting in 4 nodes. Hence it has arithmetic genus 5. Riemann-Roch now gives

$$\chi(\rho_* N \otimes_{T'} \rho_* N') = 8 - 4 = 4.$$

Also

$$\begin{aligned}\chi(\rho_* N_2) &= \chi(\mathcal{O}_{\mathrm{Y}}(E + E')) + \chi(\mathcal{O}_{\mathrm{Y}}(F_1 - F_2 + E + E')) \\ &= 2\chi(\mathcal{O}_{\mathrm{Y}}) + \frac{1}{2}(E + E').(E + E' - K) + \frac{1}{2}(F_1 - F_2 + E + E').(F_1 - F_2 + E + E' - K) \\ &= 3.\end{aligned}$$

This gives $\chi(Q) = 4 - 3 = 1$ and consequently $\deg_T Q = \chi(Q) - \chi(\mathcal{O}_T) = 0$. We deduce that $\rho_* N^* \otimes_T Q$ and $\rho_* N'$ are both line bundles of degree 2 on C.

It remains only to prove that the map $\phi : \rho_* N^* \otimes_T Q \longrightarrow \rho_* N'$ is non-zero. This can be checked generically on $\overline{\mathrm{T}}$ where it follows from general principles.

We recall here the relative Artin-Zhang theorem of Van den Bergh [VdB01, lemma 7.4.9].

Theorem 8.5 *(Van den Bergh) Let X be a locally noetherian category and suppose that there are functors $s_{n*} : X \longrightarrow Mod(T)$ for n even and $s_{n*} : X \longrightarrow Mod(T')$ for n odd. Suppose also that there exist left adjoint functors s_n^* and let $\mathcal{A}$ be the $\mathbb{Z}$-algebra $\oplus_{n \geq m} s_{n*} s_m^*$. Assume that the following conditions are satisfied.*

i. The functors s_{n*}, s_n^* preserve noetherian objects.

ii. The functor $s_{n*}s_n^*$ is right exact.

iii. For any $M \in X$ and l the canonical map $\oplus_{n>l} s_n^* s_{n*} M \longrightarrow M$ is surjective.

iv. Given any surjective map $M \longrightarrow M'$ of noetherian objects in X, we obtain surjective morphisms $s_{n*}M \longrightarrow s_{n*}M'$ for all n sufficiently large.

Then Proj $\mathcal{A} \simeq X$.

Lemma 8.6 *There is a category equivalence Proj $\mathcal{A} \simeq$ Proj A.*

Proof. It suffices to check the hypotheses of the preceding lemma apply in our case. We observe first that s_{n*}, s_n^* both preserve noetherian objects since ρ_*, ρ^* do. The projection formula gives $s_{n*}s_n^* = - \otimes_{\mathcal{O}_T} \rho_* A$ which is right exact. Lastly, we note that

$$(s_{2j})_* = \rho_*(- \otimes_Y \mathcal{O}_{\mathrm{Y}}(jE + jE')), (s_{2j+1})_* = \rho_*(- \otimes_A N \otimes_Y \mathcal{O}_{\mathrm{Y}}(jE + jE'))$$

so the last two conditions follow from the fact that $\mathcal{O}_{\mathrm{Y}}(E + E')$ is ρ-ample.

9 Numerically Calabi-Yau Orders

We seek to construct numerically Calabi-Yau orders as noncommutative cyclic covers. Unfortunately, it is not easy to determine completely which can be constructed in this fashion because the Brauer group of Kodaira dimension 0 surfaces are non-zero and are often quite large. We content ourselves with mentioning some interesting examples.

9.1 $Z = \mathbb{P}^1 \times \mathbb{P}^1, D =$ invariant (4,4)-divisor

We recall the following construction of Enriques surfaces which can be found in [BPV, chapter 4, §23 and chapter 8]. Let $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and consider the involution $\tau : ((a : b), (c : d)) \mapsto ((a : -b), (c : -d))$. Let D be a smooth τ-invariant (4,4)-divisor not passing through any of the 4 fixed points of τ. Let Y be the double cover of Z ramified along D and σ the covering involution. Recall that Y is a K3-surface and that τ lifts to a fixed point free involution (also denoted τ) of Y. The quotient $\overline{\mathrm{Y}} := Y/\langle\tau\rangle$ is an Enriques surface. Let $\overline{\mathrm{Z}} := Z/\langle\tau\rangle, \overline{\mathrm{D}}$ be the image of D in $\overline{\mathrm{Z}}$ and P be the 4 ordinary double points of $\overline{\mathrm{Z}}$. Hence, $\pi : \overline{\mathrm{Y}} \longrightarrow \overline{\mathrm{Z}}$ is ramified above $\overline{\mathrm{D}}$ and P. Let $\tilde{\mathrm{Z}}$ be the minimal resolution of $\overline{\mathrm{Z}}$ which is $\mathbb{P}^1 \times \mathbb{P}^1$ with 4 points blown-up.

$$\begin{array}{ccccc} & Y & \xrightarrow{\langle\tau\rangle} & \overline{\mathrm{Y}} & \\ & \downarrow{\scriptstyle\langle\sigma\rangle} & & \downarrow{\scriptstyle\pi} & \\ D \subset & Z = \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\langle\tau\rangle} & \overline{\mathrm{Z}} & \longleftarrow \tilde{\mathrm{Z}} \end{array}$$

We are interested in constructing rank 4 orders on Z ramified on D. In this case, $H^1(G, \mathrm{Pic}\, Y)$ is easy to compute. We have Pic $Y \simeq$ Pic $\overline{\mathrm{Y}}$/tors is isomorphic to the direct sum of the lattice E_8 and an hyperbolic plane U. Then by [Dol, remark 3.8 and proof of proposition 3.7], σ acts on Pic Y by the Bertini involution, that is, via the identity on U and via $-\mathrm{id}$ on the E_8 lattice. Hence $H^1(G, \mathrm{Pic}\, Y) \simeq (\mathbb{Z}/2)^8$ with generators given by a basis of the E_8 lattice. These give rise to noncommutative cyclic algebras by proposition 4.1.

It is also fairly easy to describe exactly which Brauer classes can be constructed this way as follows. Since $D \longrightarrow \overline{\mathrm{D}}$ is étale, Riemann-Hurwitz shows that $g(\overline{\mathrm{D}}) = 5$ and hence there are 2^{10} double covers of $\overline{\mathrm{D}}$. Let $\alpha \in \mathrm{Br}\,(\overline{\mathrm{Z}} - P - \overline{\mathrm{D}})_2 = \mathrm{Br}\,(\tilde{\mathrm{Z}} - E - \overline{\mathrm{D}})_2$ where E is the exceptional set of $\tilde{\mathrm{Z}} \longrightarrow \overline{\mathrm{Z}}$ and we have abused notation and written $\overline{\mathrm{D}}$ to also denote its strict transform in $\tilde{\mathrm{Z}}$. Maximal orders cannot ramify on E, so the Artin-Mumford sequence shows that α is determined by a double cover of $\overline{\mathrm{D}}$. Now theorem 6.1 and purity of the branch locus imply that $\pi^*\alpha \in \mathrm{Br}\ \overline{\mathrm{Y}} \simeq \mathbb{Z}/2$ where the last isomorphism follows from the fact that $\overline{\mathrm{Y}}$ is an Enriques surface. Hence if

$$\alpha \in \ker(\mathrm{Br}\,(\overline{\mathrm{Z}} - P - \overline{\mathrm{D}})_2 \longrightarrow \mathrm{Br}\ \overline{\mathrm{Y}}) \simeq (\mathbb{Z}/2)^9$$

then the pull-back β of α to $\mathrm{Br}\ K(Z)$ can be represented by a noncommutative cyclic cover. Note that β is trivial if and only if α is trivial or corresponds to the double cover $D \longrightarrow \overline{\mathrm{D}}$. We obtain consequently the desired 2^8 double covers of D.

9.2 $Z = \mathbb{P}^2$ and $D =$ sextic curve

Let $Z = \mathbb{P}^2$. Let H_1, H_2 be two lines in $\mathbb{P}^2$ and pick a sextic D which is tritangent to both lines. There is a 21-dimensional family of such sextics. Let $\pi : Y \longrightarrow Z$ be the double cover of Z ramified along D so that Y is a K3 surface. Computing intersection numbers we see that $\pi^{-1}(H_i) = E_i \cup E_i'$ where E_i, E_i' are nodal curves, that is, smooth rational curves with self-intersection number -2. If σ is the covering involution as usual, then $(1+\sigma)(E_1 - E_2) = 0$ so $L := \mathcal{O}(E_1 - E_2) \in H^1(G, \mathrm{Pic}\ Y)$. We wish to show it is non-trivial by computing the ramification of the corresponding noncommutative cyclic algebra using proposition 4.5. Let $P_i = D \cap H_i$. The ramification of $A(Y; L_\sigma)$ is the double cover of D determined by the 2-torsion divisor $P_1 - P_2$. We need to show that P_1, P_2 are not linearly equivalent. If they are then $P_1 + P_2 \in |2P_1| = |H_1|_D|$ which contradicts the fact that $P_1 + P_2$ is not a hyperplane section.

9.3 $Z = \mathbb{P}^2$ and $D =$ quartic curve

Let $Z = \mathbb{P}^2$. Suppose D is a quartic with two tangents H_1, H_2 with four fold intersection. An example of such a curve D is the Fermat quartic. Now the 4:1 cover $\pi : Y \longrightarrow Z$ totally ramified on D is a K3-surface and $\pi^{-1}(H_i)$ is the union of 4 nodal curves. If E_i denotes one such nodal curve then as before, $L := \mathcal{O}(E_1 - E_2) \in H^1(G, \mathrm{Pic}\ Y)$ and gives a noncommutative cyclic algebra which is non-trivial in the Brauer group.

10 Invariant Rings and Brauer-Severi Varieties

Artin's theorem (recalled in theorem 6.1) together with theorem 3.9, suggests that noncommutative cyclic covers, at least in the case where the relation is of the form $L_\sigma^e \xrightarrow{\sim} \mathcal{O}$, can be obtained as invariant rings of $\mathcal{E}nd\, V$ where V is some vector bundle on Y. In this section we describe explicitly both V and the group action and use it to study Brauer-Severi varieties.

Assume now that Y is a normal integral Cohen-Macaulay scheme. Let $\phi : L_\sigma^e \xrightarrow{\sim} \mathcal{O}_{\mathrm{Y}}$ be a relation satisfying the overlap condition and $A = A(Y; L_\sigma, \phi)$ the corresponding cyclic algebra. For $i = 0, \ldots, e-1$, let

$$L_i = L \otimes \sigma^* L \otimes \ldots \otimes \sigma^{(i-1)*} L$$

which is the left module structure on L_σ^i. Let $V = \bigoplus L_i = {}_{\mathcal{O}_{\mathrm{Y}}}A$ and $B = \mathcal{E}nd_{\mathcal{O}_{\mathrm{Y}}} V$. The group $G = \langle \sigma \rangle$ acts on B by algebra automorphisms in the following way. There are natural isomorphisms $\sigma^* \mathcal{H}om(L_i, L_j) \xrightarrow{\sim} \mathcal{H}om(L_{i+1}, L_{j+1})$ which sum together to give the action of $\sigma : \sigma^* B \xrightarrow{\sim} B$.

Proposition 10.1 *The invariant ring $B^G \simeq A$.*

Proof. Right multiplication by sections of A are endomorphisms of V so we obtain in this fashion a monomorphism $A \longrightarrow B$. We need to show that the image is B^G. Let $B_{ij} := \mathcal{H}om(L_i, L_{i+j})$ so that $\oplus_i B_{ij}$ is a G-submodule of B. Also σ maps B_{ij} to $B_{i+1,j}$ so the averaging operator shows that any G-invariant section in $\oplus_i B_{ij}$ has the form $s + \sigma^*(s) + \ldots + \sigma^{(e-1)*}(s)$ where $s \in B_{0j} \simeq L_j$. This is just right multiplication by $s \in L_j \subset A$.

From now on we assume that Y and $Z := Y/G$ are smooth quasi-projective varieties. We let $BS(A)$ denote the Brauer-Severi variety of A parametrising flat quotients of A of rank $\deg A$. The action of G on B induces an action on $BS(B) = \mathbb{P}(V)$ which can be nicely described as follows. A direct computation gives $\sigma^* L_i \simeq L^{-1} \otimes_Y L_{i+1}$ so G acts naturally on the projective bundle $\mathbb{P}(V)$ via the isomorphism $\sigma^* V \simeq L^{-1} \otimes_Y V$. The next result follows immediately from the previous proposition.

Corollary 10.2 *The Brauer-Severi variety of A is birational to $\mathbb{P}(V)/G$.*

Unfortunately the rational map $\mathbb{P}(V)/G - \to BS(A)$ is not regular so it is interesting to ask how to resolve the indeterminacy. We give a nice answer when $G = \mathbb{Z}/2$ and assume from now on that this is the case. Let $X = \mathbb{P}(V)$. We use the notion of G-Hilbert schemes. We refer the reader to [IN] and [BKR] for more details about this notion. We let G-Hilb X denote the subscheme of the Hilbert scheme of X which parametrises G-invariant length 2 subschemes P of X with $\mathcal{O}_P$ isomorphic to kG as a G-module. If $I_P \lhd \mathcal{O}_{\mathrm{X}}$ is the ideal sheaf of P then $I_P^G \lhd {\mathcal{O}_{\mathrm{X}}}^G$ is a maximal ideal and we obtain a regular morphism G-Hilb $X \longrightarrow X/G$ which is an isomorphism wherever the G action is free. Let $\pi : Y \longrightarrow Z$ be the quotient map as usual.

Theorem 10.3 *Suppose that $G = \mathbb{Z}/2$ so that A has degree two. Let $D \subset Z$ be the discriminant locus of $Y \longrightarrow Z = Y/G$. The rational map $BS(B)/G - \to BS(A)$ lifts to a regular map Φ : G-Hilb $BS(B) \longrightarrow BS(A)$. If W denotes the $\mathbb{P}^1$-bundle on $\pi^{-1}(D)$ obtained by restricting $BS(B)$ to $\pi^{-1}(D)$, then Φ contracts the strict transform of W/G.*

Proof. In this proof all schemes will be over Z. Let T be a test scheme and $X := BS(B)$. We construct set maps $\mathrm{Hom}(T, \text{G-Hilb}\,(X)) \longrightarrow \mathrm{Hom}(T, BS(A))$ which are functorial in T. A morphism $T \longrightarrow$ G-Hilb (X) corresponds to a subscheme $P \subseteq X \times_Z T$ satisfying the following conditions:

i. The ideal sheaf of P is G-invariant.

ii. P is flat over T.

iii. If $\mathcal{O}_P^+, \mathcal{O}_P^-$ denote the σ-eigensheaves of the $\mathcal{O}_T$-module $\mathcal{O}_P$ then $\mathcal{O}_P^+, \mathcal{O}_P^-$ each are rank 1 $\mathcal{O}_T$-modules.

The subscheme P gives a map $P \longrightarrow X = BS(B)$ which corresponds to an exact sequence of $B \otimes_Z \mathcal{O}_P$ modules

$$0 \longrightarrow I \longrightarrow B \otimes_Z \mathcal{O}_P \longrightarrow Q \longrightarrow 0.$$

Note that since the ideal sheaf of P is G-invariant, G also acts on P and the morphism $P \longrightarrow BS(B)$ is G-equivariant. Re-interpreting in terms of the above exact sequence, we see that if G acts diagonally on $B \otimes_Z \mathcal{O}_P$ then the ideal I is G-invariant and so, G acts on Q in a manner compatible with its action on $B \otimes_Z \mathcal{O}_P$. To construct a morphism $T \longrightarrow BS(A)$ it suffices to prove

Lemma 10.4 *The image of the composite map $A \otimes_Z \mathcal{O}_T \longrightarrow B \otimes_Z \mathcal{O}_P \longrightarrow Q$ is Q^G which is a flat $\mathcal{O}_T$-module of rank 2.*

Proof. We first show that the composite map

$$\Psi : B \otimes_Z \mathcal{O}_T \longrightarrow B \otimes_Z \mathcal{O}_P \longrightarrow Q$$

is surjective. To do this it suffices to assume that $T = \text{Spec } F$ for some field F. Now B is Azumaya of degree 2 so we need only show the image of Ψ is not dimension 2 over F. If this is the case then let $J = \ker \Psi$. Rank considerations force $Q \simeq (B \otimes_Z \mathcal{O}_T)/J \ \otimes_T \mathcal{O}_P$. Consequently, the map $P \longrightarrow X$ factors through $T \longrightarrow X$ contradicting the fact that P is a subscheme of $X \times_Z T$.

Now the action of G restricts to an action on $B \otimes_Z \mathcal{O}_T$ so taking G-invariants of Ψ we see that the image of the map in the lemma is indeed Q^G. If Q^- denotes the (-1)-eigenspace of σ on Q then $Q = Q^G \oplus Q^-$ as $\mathcal{O}_T$-modules so Q^G is flat over T too. To check that the rank of Q^G is 2 we need only do so for $T = \text{Spec } F$ where F is some field. Then $\mathcal{O}_P$ is either a) a quadratic field extension of F, b) $F \times F$ or c) $F[\varepsilon]$ where $\varepsilon^2 = 0$. By condition iii) on $\mathcal{O}_P$ above, we see that G acts non-trivially on $\mathcal{O}_P$. Hence it acts by the non-trivial element of the Galois group of $\mathcal{O}_P/F$ in case a), in case b) by switching factors and in case c) by sending $\varepsilon \mapsto -\varepsilon$. In cases a) and b), multiplication by any non-zero element in $\mathcal{O}_P^-$ switches Q^G and Q^- so the rank of Q^G must be 2. In case c), we consider the exact sequence of F-modules

$$0 \longrightarrow \varepsilon Q \longrightarrow Q \xrightarrow{p} Q/\varepsilon Q \longrightarrow 0.$$

Since $\sigma(\varepsilon) = -\varepsilon$ this is even an exact sequence of G-modules. If $s : Q/\varepsilon Q \longrightarrow Q$ is a G-module section of p then we have a G-module isomorphism $Q \simeq \text{im } s \otimes_F F[\varepsilon]$. Since the G-module $F[\varepsilon]$ is isomorphic to the regular representation FG and Q is flat over $F[\varepsilon]$, we must have $\dim_F Q^G = \dim_F Q/\varepsilon Q = 2$.

We now determine what happens over a closed point $T = p$ of the discrimant locus D. Let q be the point of Y lying over p. We may work complete locally at q and even restrict to a curve containing p. Consequently, we shall assume that $Y = \text{Spec } S$ where $S = k[[w]]$, G acts on Y by $\sigma : w \mapsto -w$. Hence if $v = w^2$ we have $R := \mathcal{O}_Z = S^G = k[[v]]$. Furthermore, we have

$$B = \begin{pmatrix} S & Sw^{-1} \\ Sw & S \end{pmatrix} \ , \ A = \begin{pmatrix} R & R \\ Rv & R \end{pmatrix}$$

where the action of G on B is via the action of G on each of the matrix entries. Note that $B \otimes_{\mathcal{O}_Z} \mathcal{O}_T = B/w^2B$. To simplify notation, we will describe T-points of $BS(B)$ as left ideals of B containing w^2B and similarly for $BS(A)$. We consider the point of $BS(B)$ above q given by the left ideal

$$I_\alpha := Bw + k \begin{pmatrix} 1 & \alpha w^{-1} \\ 0 & 0 \end{pmatrix} + k \begin{pmatrix} 0 & 0 \\ w & \alpha \end{pmatrix}$$

where $\alpha \in k \cup \infty$. Given the action of G on B we see that $\sigma(I_\alpha) = I_{-\alpha}$. The generic point of the strict transform of W/G can be described as follows. Let $\mathcal{O}_P = k \times k$ so that $B \otimes_{\mathcal{O}_Z} \mathcal{O}_P \simeq B/w^2B \times B/w^2B$. For $\alpha \neq 0, \infty$, the ideal $I_\alpha \times I_{-\alpha}$ is on the strict transform of W/G. Its image in $BS(A)$ is given by the ideal

$$\ker(A \longrightarrow (B \times B)/(I_\alpha \times I_{-\alpha})) = A \cap I_\alpha \cap I_{-\alpha} = \begin{pmatrix} Rv & R \\ Rv & Rv \end{pmatrix}.$$

This is independent of α and so completes the proof of the theorem.

We shall conclude this section by showing how the two lines in $BS(A)$ lying over $p \in D$ are obtained. Let $P = \text{Spec } k[\varepsilon]$ where $\varepsilon^2 = 0$. Consider the P-point of $BS(B)$ lying over q defined by the left ideal in $B[\varepsilon] := B \otimes k[\varepsilon]$

$$J_\alpha = B[\varepsilon](w - \alpha\varepsilon) + k[\varepsilon] \begin{pmatrix} 1 & \varepsilon w^{-1} \\ 0 & 0 \end{pmatrix} + k[\varepsilon] \begin{pmatrix} 0 & 0 \\ w & \varepsilon \end{pmatrix}.$$

We let G act on P by $\sigma(\varepsilon) = -\varepsilon$ so that the ideal above is G-invariant and defines a point of G-Hilb$\,(BS(B))$. Note that scaling ε changes the ideal and hence the corresponding map $P \longrightarrow BS(B)$ but not the subscheme of $BS(B)$ it defines. Considering J_α as a point of G-Hilb$\,(BS(B))$, its image in $BS(A)$ is given by the ideal

$$A \cap J_\alpha = Av + k \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 0 \end{pmatrix} + k \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}.$$

As α varies we obtain the line in $BS(A)$ over p corresponding to the fixed point I_0 of $BS(B)$. There is a similar line corresponding to the fixed point I_∞.

Unfortunately, this method will need modification when $e > 2$. The map G-Hilb$\,(BS(B)) - \to BS(A)$ is still not regular.

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