MODULI STACKS OF SERRE STABLE REPRESENTATIONS IN TILTING THEORY

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Abstract. We introduce a new moduli stack, called the Serre stable moduli stack, which corresponds to studying families of point objects in an abelian category with a Serre functor. This allows us in particular, to re-interpret the classical derived equivalence between most concealed-canonical algebras and weighted projective lines by showing they are induced by the universal sheaf on the Serre stable moduli stack. We explain why the method works by showing that the Serre stable moduli stack is the tautological moduli problem that allows one to recover certain nice stacks such as weighted projective lines from their moduli of sheaves. As a result, this new stack should be of interest in both representation theory and algebraic geometry.

Contents

Throughout, we work over an algebraically closed base field k of characteristic zero.

1. Introduction

Tilting theory has proved to be an extremely fruitful avenue of research linking the theory of algebraic geometry to representation theory. In particular, it has produced derived equivalences between certain classes of projective stacks and certain classes of finite dimensional algebras.

The usual way to set up a derived equivalence is to start with a projective stack $\mathbb{Y}$ and look for a tilting complex $T^\bullet \in \mathsf{D}^b(\mathbb{Y})$. Then $\mathbb{Y}$ will be derived equivalent to the algebra $A = \operatorname{End} T^\bullet$. For example, if $\mathbb{Y}$ is a weighted projective line as defined by Geigle-Lenzing [GL] and T is a tilting bundle, then the endomorphism algebras A are the concealed-canonical algebras of Lenzing-Meltzer [LM] which include Ringel's canonical algebras [R] as examples. From this perspective, the main question is, how to find the tilting complex. The philosophy of Mukai [Mu1], [Mu2] and Bondal-Orlov [BO2] however, is that derived equivalences in algebraic geometry come stereotypically from moduli problems, the equivalence being given by a Fourier-Mukai transform with kernel the dual universal family. The tilting condition is then elegantly explained through orthogonality of members of the universal family.

From this point of view, it is more natural to start from the other side, in our case, a finite dimensional algebra A. This is also the natural starting point for the representation theorist, who may be "given" an

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algebra to study. Now the main question becomes: Which moduli problem should you pose to obtain a derived equivalent stack? The traditional approach (see for example [K1]) is to use quiver GIT and works well enough in the case when A is derived equivalent to a projective scheme $\mathbb{Y}$. You choose some discrete invariant $\vec{d} \in K_0(A)$ (i.e. a dimension vector) and start with the rigidified moduli stack $\mathbb{X}$ of A-modules with dimension vector $\vec{d}$ (see Section 2). It turns out that the stack $\mathbb{X}$ is naturally represented as the quotient stack $[\mathcal{R}/PG]$ where $\mathcal{R}$ is the space of representations (with chosen basis) and PG is the group of basis change modulo scalars. Thus on choosing a stability condition (which roughly corresponds to choosing a nice open substack of $\mathbb{X}$), one can take a GIT quotient of $\mathcal{R}$ to produce a $\mathbb{G}_m$-quotient stack.

It is tempting to guess that whenever A is derived equivalent to a projective stack $\mathbb{Y}$ via a tilting bundle, that $\mathbb{Y}$ can be recovered as some open substack of $\mathbb{X}$ (for some dimension vector and stability condition). However, an elementary computation in the case where A is a canonical algebra other than the Kronecker algebra (and hence derived equivalent to a weighted projective line $\mathbb{Y}$ which is not a scheme) then there is no open substack of $\mathbb{X}$ which is isomorphic to $\mathbb{Y}$. The key problem is that the (rigidified) automorphism groups of modules do not match up with the inertia groups of the derived equivalent stack. In this paper, we introduce a new moduli stack $\mathbb{X}^S$ called the moduli stack of Serre stable representations, which overcomes these problems for most concealed-canonical algebras. Morally speaking, it is described as follows. The shifted Serre or Nakayama functor ν_s induces a rationally defined self map on $\mathbb{X}$ and $\mathbb{X}^S$ is the fixed point stack of this self map. Alternately, one can motivate this new stack using Bondal-Orlov's [BO1] concept of a point object. These objects are *Serre stable*, which in the context of finite dimensional A-modules M means $M \simeq \nu_s M$. From this perspective, $\mathbb{X}^S$ parametrises flat families of Serre stable modules. Now ν_s induces a linear endomorphism Φ of the partially ordered group $K_0(A)$. If $\mathbb{X}^S$ is to be non-empty, we need $\vec{d}$ to be fixed by Φ, in which case we say it is *Coxeter stable*. We will mainly be interested in the case where $\vec{d}$ is *minimal* amongst Coxeter stable vectors $\vec{d'}$ with $\vec{d'} > \vec{0}$. The Serre stability condition also arises naturally in Bridgeland-King-Reid's criterion for an exact functor to be an equivalence [BKR, Theorem 2.4].

The correct setting for our results are smoothly weighted projective varieties (defined in Section 9), a notion which generalises weighted projective lines. Essentially, these are stacks which are generically varieties and stacky behaviour is confined to smooth non-intersecting divisors. The first result concerns the Fano or anti-Fano case (defined Section 10), which for concealed-canonical algebras corresponds to the non-tubular case.

Theorem 1.1 (Theorem 10.6). *Let $\mathbb{Y}$ be a smoothly weighted projective variety which is either Fano or anti-Fano and $\mathcal{T}$ be a tilting bundle on $\mathbb{Y}$ with non-isomorphic indecomposable summands. If $\mathbb{X}^S$ is the Serre stable moduli stack of representations for the endomorphism algebra $A = \operatorname{End}_{\mathbb{Y}} \mathcal{T}$ corresponding to the dimension vector $\vec{d}$ of $\mathcal{T}$, then $\mathbb{Y} \simeq \mathbb{X}^S$ and the dual $\mathcal{T}^\vee$ is the universal representation.*

To understand why such a result should hold, we first note that the Serre stable moduli stack can be defined in fairly general contexts, essentially whenever one works in an abelian category with a Serre functor. In particular, one can start with a stack $\mathbb{Y}$ and ask if there is some tautological moduli problem in $\mathsf{coh}(\mathbb{Y})$ whose solution is $\mathbb{Y}$ itself. In general, this should not be possible as there are non-isomorphic stacks with equivalent categories of coherent sheaves. However, if $\mathbb{Y}$ is a projective scheme, then we can look at the rigidified moduli stack $\mathbb{W}$ of skyscraper sheaves which in this case, coincides with the Serre stable moduli stack $\mathbb{W}^S$, and this of course recovers $\mathbb{Y}$. When $\mathbb{Y}$ is a smoothly weighted projective variety, then $\mathbb{W}$ and $\mathbb{W}^S$ are no longer isomorphic for there are Serre unstable skyscraper sheaves (see Example 7.3). In this case it is $\mathbb{W}^S$ which recovers the original stack $\mathbb{Y}$.

Theorem 1.2 (Theorem 9.6). *Let $\mathbb{Y}$ be a smoothly weighted projective variety and $\mathbb{W}^S$ be the Serre stable moduli stack of "skyscraper" sheaves on $\mathbb{Y}$. Then $\mathbb{W}^S \simeq \mathbb{Y}$*

To a large extent, this explains why the Serre stable moduli stack is the correct stack to look at when we have a concealed-canonical algebra. Indeed, we use Theorem 1.2 to prove Theorem 1.1.

Theorem 1.1 suggests a first approach to answering the question: given a finite dimensional algebra, how do you find a derived equivalent stack? You pick a Coxeter stable dimension vector, compute the Serre stable moduli stack and then check if the dual of the universal representation is tilting. However, one might hope for more. Indeed in [BKR], the tilting condition comes out of the theory and there is no need to check it case by case. Emulating this, we seek module-theoretic criteria for the dual universal sheaf to be tilting. This has the potential for answering questions such as: given a class $\mathcal{C}$ of stacks, characterise the endomorphism algebras of tilting bundles on objects of $\mathcal{C}$. For example, we have the following characterisation of non-tubular concealed-canonical algebras.

Theorem 1.3 (Theorem 8.2 + Remark to Theorem 10.6). *Let A be a basic connected finite dimensional algebra of finite global dimension and $\vec{d} \in K_0(A)$ be a minimal Coxeter stable dimension vector. Suppose that*

(i) *the Serre stable moduli stack $\mathbb{X}^S$ is a weighted projective curve, and*
(ii) *any Serre stable module M of dimension vector $\vec{d}$ is the direct sum of modules M_i such that every proper submodule N of M_i satisfies $\sum_{i\geq 0}(-1)^i \dim \mathrm{Ext}^i_A(M,N) < 0$.*

Then the dual of the universal representation is a tilting bundle giving a derived equivalence between A and $\mathbb{X}^S$. In particular, a basic finite dimensional algebra A is non-tubular concealed-canonical if and only if it satisfies the hypotheses above and furthermore, $\ker(\Phi - \mathrm{id}_{K_0(A)})$ *is 1-dimensional, where Φ is the Coxeter transformation.*

We remark that if A is a concealed-canonical algebra which is not tubular, then the minimal Coxeter stable dimension vector is unique and there is no problem in choosing the correct dimension vector. Hypothesis (ii) of Theorem 1.3 is a module-theoretic condition related to classical stability. The moduli-theoretic condition (i) is unfortunate and current work aims to replace it with a module-theoretic condition, that is, one involving A-modules as opposed to flat families of them. We are forced to include it as we don't have the required stack technology. The main obstruction is that we don't have a stable reduction theorem to guarantee that $\mathbb{X}^S$ is proper.

Nevertheless, we do show from first principles that the hypotheses of Theorem 1.3 hold for all canonical algebras. In this case, a similar result has been reached by Abdelghadir-Ueda [AU] using quiver GIT. However, they consider an ad hoc moduli space of enriched quiver representations instead of our Serre stable moduli stack. Our approach also does not require the choice of a separate stability condition. The Serre stability condition in this case is enough to remove modules which would otherwise cause the moduli stack to be badly behaved e.g. non-separated.

We hope the contents of this paper reinforce the following not so well advertised theme in non-commutative algebraic geometry: moduli spaces are an interesting and fruitful way to study non-commutative algebras. We see this theme already appearing in Artin-Tate-Van den Bergh's paper [ATV] which kicked off the study of non-commutative projective geometry by looking at moduli spaces of point modules to unlock secrets in the Sklyanin algebra. In general, given any moduli stack $\mathbb{M}$ of A-modules, the universal sheaf $\mathcal{U}$ can be considered an $(\mathcal{O}_{\mathbb{M}}, A)$-bimodule and Hom, $\otimes$ can be used to relate the categories of quasi-coherent sheaves on $\mathbb{M}$ and A-modules. The question is which moduli stacks will easily give interesting information and the point of this paper is to see how the Serre stable moduli stack is a good candidate in many contexts.

The outline of this paper is as follows. Section 2 reviews aspects of stack theory as it relates to the moduli of A-modules. It is aimed at representation theorists. In Section 3, we introduce the Serre stable moduli stack. To understand this stack, it is instructive to study its k-points, something we do in Section 4. Condition (ii) of Theorem 1.3 naturally arises here. In Section 5, we study canonical algebras "afresh" via moduli spaces. In particular, we compute from first principles the Serre stable moduli stack for canonical algebras and dimension vector $\vec{d} = \mathbf{1}$ and show that it satisfies the hypotheses of Theorem 1.3. In Section 6, we also compute the Serre stable moduli stack for the Beilinson algebra and dimension vector $\mathbf{1}$, comparing our result with the traditional approach via quiver GIT. In both these examples, we'll observe a nice feature of the Serre moduli stack, that we do not need to choose a separate stability condition as occurs for quiver GIT, and that the choice of dimension vector is essentially locked in. We review cyclic quotient stacks in Section 7 and the "Serre" functor in this case, in preparation for studying the Serre stable moduli stack of "skyscraper" sheaves. The local computations in this section will also clarify the Serre stability condition. Section 8 is devoted to proving Theorem 1.3 and hence, that the dual of the universal sheaf is tilting in the canonical algebra case. This reproves Geigle-Lenzing's derived equivalence. Theorems 1.2 and 1.1 are then proved in Sections 9 and 10 respectively.

Conventions Throughout this paper, k will denote an algebraically closed field of characterisitic zero and A will denote a finite dimensional k-algebra. The symbol $\mathbb{G}_m$ denotes the multiplicative group of non-zero elements in k. Stacks will be denoted using the blackboard bold font such as $\mathbb{Y}, \mathbb{X}, \mathbb{W}$. The j-th homology and cohomology group of a complex will be denoted by h_j and h^j respectively. By default, A-modules will be right modules, though occasionally, we will need to look at left modules, for example when looking at duals of these modules. Similarly, modules $\mathcal{M}$ over $\mathcal{O}_{\mathbb{Y}} \otimes_k A$ will usually be viewed as $(\mathcal{O}_{\mathbb{Y}}, A)$-bimodules with A acting on the right and "functions" in $\mathcal{O}_{\mathbb{Y}}$ acting on the left. The unadorned tensor symbol $\otimes$ will denote the tensor product over k.

A definition of a projective stack has been proposed by Kresch [Kr] which we briefly recall here. Let $\mathbb{X}$ be a Deligne-Mumford stack of finite type with finite stabilisers and $c\colon \mathbb{X} \to X$ be the map to its coarse moduli scheme. We say $\mathbb{X}$ is *(quasi)-projective* if its coarse moduli space X is (quasi)-projective and it has a *generating sheaf* $\mathcal{E}$, that is, a locally free coherent sheaf $\mathcal{E}$ on $\mathbb{X}$ such that the natural map

$$c^*(c_*\,\mathcal{H}om_{\mathbb{X}}(\mathcal{E},\mathcal{F}))\otimes_{\mathbb{X}}\mathcal{E}\longrightarrow\mathcal{F}$$

is surjective for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{X}$. Thus $c_*\,\mathcal{H}om_{\mathbb{X}}(\mathcal{E},-)$ induces a Morita equivalence between $\mathbb{X}$ and the sheaf of algebras $c_*\,\mathcal{E}nd_{\mathbb{X}}\,\mathcal{E}$ on X. We will thus define the *support* of a coherent sheaf $\mathcal{F}$ on $\mathbb{X}$ to be the support of $c_*\,\mathcal{H}om_{\mathbb{X}}(\mathcal{E},\mathcal{F})$ on X.

Given an object in an abelian category, we say that it is *simple* if it has two subobjects, zero and itself. This applies in particular to sheaves, where simplicity often has a different meaning in the literature.

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2. The rigidified moduli stack of modules

In this section, we recall the moduli stack of modules and its description as a quotient stack. We also recall the less well-known procedure of rigidification. This overview is aimed at representation theorists with only a passing acquaintance with stacks. Although it is too brief to allow such readers to follow all the proofs in this paper, it should allow them to understand and appreciate the results. The reader who wishes to see more details about stacks should consult standard texts such as [LM-B], [BCEFFK], [DM] and [Stacks]. Brief surveys can also be found in [Beh, Lecture I], [Fan], [Gom] and[V, Appendix].

Let A be a basic finite dimensional k-algebra, so we may write $A = kQ/I$ for some quiver $Q = (Q_0, Q_1)$ and admissible ideal I (see [ASS] page 53). We let $e_v \in A$ be the idempotent corresponding to the vertex $v \in Q_0$. We fix a dimension vector $\vec{d}\colon Q_0 \to \mathbb{N}, v \mapsto d_v$ which can also be viewed as an element of the Grothendieck group $K_0(A)$. Consider the affine space

$$\mathbb{A}^Q := \prod_{v\to w\in Q_1}\mathrm{Hom}_k(k^{d_v},k^{d_w})$$

whose k-points correspond to the representations of Q of the form $V = k^{\vec{d}} := \bigoplus_v k^{d_v}$ and hence isomorphism classes of kQ-modules M with a chosen ordered basis for each Me_v. The ideal I determines a closed subscheme $\mathcal{R}$ of $\mathbb{A}^Q$ corresponding to the A-modules. Now $\mathcal{R}$ is a fine moduli space parametrising A-modules of dimension vector $\vec{d}$ with a chosen basis. Informally, this means that for any commutative ring R, the R-points of $\mathcal{R}$ correspond to isomorphism classes of $R\otimes A$-modules with given dimension vector and R-basis. More precisely, let $\mathcal{U} = \mathcal{O}_{\mathbb{A}^Q}^{\vec{d}}$ be the universal representation of Q on $\mathbb{A}^Q$. The fibre above $q \in \mathbb{A}^Q$ is simply the representation $V_q = \mathcal{O}_q \otimes_{\mathbb{A}^Q}\mathcal{U}$ corresponding to q. Then $\mathcal{U}|_{\mathcal{R}}$ is an $\mathcal{O}_{\mathcal{R}}\otimes A$-module and for any R-point $f\colon \mathrm{Spec}(R) \longrightarrow \mathcal{R}$, the pullback $f^*\mathcal{U}|_{\mathcal{R}}$ is an $R\otimes A$-module of dimension vector $\vec{d}$ with a chosen ordered basis. Furthermore, the isomorphism classes of such modules with chosen basis are given by a unique R-point in this fashion. We call $\mathcal{U}|_{\mathcal{R}}$ the *universal representation* because of this universal property.

Unfortunately, if we try to parametrise isomorphism classes of A-modules (without chosen basis), we find that there is in general no such universal A-module and one main obstruction is that A-modules have automorphisms (see [HM, Chapter 2, Section A] for an explanation of this phenomenon). Algebraic geometers can often circumvent this obstacle by enlarging the category of schemes to stacks.

2.1. Review of stacks.

For us, we will view a stack $\mathbb{X}$ as a sheaf of groupoids. The data involved in defining such a stack thus consists of:

(i) for each noetherian test scheme T, a groupoid $\mathbb{X}(T)$ viewed as a category, all of whose morphisms are isomorphisms, and
(ii) for each map of test schemes $f\colon T' \longrightarrow T$, a *pullback* functor $f^*\colon \mathbb{X}(T) \longrightarrow \mathbb{X}(T')$.

We omit the long list of axioms these data must satisfy (see [BCEFFK, Definition 4.3] for a complete definition). Informally, the isomorphism classes in $\mathbb{X}(T)$ should be thought of as morphisms $T \longrightarrow \mathbb{X}$. Categorifying the set of T-points allows us to remember automorphisms which prevented the existence of universal families. Hence we will refer to $\mathbb{X}(T)$ as the *category of T-points of* $\mathbb{X}$. A morphism of stacks is simply a functor which respects pullback. Any quasi-separated scheme X gives rise to a stack (also denoted X) defined as follows: $X(T)$ is the category of T-points of X, with only identity morphisms. In this way, the category of quasi-separated schemes embeds in the category of stacks. One might wonder if the image of this embedding is the stacks whose category of T-points (for all T) only have identity morphisms (and so are essentially sets). This is almost true (one needs to include algebraic spaces).

We now describe the moduli stack $\tilde{\mathbb{X}}$ of A-modules (of dimension vector $\vec{d}$). Following [K1] we first make the following definition of a family of A-modules:

Definition 2.1. Let T be scheme. A *flat family of A-modules over T* is a finitely generated $\mathcal{O}_T \otimes A$-module $\mathcal{M}$ which is locally free over T. If $T = \operatorname{Spec}(R)$ then we simply say *a flat family over R*. If A is described by a quiver with relations then we also call a flat family of modules a *flat family of representations*. This is just a representation of Q with the given relations in the category of locally free sheaves over T. The dimension vector of $\mathcal{M}$ is given by $d_v = \operatorname{rank}_T \mathcal{M} e_v$.

Let $\tilde{\mathbb{X}}$ be the stack defined as follows. For a test scheme T, objects of $\tilde{\mathbb{X}}(T)$ are the flat families of A-modules over T of dimension vector $\vec{d}$ and the morphisms are the isomorphisms in the category of $\mathcal{O}_T \otimes A$-modules. Given a morphism $f \colon T' \longrightarrow T$ and $\mathcal{M} \in \tilde{\mathbb{X}}(T)$, we have the usual pullback of sheaves which defines $f^*\mathcal{M} \in \tilde{\mathbb{X}}(T')$. This gives the pullback functor of the stack. To show these data do indeed satisfy all the stack axioms, it is usual to identify it with another well-known stack.

For this, we need to introduce the quotient stack construction, which will be vital for us. Let X be a quasi-separated scheme and G an algebraic group acting on X. We will need to use the notion of a *G-torsor* (also called a G-bundle or principal homogeneous space for G) whose definition can be found in [Mil, Chapter III, §4]. We define the *quotient stack* $[X/G]$ whose category of T-points consists of diagrams

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{\tilde{f}} & X \\ \downarrow{\scriptstyle \pi} & & \\ T & & \end{array}$$

where $\tilde{T} \longrightarrow T$ is a G-torsor and $\tilde{f} \colon \tilde{T} \longrightarrow X$ is a G-equivariant morphism. The morphisms are precisely the isomorphisms of this diagram compatible with given structure. There is a natural quotient morphism $X \longrightarrow [X/G]$ defined by the functor $X(T) \longrightarrow [X/G](T)$ which sends the T-point $f \colon T \longrightarrow X$ to the trivial G-torsor $G \times T \longrightarrow T$ and G-equivariant map

$$G \times T \xrightarrow{1_G \times f} G \times X \xrightarrow{\alpha} X$$

where α is the action of G on X. For quotient stacks, automorphism groups are easily computed. If X is a k-variety and $q \in X$ a k-point, the automorphism group of its image in $[X/G]$ is simply the stabiliser group $\operatorname{Stab}_G(q)$. For this reason, we will usually refer to these automorphism groups as *inertia groups*.

Recall that $\mathcal{R} \subseteq \mathbb{A}^Q$ is defined by the admissible ideal I bounding the quiver (Q_0, Q_1). The stack $\tilde{\mathbb{X}}$ of A-modules of dimension vector $\vec{d}$ turns out as one would want, to be the quotient stack of $\mathcal{R}$ by the group of change of bases. More precisely, let $G = \prod_{v \in Q_0} GL_{d_v}$ which acts naturally on $\mathcal{R}$. We briefly describe the isomorphism $\tilde{\mathbb{X}} \simeq [\mathcal{R}/G]$. Given a flat family of A-modules $\mathcal{M} = \bigoplus_{v \in Q_0} \mathcal{M}_v \in \tilde{\mathbb{X}}(T)$, the frame bundle $\pi_v \colon \tilde{T}_v \longrightarrow T$ of the rank d_v vector bundle $\mathcal{M}_v$ is a GL_{d_v}-torsor whose fibre above a k-point q is just the group of vector space isomorphisms $k^{d_v} \longrightarrow \mathcal{O}_q \otimes_T \mathcal{M}_v$. The fibre product of these frame bundles $\tilde{T}_v, v \in Q_0$ over T gives a G-torsor $\pi \colon \tilde{T} \longrightarrow T$ whose k-points are just the A-modules of form $\mathcal{O}_q \otimes_T \mathcal{M}$ together with a choice of basis. There is hence a G-equivariant morphism $\tilde{f} \colon \tilde{T} \longrightarrow \mathcal{R}$ and the pair $(\pi, \tilde{f})$ defines an element of $[\mathcal{R}/G](T)$. This turns out to define the isomorphism $\tilde{\mathbb{X}} \longrightarrow [\mathcal{R}/G]$.

The inverse isomorphism is given as follows. Consider a G-torsor $\pi \colon \tilde{T} \longrightarrow T$ and G-equivariant morphism $\tilde{f} \colon \tilde{T} \longrightarrow \mathcal{R}$ defining an object in $[\mathcal{R}/G](T)$. Note first that the universal sheaf $\mathcal{U}|_{\mathcal{R}}$ is naturally a G-equivariant sheaf, so $f^*\mathcal{U}|_{\mathcal{R}}$ is a G-equivariant sheaf on $\tilde{T}$. This descends (via descent along a torsor) to a flat family of A-modules $(f^*\mathcal{U}|_{\mathcal{R}})^G$ over T. This defines a functor $[\mathcal{R}/G](T) \longrightarrow \tilde{\mathbb{X}}(T)$ and yields the inverse functor. For this reason, we will refer to $\mathcal{U}|_{\mathcal{R}}$ together with its G-action as the *universal sheaf* on $[\mathcal{R}/G]$.

2.2. **Weighted projective curves.** The weighted projective line as studied by representation theorists is usually viewed as the quotient stack of a punctured surface by a 1-dimensional group. However, it is more geometrically meaningful to define it by gluing together quotient stacks of the form $[U/\mu_p]$ where U is a 1-dimensional variety and μ_p is the cyclic group of p-th roots of unity. We will use this latter formulation. Furthermore, we will define weighted projective curves since this involves no more work. The relation between the two approaches is made explicit in the appendix.

We start with a smooth projective curve C. Let $q_1, \dots, q_n \in C$ be distinct points where we "weight" the curve C, that is introduce stacky behaviour. Let $p_1, \dots, p_n$ be integers ≥ 2 called the *weights*. To these data, we define the weighted projective curve $\mathbb{Y} = \mathbb{Y}(\sum p_i q_i)$ together with a morphism $\psi\colon \mathbb{Y} \longrightarrow C$ as follows. Above $U_0 := C - \{q_1, \dots, q_n\}$, ψ is an isomorphism, so $\mathbb{Y}$ has an open substack which is a scheme and in fact, an open subset of the original curve C. Above the point q_i, we pick a sufficiently small open neighbourhood $U_i \subset C$ disjoint from all the other q_j's. Let $t \in \mathcal{O}_{U_i}$ be a local parameter defining q_i and $\tilde{U}_i = \operatorname{Spec}(\mathcal{O}_{U_i}[u]/(u^{p_i} - t))$, so $\tilde{U}_i \longrightarrow U_i$ is a μ_{p_i}-cover of U_i which is totally ramified above q_i and unramified elsewhere. We define $\psi^{-1}(U_i) = [\tilde{U}_i/\mu_{p_i}]$. Note that $\tilde{U}_i \longrightarrow U_i$ factors as

$$\tilde{U}_i \xrightarrow{\rho} [\tilde{U}_i/\mu_{p_i}] \xrightarrow{\psi_i} U_i$$

where ρ is the quotient map described in Subsection 2.1. To define ψ_i, consider an object of $[\tilde{U}_i/\mu_{p_i}](T)$ given by the G-torsor $\pi\colon \tilde{T} \longrightarrow T$ and G-equivariant map $\tilde{T} \longrightarrow U_i$. Then ψ_i of this object is the unique morphism $\beta \in U_i(T) = \operatorname{Hom}(T, U_i)$ which makes the diagram below commute

$$\begin{array}{ccc} \tilde{T} & \longrightarrow & \tilde{U}_i \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ T & \xrightarrow{\beta} & U_i \end{array}$$

Since μ_{p_i} acts freely away from the ramification locus, the map $\psi_i\colon \psi^{-1}(U_i) \longrightarrow U_i$ is an isomorphism away from q_i. However, the inertia group of the point above q_i is the stabiliser group μ_{p_i}. Hence, in the weighted projective curve $\mathbb{Y}$, the point q_i is replaced with a "stacky" point. We call C the *coarse moduli scheme* of $\mathbb{Y}$ since it is, in a sense that can be made precise, the "best" scheme approximation to $\mathbb{Y}$. When $C = \mathbb{P}^1$, we call $\mathbb{Y}$ a *weighted projective line.*

We recall that if T is a tilting bundle on $\mathbb{Y}$ and A is the concealed-canonical algebra $\operatorname{End}_{\mathbb{Y}} T$, then the non-stacky k-points of $\mathbb{Y}$ correspond to the tubes of A of rank 1 and a stacky point with inertia group μ_{p_i} corresponds to a tube of rank p_i.

The inertia groups of the weighted projective line are generically trivial. Every non-zero A-module has at least a copy of $\mathbb{G}_m$ in its automorphism group, so the moduli stack $\tilde{\mathbb{X}}$ is never a weighted projective line. There is however, an easy way to remove this common $\mathbb{G}_m$ from the inertia group which we describe in the next subsection.

2.3. **Rigidification.** We describe here the process of rigidification in our specialised context, as one might find for example in [ACV, Section 5]. One manifestation of the common copy of $\mathbb{G}_m$ in the automorphism groups of A-modules is that the diagonal copy of $\mathbb{G}_m$ in G acts trivially on $\mathcal{R}$. Thus the easy way to remove the common copy of $\mathbb{G}_m$ is to replace $[\mathcal{R}/G]$ with $[\mathcal{R}/PG]$ where $PG = G/\mathbb{G}_m$.

We now define a moduli stack $\mathbb{X}$ which gives a module-theoretic interpretation of this new quotient stack. We start by defining a pre-stack $\mathbb{X}^{pre}$ (pre-stacks are defined by the same data as a stack, and obey the same axioms as a stack except the sheaf axiom, see [BCEFFK, Definition 4.2]). For a noetherian test scheme T, we let the objects of $\mathbb{X}^{pre}(T)$ be the objects of $\tilde{\mathbb{X}}(T)$. However, given objects $\mathcal{M}, \mathcal{N} \in \mathbb{X}^{pre}(T)$ the morphisms from $\mathcal{M}$ to $\mathcal{N}$ in $\mathbb{X}^{pre}(T)$ will consist of equivalence classes of isomorphisms $\theta\colon\ \mathcal{M} \to \mathcal{L} \otimes_T \mathcal{N}$ of $\mathcal{O}_T \otimes A$-modules for some line bundle $\mathcal{L}$ on T. If $\theta'\colon\ \mathcal{M} \to \mathcal{L}' \otimes_T \mathcal{N}$ is another such morphism, we say θ, θ' are *equivalent* if and only if there is an isomorphism $\ell\colon\ \mathcal{L} \to \mathcal{L}'$ such that $\theta' = (\ell \otimes \mathrm{id})\theta$. Note that if $\mathcal{L} = \mathcal{L}'$ then this just means isomorphisms differ by an element of $\mathbb{G}_{m,T}$. Given another isomorphism $\phi\colon\ \mathcal{N} \to \mathcal{L}'' \otimes \mathcal{P}$ with $\mathcal{P} \in \mathbb{X}^{pre}(T)$ and $\mathcal{L}''$ a line bundle on T, we obtain an isomorphism $(\mathcal{L} \otimes \phi)\theta\colon\ \mathcal{M} \to \mathcal{L} \otimes_T \mathcal{N} \to \mathcal{L} \otimes_T \mathcal{L}'' \otimes_T \mathcal{P}$. The equivalence class of this isomorphism remains unchanged if we replace θ and ϕ with equivalent morphisms, so we obtain a composition law on $\mathbb{X}^{pre}(T)$. Unfortunately, $\mathbb{X}^{pre}$ may fail the sheaf axiom for a stack, but sheafifying or "stackifying" it will yield a stack $\mathbb{X}$.

It will be instructive to look at the special case where there is a vertex v_0 such that $d_{v_0} = 1$. This means that if $\mathcal{M} = \oplus \mathcal{M}_v \in \mathbb{X}^{pre}(T)$, then $\mathcal{M}_{v_0}$ is a line bundle on T, so we may, up to isomorphism, replace $\mathcal{M}$ with $\mathcal{M}_{v_0}^{-1} \otimes_T \mathcal{M}$ and so assume $\mathcal{M}_{v_0} \simeq \mathcal{O}_T$. Performing this replacement is called rigidification which is why $\mathbb{X}$ is called the *rigidification* of $\tilde{\mathbb{X}}$. When we introduce the moduli stack of Serre stable representations, we will see it will be convenient to include families which are not necessarily rigidified. There will be many interesting cases where the $d_{v_0} = 1$ assumption holds. For example, if A is canonical, the dimension vector of interest for us will be $\vec{d} = \mathbf{1}$ where all entries are 1, whilst if A is tame hereditary, the dimension vector $\vec{d}$ of interest for us will be the purely imaginary root, which also always has at least one 1 associated to a vertex.

Proposition 2.2. *With the above definitions, we have $\mathbb{X} \simeq [\mathcal{R}/PG]$.*

Proof. (Sketch) We will carry out the proof under the assumption that $v_0 \in Q_0$ is a vertex with $d_{v_0} = 1$ and indicate afterwards how to modify the proof in general. We use étale cohomology below. Consider the exact sequence of group schemes

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G \longrightarrow PG \longrightarrow 1.$$

Note that $\mathbb{G}_m$ lies in the centre of G. Furthermore, this sequence is split since we may project G onto $GL_{d_{v_0}} \simeq \mathbb{G}_\mathrm{m}$ and thus identify PG with the subgroup $G' = \{(g_v)_{v\in Q_0} \mid g_{v_0} = 1\}$. Hence in the exact sequence below

$$0 \longrightarrow H^1(T, \mathbb{G}_m) \longrightarrow H^1(T, G) \xrightarrow{\psi} H^1(T, PG) \xrightarrow{\beta} H^2(T, \mathbb{G}_m),$$

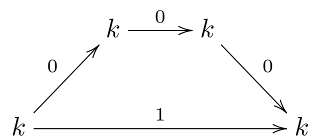
ψ is a split surjection.

An object of $[\mathcal{R}/PG](T)$ consists of a PG-torsor $\tilde{T} \longrightarrow T$ and a PG-equivariant map $\phi\colon \tilde{T} \to \mathcal{R}$. Now $H^1(T, \mathbb{G}_m), H^1(T, G)$ and $H^1(T, PG)$ classify $\mathbb{G}_m$-torsors, G-torsors and PG-torsors over T respectively by [Mil, Chapter III, Corollary 4.7]. Hence the split sequence above shows that $\tilde{T}$ comes from a G-torsor $T' \longrightarrow T$ in the sense that $T'/\mathbb{G}_m = \tilde{T}$. This G-torsor is unique up to a $\mathbb{G}_m$-torsor and, together with the G-equivariant map $T' \longrightarrow \tilde{T} \longrightarrow \mathcal{R}$ defines an object of $\mathcal{M} \in \tilde{\mathbb{X}}(T)$ by the isomorphism $\tilde{\mathbb{X}} \simeq [\mathcal{R}/G]$. (In fact, our choice of splitting ensures that $\mathcal{M}$ is rigidified). From the above discussion, we see that the morphisms in $[\mathcal{R}/PG](T)$ and $\mathbb{X}^{pre}(T)$ correspond, so there is a fully faithful functor $[\mathcal{R}/PG](T) \longrightarrow \mathbb{X}^{pre}(T)$. It is dense since any object $\mathcal{M} \in \mathbb{X}^{pre}(T)$ can be assumed to be rigidified and hence the frame bundle of $\mathcal{M}_{v_0}$ is trivial. The corresponding G-torsor thus comes from a G'-torsor and we are done in this special case. Moreover, $\mathbb{X}^{pre}$ is already a stack and we do not need to stackify it, that is, $\mathbb{X}^{pre} = \mathbb{X}$.

If we do not make our assumption on the special vertex v_0, then ψ may not be split surjective, so a PG-torsor, say corresponding to $\gamma \in H^1(T, PG)$ may not lift to a G-torsor. However, passing to an étale extension $U \to T$, we may split the Brauer class $\beta(\gamma)$ in the sense that its image in $H^2(U, \mathbb{G}_m)$ is now zero. Consequently, the resulting PG-torsor $\gamma|_U$ on U now lifts to a G-torsor on U. □

We call $\mathbb{X}$ the *rigidified moduli stack of A-modules* with dimension vector $\vec{d}$. If the dimension vector needs to be noted, we will denote this stack by $\mathbb{X}_{\vec{d}}$.

Given an A-module M of dimension vector $\vec{d}$ which is a *brick* in the sense that $\operatorname{End}_A M = k$, the inertia group of the corresponding point of $\mathbb{X}$ is now trivial as desired. However, automorphism groups of regular A-modules like



are typically products of copies of $\mathbb{G}_m$, so in general, the rigidified moduli stack $\mathbb{X}$ is still not a weighted projective line. In the next section, we introduce the Serre stable moduli stack which rectifies this problem.

3. The Serre stable moduli stack $\mathbb{X}^S$

In this section, we introduce the main new object of study, the *moduli stack of Serre stable representations.* Morally speaking, the Serre functor induces a rational self map on the rigidified moduli stack $\mathbb{X}$ of A-modules of dimension vector $\vec{d}$, and we look at the fixed point stack of this map.

Let A be a finite dimensional k-algebra with finite global dimension and $D = \operatorname{Hom}_k(-, k)$. The *Nakayama functor*

$$\nu := - \otimes^{\mathbf{L}}_A DA\colon \mathsf{D}^b(\mathsf{mod}\, A) \longrightarrow \mathsf{D}^b(\mathsf{mod}\, A),$$

is a Serre functor in the sense that for any $M, N \in \mathsf{D}^b(\mathsf{mod}\, A)$ we have

$$\operatorname{Hom}_{\mathsf{D}^b}(M, N) \simeq D \operatorname{Hom}_{\mathsf{D}^b}(N, \nu M)$$

(see [Hap]). Our convention for shifting in the triangulated category $\mathsf{D}^b(\mathsf{mod}\, A) \ni K_\bullet$ is given by $(K_\bullet[-s])_j = K_{j-s}$.

Definition 3.1. We choose an integer s called the *shift parameter* and let $\nu_s := \nu \circ [-s]$. An A-module M which satisfies $\nu_s M \simeq M$ is said to be *s-Serre stable* or just *Serre stable* if the shift parameter s is understood.

Note that the actions of ν and ν_s extend to $\mathsf{D}^b(\mathsf{mod}\,\mathcal{O}_T \otimes_k A)$, however ν is no longer a Serre functor in this category. We wish to show families of Serre stable modules are stable under base change and so need the following:

Lemma 3.2. *Let $\mathcal{M}$ be a flat family of A-modules over R and $R \to S$ be a morphism of commutative noetherian rings. Suppose further that the R-modules $h_i(\mathcal{M} \otimes^{\mathbf{L}}_A DA)$ are flat for $i < n$. Then for all $i \leq n$, there is a natural isomorphism*

$$S \otimes_R h_i(\mathcal{M} \otimes^{\mathbf{L}}_A DA) \simeq h_i(S \otimes_R \mathcal{M} \otimes^{\mathbf{L}}_A DA).$$

Proof. We use the hypertor homology spectral sequence [W, Application 5.7.8] with E^2 page:

$$E^2_{ij} = \operatorname{Tor}^R_i(N, h_j(K_\bullet)) \Rightarrow h_{i+j}(N \otimes^{\mathbf{L}}_R K_\bullet)$$

with $N = S$ and $K_\bullet = \mathcal{M} \otimes^{\mathbf{L}}_A DA$ to see that for $i \leq n$

$$S \otimes_R h_i(\mathcal{M} \otimes^{\mathbf{L}}_A DA) \simeq h_i(S \otimes^{\mathbf{L}}_R \mathcal{M} \otimes^{\mathbf{L}}_A DA) = h_i(S \otimes_R \mathcal{M} \otimes^{\mathbf{L}}_A DA). \qquad \square$$

We now modify the stack $\mathbb{X}$ by considering only A-modules which are stable under ν_s. To set things up properly, we need the following:

Proposition 3.3. *Fix dimension vectors $\vec{d}_0, \ldots, \vec{d}_n \in \mathbb{N}^{Q_0}$. Let $\mathbb{X}^{\vec{d}_\bullet}$ be the full subcategory of $\mathbb{X}$ which, over a commutative noetherian ring R, consists of flat families $\mathcal{M}$ of A-modules over R of dimension vector $\vec{d}$ such that $h_i(\nu\,\mathcal{M})$ is a flat family of A-modules over R with dimension vector $\vec{d}_i$ for all $i = 0, \ldots, n$. Then $\mathbb{X}^{\vec{d}_\bullet}$ is a locally closed substack of $\mathbb{X}$.*

Proof. We prove this by induction on n. Let e_v be the idempotent corresponding to the vertex $v \in Q_0$. Suppose $n = 0$ and let $\mathcal{E}_v := h_0(\nu\,\mathcal{M})e_v$. Each $\mathcal{E}_v$ is a finitely generated R-module and hence the locus $Z_v \subseteq \operatorname{Spec}(R)$, where it is locally free of rank $d_{0,v}$, is locally closed in $\operatorname{Spec}(R)$; in fact, it is given by the intersection of the closed condition determined by the $(d_{0,v} - 1)$-st Fitting ideal of $\mathcal{E}_v$ and the open condition determined by its $d_{0,v}$-th Fitting ideal (see [Stacks, Tag 07Z6]). For $\mathbb{X}^{\vec{d}_\bullet}$ to be locally closed we need to show that given any base change $R \to S$, we have that $S \otimes_R \mathcal{M} \in \mathbb{X}^{\vec{d}_\bullet}(S)$ if and only if the map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ factors through the locally closed subscheme $\cap_v Z_v$. However, Fitting ideals commute with arbitrary base change and the formation of $\mathcal{E}_v$ also commutes with base change by Lemma 3.2. Thus $h_0(\nu\,\mathcal{M})e_v$ is locally free of rank $d_{0,v}$ on a locally closed subscheme of $\operatorname{Spec}(R)$.

Suppose now the proposition is true for $i = 0, \ldots, n-1$ i.e. there exists a locally closed subscheme of $\operatorname{Spec}(R)$ on which $h_i(\nu\,\mathcal{M})e_v$ are locally free of rank $d_{i,v}$ for $i = 0, \ldots, n-1$. By the same argument as in the base case, there exists a locally closed subscheme of this (locally closed) subscheme on which $h_n(\nu\,\mathcal{M})e_v$ are locally free for all $v \in Q_0$. Since $h_i(\nu\,\mathcal{M})e_v$ are locally free, and hence flat, Lemma 3.2 once again ensures that the formation of these modules commutes with base change. Finally, as before, Fitting ideals commute with the base change as does the formation of subschemes. $\square$

We now fix the *shift parameter* s which will in examples be the dimension of the moduli space considered. Since we are interested in families $\mathcal{M}$ such that $\nu_s\,\mathcal{M} \simeq \mathcal{M}$, we will need $\vec{d}$ to be a fixed point of the *(shifted) Coxeter transformation* $\Phi\colon K_0(A) \to K_0(A)$ which the shifted Serre functor ν_s induces. We call such a dimension vector *Coxeter stable*. It is also natural to restrict to only those families $\mathcal{M}$ of A-modules such that $h_i(\nu\,\mathcal{M}) = 0$ for $i \neq s$ and $h_s(\nu\,\mathcal{M})$ is locally free of rank vector $\vec{d}$. Proposition 3.3 ensures that the subcategory $\mathbb{X}^0 \subseteq \mathbb{X}$ of such A-modules is a locally closed substack.

Corollary 3.4. *Suppose that $\operatorname{pd}_{\mathrm{A}} DA < \infty$. If $s = \operatorname{pd}_{\mathrm{A}} DA - 1$ or $\operatorname{pd}_{\mathrm{A}} DA$ and $\vec{d}$ is Coxeter stable, then $\mathbb{X}^0$ is an open substack of $\mathbb{X}$.*

Proof. We need to examine the proof of Proposition 3.3 a little more carefully. For $i < \operatorname{pd}_{\mathrm{A}} DA - 1$, the locally closed condition given by the vanishing of $h_i(\nu\,\mathcal{M})$ is open. Let us suppose that $\mathcal{M}$ lies in this open substack. Then for every residue field κ of R, we know that $\nu_s(\kappa \otimes_R \mathcal{M})$ has homology only in degrees 0 and 1. Hence the dimension vector of $h_0(\nu_s(\kappa \otimes_R \mathcal{M}))$ must be at least $\vec{d}$. Hence the locus where it is locally free of rank vector $\vec{d}$ is open. Furthermore, if $s = \operatorname{pd}_{\mathrm{A}} DA - 1$ then on this open locus $h_{s+1}(\kappa \otimes_R \mathcal{M} \otimes^{\mathbf{L}}_A DA) = 0$ since the dimension vector is fixed. $\square$

We now define the shifted Serre functor in families. Let T be a noetherian test scheme and $\mathcal{M} \in \mathbb{X}^0(T)$. The functorial assignment $\mathcal{M} \mapsto h_0(\nu_s \mathcal{M})$ is compatible with base change by Lemma 3.2 , so it defines a morphism of stacks $\nu_s \colon \mathbb{X}^0 \to \mathbb{X}$. We thus have both a diagonal map $\Delta \colon \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ and a graph of ν_s morphism $\Gamma_{\nu_s} \colon \mathbb{X}^0 \to \mathbb{X} \times \mathbb{X}$. The natural definition of the moduli stack of representations fixed by the Serre functor is of course the fibre product

$$\mathbb{X}^S_{\vec{d},s} = \mathbb{X} \times_{\mathbb{X} \times \mathbb{X}} \mathbb{X}^0 \,.$$

We call this the *moduli stack of Serre stable modules* or *Serre stable representations* of dimension vector $\vec{d}$ and shift parameter s. If $\vec{d}, s$ are understood, we will drop the subscripts $\vec{d}, s$ and call this the *Serre stable moduli stack*. Note that $\mathbb{X}$ is not usually quasi-separated, so Δ may not even be quasi-compact let alone a closed immersion. This gives us hope that we may be dealing with a more useful stack than $\mathbb{X}$. Note that given a morphism $\mathcal{M} \to \mathcal{N}$ in $\mathbb{X}^0(T)$, we obtain a morphism $\nu_s \mathcal{M} \to \nu_s \mathcal{N}$ in $\mathbb{X}(T)$. Also, fibre products of Artin stacks are Artin stacks. Unravelling the definition of fibre products of stacks gives

Proposition 3.5. *The objects of* $\mathbb{X}^S(T)$ *are equivalences classes of isomorphisms* $\mathcal{M} \to \mathcal{L} \otimes_T \nu_s \mathcal{M}$ *where* $\mathcal{M}$ *is a flat family of* A*-modules over* T*,* $\mathcal{L}$ *is a line bundle on* T*, and equivalence classes are defined as for* $\mathbb{X}$ *above in Subsection 2.3. The morphisms in* $\mathbb{X}^S(T)$ *consist of isomorphisms* $\phi \colon \mathcal{M} \to \mathcal{M}'$ *which are compatible in* $\mathbb{X}(T)$ *with the isomorphisms* $\theta \colon \mathcal{M} \to \mathcal{L} \otimes_T \nu_s \mathcal{M}, \theta' \colon \mathcal{M}' \to \mathcal{L}' \otimes_T \nu_s \mathcal{M}'$*, that is, there is an isomorphism of line bundles* $\lambda \colon \mathcal{L} \longrightarrow \mathcal{L}'$ *such that the diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\theta} & \mathcal{L} \otimes_T \nu_s \mathcal{M} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \lambda \otimes_T \nu_s \phi} \\ \mathcal{M}' & \xrightarrow{\theta'} & \mathcal{L}' \otimes_T \nu_s \mathcal{M}' \end{array}$$

commutes up to a scalar in $\mathcal{O}_T^\times$*. Furthermore,* $\mathbb{X}^S$ *is an Artin stack of finite type over* k*.*

Remark 3.6. Suppose that we have a flat family of A-modules $\mathcal{M} \in \mathbb{X}(T)$ and an isomorphism $\theta \colon \mathcal{M} \to \mathcal{L} \otimes_T \nu_s \mathcal{M}$ representing an object of $\mathbb{X}^S(T)$. Given an automorphism $\psi \colon \mathcal{M} \to \mathcal{M}$, we obtain a new object $(\mathcal{L} \otimes_T \nu_s \psi^{-1})\theta\psi \colon \mathcal{M} \to \mathcal{L} \otimes_T \nu_s \mathcal{M}$ which is isomorphic to θ in $\mathbb{X}^S(T)$. It will frequently be useful to pass to such an isomorphic family.

4. The k-points of $\mathbb{X}^S$

In this section, we study the category of k-points $\mathbb{X}^S(k)$ of the Serre stable moduli stack $\mathbb{X}^S$. This will not only elucidate the stacky structure, but will also be invaluable for invoking Bridgeland-King-Reid theory [BKR] in our goal of finding module-theoretic characterisations of endomorphism algebras of tilting bundles on smooth projective stacks. We also introduce the notion of regular semisimplicity which plays an important role in our criterion for tilting proved in Section 8.

We continue the notation of Section 3. In particular, we will have fixed a dimension vector $\vec{d} \in K_0(A)$ and shift parameter s and let $\mathbb{X}^S$ denote the corresponding Serre stable moduli stack. By Proposition 3.5, an object of $\mathbb{X}^S(k)$ consists of a Serre stable module M of dimension vector $\vec{d}$ together with an isomorphism $\theta \colon M \xrightarrow{\sim} \nu_s M$. A priori, the object depends on θ, but the next result gives a sufficient criterion for this not to be the case. This allows us to think of the k-points of $\mathbb{X}^S$ as being parametrised by the Serre stable modules themselves in this case.

Recall that $K_0(A)$ has a natural partial ordering $\leq$ where modules induce non-negative elements and that $\vec{d} \in K_0(A)$ is Coxeter stable if it is fixed by the shifted Coxeter transformation. We say $\vec{d} \in K_0(A)$ is *minimal Coxeter stable* if $\vec{d}$ is Coxeter stable but the only other Coxeter stable $\vec{c} \in K_0(A)$ with $\vec{0} \leq \vec{c} \leq \vec{d}$ is $\vec{c} = \vec{0}$.

Proposition 4.1. *Let* $\vec{d}$ *be a minimal Coxeter stable dimension vector and* M *be a Serre stable module with dimension vector* $\vec{d}$*. Suppose that* $\operatorname{End}_A M$ *is a semisimple* k*-algebra with* n *Wedderburn components.*

(i) $\operatorname{End}_A M \simeq k^n$ *and the indecomposable summands* $M_1, \dots, M_n$ *of* M *form a single* ν_s*-orbit.*
(ii) *Any two isomorphisms* $\theta, \theta' \colon M \longrightarrow \nu_s M$ *define isomorphic objects in* $\mathbb{X}^S(k)$*.*
(iii) *Let* $\theta \colon M \longrightarrow \nu_s M$ *be an isomorphism so* (M, θ) *defines an object of* $\mathbb{X}^S(k)$*. Then the automorphism group of* (M, θ) *in* $\mathbb{X}^S(k)$ *is* μ_n*.*

In particular, if every Serre stable module with dimension vector $\vec{d}$ *has semisimple endomorphism ring, then* $\mathbb{X}^S$ *is a Deligne-Mumford stack whose* k*-points are parametrised by Serre stable modules.*

Proof. The Deligne-Mumford criterion [LM-B, Théorème 8.1] will show that $\mathbb{X}^S$ is a Deligne-Mumford stack once we have proven parts (i), (ii) and (iii). Consider an object $\theta \colon M \xrightarrow{\sim} \nu_s M$ of $\mathbb{X}^S(k)$. Let $M = \bigoplus_{i=1}^n M_i$ be the decomposition into indecomposable summands. The Serre functor is additive, so $\nu_s M_i$ is a module, which given the isomorphism θ must be isomorphic to one of the M_j's. Minimality of $\vec{d}$ ensures that the M_i are non-isomorphic and form a single ν_s-orbit, so we can re-order them so that $M_{i+1} \simeq \nu_s M_i$. Hence $\operatorname{End}_A M = k^n$ and the group of A-module automorphisms of M is $\mathbb{G}_m^n$. Part (i) is proved.

We now prove part (ii). If $\lambda = (\lambda_1, \dots, \lambda_n) \in k^n$ defines an endomorphism of M then $\nu_s \lambda = (\lambda_n, \lambda_1, \dots, \lambda_{n-1})$. An easy computation now shows that as λ ranges over $\mathbb{G}_m^n$, $(\nu_s\lambda)\lambda^{-1}$ ranges over all $(\beta_1, \dots, \beta_n) \in \mathbb{G}_m^n$ with $\beta_1\beta_2\dots\beta_n = 1$. Up to scalars, this covers all of $\mathbb{G}_m^n$, so any other isomorphism $\theta' \colon M \xrightarrow{\sim} \nu_s M$ gives an object isomorphic to $\theta \colon M \xrightarrow{\sim} \nu_s M$.

Finally, to prove (iii), note that $\nu_s \colon k^n \to k^n$ has n 1-dimensional eigenspaces, the eigenvalues being the elements of μ_n. Each eigenspace gives a unique automorphism of the object $\theta \colon M \simeq \nu_s M$ in $\mathbb{X}^S(k)$ and there are no other automorphisms. Composition of automorphisms corresponds to multiplication of eigenvalues, so we are done. □

For canonical algebras, we will be interested in the dimension vector $\vec{d} = \mathbf{1}$, where all components are 1. The hypotheses of the previous proposition are easily checked in this case.

Proposition 4.2. *Any module M with dimension vector $\mathbf{1}$ has endomorphism ring $\operatorname{End}_A M \simeq k^n$ where n is the number of indecomposable summands of M.*

Proof. Consider the decomposition into indecomposables $M = \bigoplus_i M_i$. The choice of dimension vector means that $\operatorname{Hom}_A(M_i, M_j) = 0$ if $j \neq i$. Furthermore, given any endomorphism $f \colon M_i \to M_i$ let $N = \operatorname{im} f$. Looking at each vertex, one sees easily that the composite $N \hookrightarrow M_i \xrightarrow{f} N$ is an isomorphism since $\vec{d}$ only has entries 1. Thus f surjects onto some direct summand and so must either be 0 or an isomorphism. □

One of our goals is to see if we can recognise endomorphism algebras of tilting bundles on smooth projective stacks. Such an algebra A has finite global dimension, so we shall assume this for the rest of this section. One case that has been studied in some detail are Lenzing-Meltzer's [LM] *concealed-canonical algebras* which are precisely the endomorphism algebras of tilting bundles on weighted projective lines $\mathbb{Y}$.

For our study of concealed-canonical algebras, we need the bilinear *Euler form* on $K_0(A)$, namely

$$\langle [M], [N] \rangle = \sum_i (-1)^i \dim \operatorname{Ext}^i_A(M, N).$$

Suppose now that $s = 1$ and $\vec{d}$ is Coxeter stable. Then by Serre duality for $\mathsf{D}^b(\mathsf{mod}\, A)$, we have $\langle \vec{d}, - \rangle = -\langle -, \vec{d} \rangle$. In particular, $\langle \vec{d}, \vec{d} \rangle = 0$ which suggests that $\langle \vec{d}, - \rangle \colon K_0(A) \longrightarrow \mathbb{Z}$ is a natural stability condition to use. This motivates

Definition 4.3. With the above hypotheses, we say that an A-module M is *regular simple* if $\langle \vec{d}, [M] \rangle = 0$ but $\langle \vec{d}, [N] \rangle < 0$ for every proper submodule N of M. We say that an A-module is *regular semisimple* if it is a direct sum of regular simples.

If A is basic, tame hereditary, that is, the quiver algebra of an extended Dynkin diagram, then the Coxeter stable dimension vectors are all multiples of the imaginary root $\vec{\delta}$ and setting $\vec{d} = \vec{\delta}$, the above definition coincides with the usual definition of regular simple (see [C-B, Sections 4 and 8]).

We are interested in regular semisimplicity because of the following standard orthogonality result from stability theory which is easily proven.

Proposition 4.4. *Let N, N' be two non-isomorphic regular simple modules. Then $\operatorname{Hom}_A(N, N') = 0$ and $\operatorname{End}_A N = k$. In particular, if M is regular semisimple, then $\operatorname{End}_A M$ is semisimple.*

Regular simple modules arise naturally in the following context.

Proposition 4.5. *Let $\mathcal{T} = \bigoplus_{v \in Q_0} \mathcal{T}_v$ be a tilting bundle on a weighted projective line $\mathbb{Y}$ where the $\mathcal{T}_v$ are non-isomorphic indecomposable summands. Let $A = \operatorname{End}_{\mathbb{Y}} \mathcal{T}$ be the associated concealed-canonical algebra whose quiver has vertices Q_0 and let $\vec{d}$ be the dimension vector of $\mathcal{T}$. For any simple sheaf S on $\mathbb{Y}$, the A-module $M = \operatorname{Hom}_{\mathbb{Y}}(\mathcal{T}, S)$ is regular simple.*

Proof. Let p be a general point of the scheme locus of $\mathbb{Y}$ so $N = \mathrm{Hom}_{\mathbb{Y}}(\mathcal{T}, k(p))$ is an A-module with dimension vector $\vec{d}$. We may assume that S and $k(p)$ have disjoint support, so the derived equivalence $\mathbf{R}\mathrm{Hom}_{\mathbb{Y}}(\mathcal{T}, -)$ shows that $\langle \vec{d}, [M] \rangle = 0$. Let M' be a proper submodule M with $\langle \vec{d}, [M'] \rangle \geq 0$. We may assume that M' is indecomposable. Let $S' = M' \otimes^{\mathbf{L}}_A \mathcal{T}$. Now $\mathsf{coh}(\mathbb{Y})$ is hereditary, so $S' = F[i]$ for some indecomposable sheaf F and $i \geq 0$. Also, $(-) \otimes^{\mathbf{L}}_A \mathcal{T}$ is a derived equivalence so the inclusion $M' \hookrightarrow M$ shows that $0 \neq \mathrm{Hom}_{\mathsf{D}^b}(F[i], S) = \mathrm{Ext}^{-i}_{\mathbb{Y}}(F, S)$. Thus $i = 0$. If F is a bundle, then $\mathrm{Ext}^1_{\mathbb{Y}}(F, k(p)) = 0$ so $\langle \vec{d}, [M'] \rangle < 0$. If F is a torsion sheaf, then applying $\mathbf{R}\mathrm{Hom}_{\mathbb{Y}}(\mathcal{T}, -)$ to the exact sequence

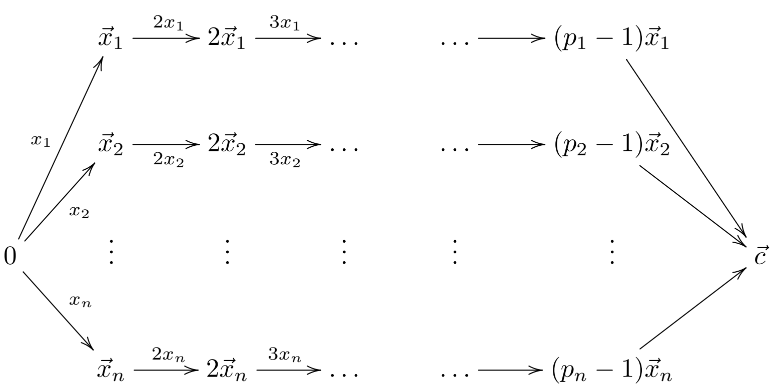
$$0 \longrightarrow \ker \iota \longrightarrow F \xrightarrow{\iota} S \longrightarrow 0$$

we see that M' is not a submodule of M. □

5. Canonical algebras

In this section, we will look at Ringel's [R] canonical algebras A afresh, starting from the generators and relations description. In other words, we will pretend that the algebra was given to us "at random" and we will not use the fact that it is the endomorphism ring of a tilting bundle on a weighted projective line. We follow instead the non-commutative algebro-geometric theme announced in the introduction, and study A through moduli spaces. In particular, we will compute the Serre stable moduli stack $\mathbb{X}^S$ for a natural choice of dimension vector $\vec{d}$ and show it is a weighted projective line.

Let A be the *canonical algebra* given by the following quiver $Q = (Q_0, Q_1)$



where $n \geq 2, p_i \geq 1$, with relations

$$x_i^{p_i} = x_1^{p_1} + \lambda_i x_2^{p_2}, \quad \text{for } i \geq 3, \lambda_i \neq 0.$$

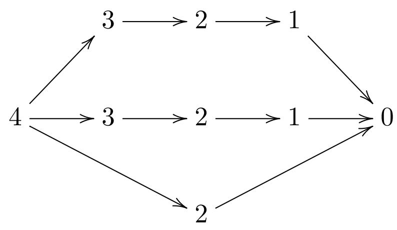
If $\mathbb{Y}$ is the weighted projective line with weight p_1 at 0, p_2 at ∞ and p_i at λ_i for $i \geq 3$, then Geigle and Lenzing show in [GL] that there is a derived equivalence $\mathsf{D}^b(\mathsf{coh}(\mathbb{Y})) \simeq \mathsf{D}^b(\mathsf{mod}\, A)$. We will reprove this result in Section 8.

The *tubular* canonical algebras are the ones whose weights satisfy $\sum_i (1 - \frac{1}{p_i}) = 2$. This occurs precisely when $\nu_1^e \simeq \mathrm{id}$ for some positive integer e (or equivalently, $\omega_{\mathbb{Y}}^{\otimes e} \simeq \mathcal{O}_{\mathbb{Y}}$).

Proposition 5.1. *$\vec{d} = \mathbf{1}$ is a minimal Coxeter stable dimension vector (for the shift parameter $s = 1$). Furthermore, it is unique if and only if A is not tubular.*

Proof. We sketch the proof of this standard result. Note first that the indecomposable projectives form a basis for $K_0(A)$ and if $P_{\vec{x}}$ denotes the projective corresponding to the vertex $\vec{x}$, then $\nu(P_{\vec{x}}) = I_{\vec{x}}$ where $I_{\vec{x}}$ is the indecomposable injective corresponding to $\vec{x}$. Hence the Coxeter transformation Φ is readily computed.

In the non-tubular case, one needs only check that $\mathbf{1}$ spans the eigenspace E_1 of Φ with eigenvalue 1. In the tubular case, one needs first to show that this eigenspace contains $\mathbf{1}$ and is 2-dimensional. If it were not minimal, then there would be another Coxeter stable dimension vector $\vec{d'}$ all of whose entries are 0 or 1. Writing $I = \{v \in Q_0 \mid d'_v = 1\}$ we see then that every vector $\vec{b} \in E_1$ has the same co-ordinates on I and the same on $Q_0 - I$. We thus need only find another Coxeter stable vector with 3 distinct coordinates. For example, the tubular canonical algebra with weights $(p_1, p_2, p_3) = (4, 4, 2)$ has a Coxeter stable dimension vector



The other tubular canonical algebras have a similar Coxeter stable dimension vector. $\square$

This proposition suggests that it might be interesting to study the moduli stack $\mathbb{X}^S$ of Serre stable A-modules with dimension vector $\vec{d} = \mathbf{1}$ and shift parameter $s = 1$. By Propositions 4.1 and 4.2, a k-point of this stack is given by a representation $M = (M_v, \varphi_\alpha)_{v\in Q_0,\alpha\in Q_1}$ of Q, satisfying the above relations, with dimension vector $\mathbf{1}$, and which is Serre stable in the sense that

$$M \simeq M \otimes^{\mathbf{L}}_A DA[-1]$$

Equivalently, Serre stability here means $\operatorname{Tor}_1^A(M, DA) = D\operatorname{Ext}^1_A(M, A) \simeq M$ and $\operatorname{Tor}_i^A(M, DA) = D\operatorname{Ext}^i_A(M, A) = 0$ if $i \neq 1$.

It will be instructive to examine closely the k-points of $\mathbb{X}^S$ in order to have a good guess as to what $\mathbb{X}^S$ is. The following lemmas will be helpful in this regard.

Lemma 5.2. *Let M be a k-point of $\mathbb{X}^S$. Then for any other representation N we have for all i:*

$$\operatorname{Ext}^i_A(M, N) \simeq D\operatorname{Ext}^{1-i}_A(N, M).$$

In particular $\operatorname{Hom}_A(M, N) \simeq D\operatorname{Ext}^1_A(N, M)$ *and* $\operatorname{Ext}^1_A(M, N) = D\operatorname{Hom}_A(N, M)$.

Proof. Writing D for $\mathsf{D}^b(\mathsf{mod}\, A)$, we have $\operatorname{Hom}_{\mathsf{D}}(M, N[i]) = D\operatorname{Hom}_{\mathsf{D}}(N[i], \nu M) = D\operatorname{Hom}_{\mathsf{D}}(N, \nu M[-i])$. Since $M \simeq \nu M[-1]$, it follows that

$$\operatorname{Hom}_{\mathsf{D}}(M, N[i]) \simeq D\operatorname{Hom}_{\mathsf{D}}(N, M[1-i])$$

and the lemma is proved. $\square$

Lemma 5.3. *Let $M = (M_v, \varphi_\alpha)_{v\in Q_0,\alpha\in Q_1}$ be a k-point of $\mathbb{X}^S$. If $\varphi_{ax_i} = 0$ for some $a = 1, \dots, p_i$ then $\varphi_{bx_i} = 0$ for all $b = 1, \dots, p_i$. Furthermore, if $\varphi_{x_i} = 0$ then $\varphi_{x_j} \neq 0$ if $i \neq j$.*

Proof. Suppose $a \neq p_i$ and that $\varphi_{ax_i} = 0$. We claim that $\varphi_{(a+1)x_i} = 0$. Note that $\operatorname{Hom}_A(M, S_{a\vec{x}_i}) \simeq k$ where $S_{a\vec{x}_i}$ is the simple representation at vertex $a\vec{x}_i$. We now compute $\operatorname{Ext}^1_A(S_{a\vec{x}_i}, M)$. We have the following projective resolution

$$0 \longrightarrow P_{(a+1)\vec{x}_i} \longrightarrow P_{a\vec{x}_i} \longrightarrow S_{a\vec{x}_i} \longrightarrow 0$$

where $P_{a\vec{x}_i}$ is the projective at vertex $a\vec{x}_i$, which gives the following long exact sequence

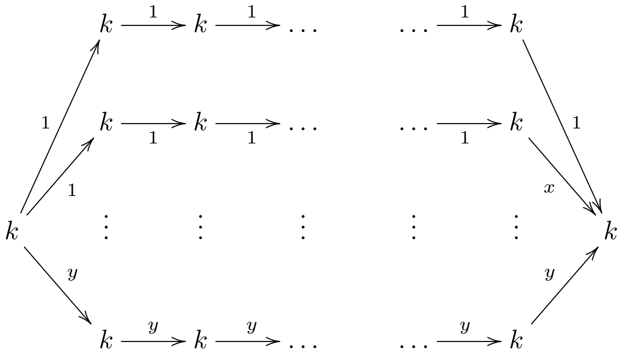
$$0 \longrightarrow \operatorname{Hom}_A(S_{a\vec{x}_i}, M) \longrightarrow \operatorname{Hom}_A(P_{a\vec{x}_i}, M) \xrightarrow{f} \operatorname{Hom}_A(P_{(a+1)\vec{x}_i}, M) \longrightarrow \operatorname{Ext}^1_A(S_{a\vec{x}_i}, M) \longrightarrow 0.$$

Now $\operatorname{Hom}_A(P_{a\vec{x}_i}, M) \simeq \operatorname{Hom}_A(P_{(a+1)\vec{x}_i}, M) \simeq k$ and so if $\varphi_{(a+1)x_i} \neq 0$, then $\operatorname{Hom}_A(S_{a\vec{x}_i}, M) = 0$ implying f is surjective and forcing $\operatorname{Ext}^1_A(S_{a\vec{x}_i}, M) = 0$. This contradicts Lemma 5.2 so $\varphi_{(a+1)x_i} = 0$.

A similar argument, using injective resolutions shows that if $a \neq 1$ and $\varphi_{ax_i} = 0$ then $\varphi_{(a-1)x_i} = 0$.

Note if $\varphi_{x_i} = \varphi_{x_j} = 0$ then the relations, and the result we just proved, imply all arrows must be 0. Such a representation cannot be Serre stable, for it contains the projective summand $P_{\vec{c}}$. $\square$

This lemma shows that $\mathbb{X}^S$ is covered by open substacks $\mathbb{U}_i$ consisting of Serre stable families where only φ_{x_i} is allowed to be 0. Furthermore, any k-point of say $\mathbb{U}_n$ is given by a module isomorphic to one of the form



(1)

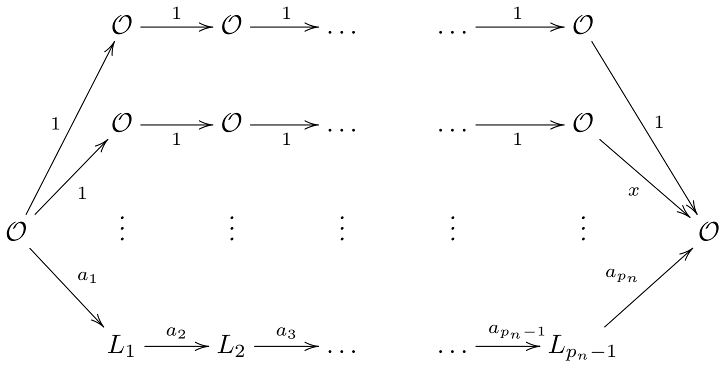
where $x \neq 0, \lambda_3, \ldots, \lambda_{n-1}$ and $y^{p_n} = x - \lambda_n$. This is indecomposable for $y \neq 0$ and the direct sum of p_n indecomposables when $y = 0$. From Proposition 4.1, we see that the corresponding inertia groups are 1 and μ_{p_n}. This immediately suggests the following

Theorem 5.4. *There is an isomorphism* $\mathbb{X}^S \simeq \mathbb{Y}$.

Proof. We first compute the open substack $\mathbb{U}_n$. The other open patches $\mathbb{U}_i$ can be computed similarly. It will then be clear from the computations below, that they glue together to form $\mathbb{Y}$ as desired.

Lemma 5.5. *Let T be a scheme. The objects of $\mathbb{U}_n(T)$, are precisely those isomorphic to $\mathcal{M} \xrightarrow{\sim} L_{p_n-1}^{-1} \otimes_T \nu_1 \mathcal{M}$ where*

$\mathcal{M} :=$

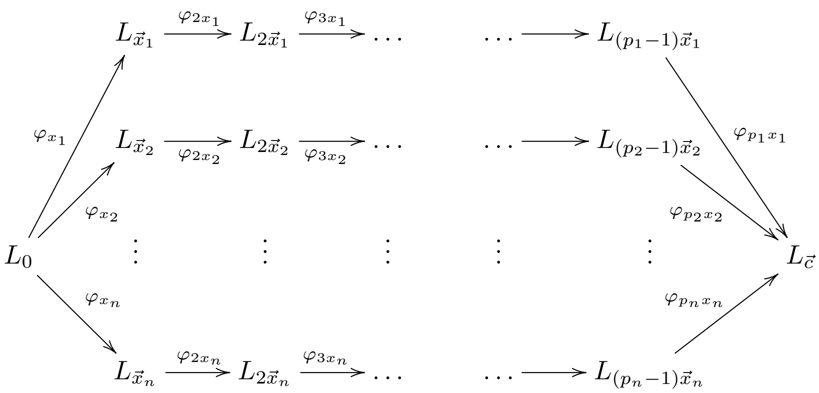


,

$\mathcal{O} = \mathcal{O}_T$ and L_i are invertible sheaves on T.

Proof. An object of $\mathbb{U}_n(T)$ consists of a representation

$\mathcal{M} :=$



together with an isomorphism $\mathcal{M} \xrightarrow{\sim} N \otimes_T \nu_1 \mathcal{M}$ where N and $L_{a\vec{x}_i}$ are invertible sheaves on T. By applying $L_0^{-1} \otimes_T -$ to obtain an isomorphic element of $\mathbb{U}_n(T)$, we can assume $L_0 = \mathcal{O}$. Furthermore, for $i = 1, \ldots, n-1$ all φ_{x_i} must be surjective for otherwise, there will be an element of $\mathbb{U}_n(k)$ with $\varphi_{x_i} = 0$ which contradicts the definition of $\mathbb{U}_n$. Thus, these φ_{x_i} must be isomorphisms and so $L_{a\vec{x}_i} \simeq \mathcal{O}$ and by further changing bases if necessary, we see that $\mathcal{M}$ is of the form given.

Now we compute $\nu_1 \mathcal{M}$. Let $P_{a\vec{x}_i}$ be the projective at vertex $a\vec{x}_i$ and recall that $P_{a\vec{x}_i} \otimes_A DA = I_{a\vec{x}_i}$ which is the injective at vertex $a\vec{x}_i$. We have the following resolution

$$0 \longrightarrow \begin{bmatrix} \mathcal{O} \otimes P_{\vec{x}_n} \\ L_1 \otimes P_{2\vec{x}_n} \\ \vdots \\ L_{p_n-1} \otimes P_{\vec{c}} \end{bmatrix} \xrightarrow{\partial} \begin{bmatrix} \mathcal{O} \otimes P_0 \\ L_1 \otimes P_{\vec{x}_n} \\ \vdots \\ L_{p_n-1} \otimes P_{(p_n-1)\vec{x}_n} \end{bmatrix} \longrightarrow \mathcal{M} \longrightarrow 0$$

where

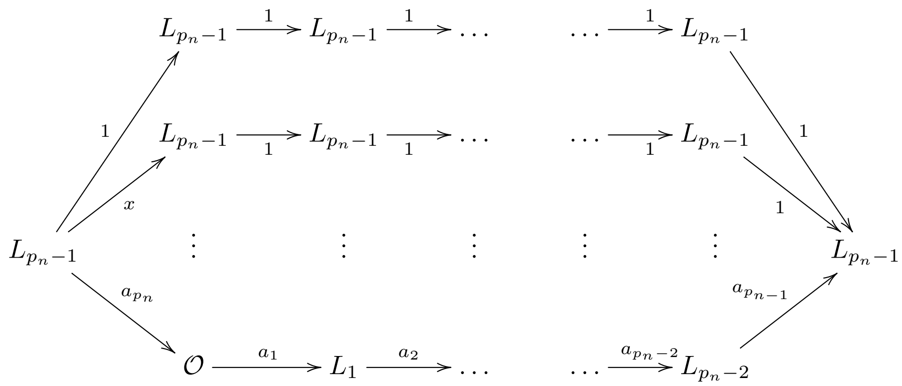
$$L_i = L_{i\vec{x}_n}, \quad \partial = \begin{bmatrix} 1 \otimes x_n & 0 & \dots & -a_{p_n} \otimes x_2^{p_2} \\ -a_1 \otimes 1 & 1 \otimes 2x_n & & \\ 0 & -a_2 \otimes 1 & & \\ \vdots & \vdots & & \\ 0 & 0 & \dots & 1 \otimes p_n x_n \end{bmatrix}$$

We now apply $- \otimes_A DA$ to the resolution to obtain the complex

$$0 \longrightarrow \begin{bmatrix} \mathcal{O} \otimes I_{\vec{x}_n} \\ L_1 \otimes I_{2\vec{x}_n} \\ \vdots \\ L_{p_n-1} \otimes I_{\vec{c}} \end{bmatrix} \xrightarrow{\partial^\vee} \begin{bmatrix} \mathcal{O} \otimes I_0 \\ L_1 \otimes I_{\vec{x}_n} \\ \vdots \\ L_{p_n-1} \otimes I_{(p_n-1)\vec{x}_n} \end{bmatrix} \longrightarrow 0$$

It is easy to see that $\partial^\vee$ is surjective and hence there is homology only in degree 1. We compute the kernel and obtain

$\nu_1 \mathcal{M} =$

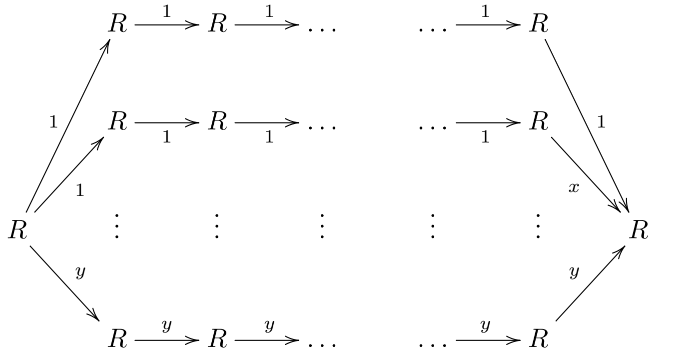


Thus, by fixing an isomorphism $\phi\colon \mathcal{M} \xrightarrow{\sim} L_{p_n-1}^{-1} \otimes_T \nu_1 \mathcal{M}$ we obtain an element of $\mathbb{X}^S(T)$. $\square$

The modules in (1) suggest what the universal representation on $\mathbb{U}_n$ looks like. First, let

$$R := \frac{k[x^{\pm 1}, (x-\lambda_3)^{-1}, \dots, (x-\lambda_{n-1})^{-1}, y]}{y^{p_n} = x - \lambda_n}.$$

This is a commutative k-algebra of Krull dimension 1. In fact, $\mathrm{Spec}(R)$ is a cyclic cover of $V = \mathbb{P}^1 - \{q_1, \dots, q_{n-1}\}$ which is totally ramified above q_n and unramified elsewhere. In particular, μ_{p_n} acts on R by multiplying y by a primitive p_n-th root of unity ζ. It follows that $[\mathrm{Spec}(R)/\mu_{p_n}]$ is the open substack of $\mathbb{Y}$ corresponding to the inverse image of V under the natural map $\mathbb{Y} \longrightarrow \mathbb{P}^1$ (see the construction of weighted projective curves in Section 2.2). We claim that $[\mathrm{Spec}(R)/\mu_{p_n}] \simeq \mathbb{U}_n$. To show this consider the following flat family $\tilde{\mathcal{U}}$ of A-modules over R:



(2)

This family is μ_{p_n}-equivariant (as replacing y with $\zeta^i y$ yields an isomorphic family) and hence is a family over $[\mathrm{Spec}(R)/\mu_{p_n}]$. From the previous lemma, we see that indeed $\nu_1 \tilde{\mathcal{U}} \simeq \tilde{\mathcal{U}}$ and so $\tilde{\mathcal{U}} \in \mathbb{X}^S(R)$ and thus we get a map $[\mathrm{Spec}(R)/\mu_{p_n}] \to \mathbb{U}_n$. We claim that the family is universal and hence the map is in fact an isomorphism. More precisely, we aim to show that any $\mathcal{M} \in \mathbb{U}_n(T)$, which must have the form as described in Lemma 5.5, is a pullback of $\tilde{\mathcal{U}}$ via a unique morphism $T \to [\mathrm{Spec}(R)/\mu_{p_n}]$. From the proof of Lemma 5.5, we see that ϕ induces isomorphisms

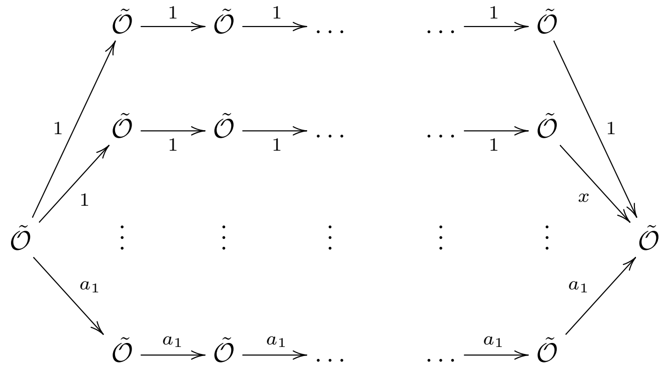
$$L_1 \otimes_T L_{p_n-1} \simeq \mathcal{O}, \quad L_2 \otimes_T L_{p_n-1} \simeq L_1, \quad \dots \quad , L_{p_n-1} \otimes_T L_{p_n-1} \simeq L_{p_n-2}$$

thus

$$L_1^{\otimes 2} \simeq L_2, \quad L_1^{\otimes 3} \simeq L_3, \dots \quad L_1^{\otimes p_n} \simeq \mathcal{O}\,.$$

Hence L_1 is a p_n-torsion line bundle, which together with the isomorphism $L_1^{\otimes p_n} \simeq \mathcal{O}$ defines an étale cyclic cover $\pi\colon \tilde{T} = \underline{\mathrm{Spec}}_T \left(\bigoplus_{i=0}^{p_n-1} L^{\otimes i} \right) \to T$. We thus get

$\pi^* \mathcal{M} =$



where $\tilde{\mathcal{O}} = \mathcal{O}_{\tilde{T}}$. This family comes as a pullback of $\tilde{\mathcal{U}}$ via the map $R \to \mathcal{O}_{\tilde{T}}$ given by $x \mapsto x$ and $y \mapsto a_1$. This verifies the isomorphism $\mathbb{U}_n \simeq [\mathrm{Spec}(R)/\mu_{p_n}]$. Patching together the $\mathbb{U}_i$ using (2) now shows that $\mathbb{X}^S \simeq \mathbb{Y}$ as desired. □

Let $\mathcal{U}$ be the universal representation on $\mathbb{X}^S$ and $\mathcal{T} := \mathcal{U}^\vee$ be the dual bundle. We will recover Geigle-Lenzing's derived equivalence in Section 8, by showing that $\mathcal{T}$ is a tilting bundle on $\mathbb{X}^S$. In preparation for this we need one more result about the k-points of $\mathbb{X}^S$.

Proposition 5.6. *Let M be a k-point of $\mathbb{X}^S$. Then M is regular semisimple.*

Proof. First note that the dimension vector $\mathbf{1}$ can be expressed as $\mathbf{1} = [P_{\vec{0}}] - [P_{\vec{c}}]$. From Lemma 5.3 or (1), we know that the indecomposable summands M_i of M are either simple or have a unique top S_0 and unique socle $S_{\vec{c}}$. Thus if N is any proper submodule of M_i, then there exists a subset $I \subseteq \{\vec{x}_1, 2\vec{x}_1, \dots, (p_n-2)\vec{x}_n, (p_n-1)\vec{x}_n\}$ such that

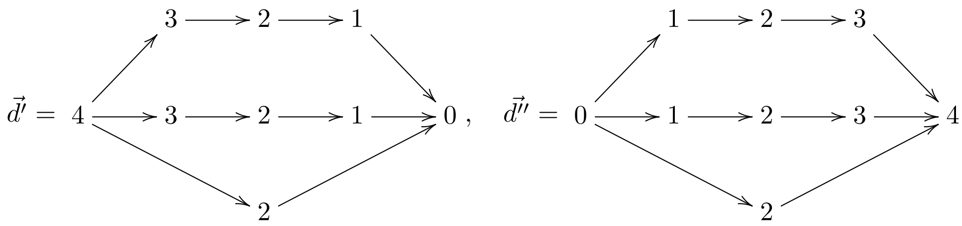
$$[N] = [S_{\vec{c}}] + \sum_{\vec{x} \in I} [S_{\vec{x}}].$$

We compute then that

$$\langle \mathbf{1}, [N] \rangle = \langle [P_{\vec{0}}] - [P_{\vec{c}}], [S_{\vec{c}}] + \sum_{\vec{x} \in I} [S_{\vec{x}}] \rangle = \langle -[P_{\vec{c}}], [S_{\vec{c}}] \rangle = -1 < 0.$$

Hence M is regular semisimple. $\square$

It would be an interesting exercise to compute the Serre stable moduli stack for other minimal Coxeter stable dimension vectors in the tubular case. The Coxeter stable dimension vectors which are ≥ 0 are easy enough to describe. For the tubular algebra with weights $(4,4,2)$, these are all $\mathbb{Q}_{\geq 0}$-linear combinations of the two vectors



The other cases are obtained similarly. Under the Geigle-Lenzing derived equivalence, the dimension vector $\vec{d}'$ above can be realised as a rank 4 vector bundle on $\mathbb{Y}$ as the following example shows.

Example 5.7. In the tubular case, $\mathbb{Y} \simeq [E/G]$ for some elliptic curve E and $G = \langle \sigma \rangle$. We will view coherent sheaves on $\mathbb{Y}$ as G-equivariant sheaves on E. If the weights are $(4,4,2)$ then $G \simeq \mathbb{Z}/4$ and there is a G-cover $\pi \colon E \longrightarrow \mathbb{P}^1$ ramified over $0, \infty, \lambda_3$ with ramification indices $4,4,2$. The ramification divisor on E thus can be written as $3q_1 + 3q_2 + q_3' + q_3''$. The corresponding canonical algebra is the endomorphism algebra of the tilting bundle

$$\mathcal{T} = \mathcal{O}_E \oplus \Big(\bigoplus_{i=1}^{3} \mathcal{O}_E(iq_1)\Big) \oplus \Big(\bigoplus_{i=1}^{3} \mathcal{O}_E(iq_2)\Big) \oplus \mathcal{O}_E(q_3' + q_3'') \oplus \pi^* \mathcal{O}_{\mathbb{P}^1}(1).$$

The canonical bundle is the G-equivariant sheaf

$$\omega_{\mathbb{Y}} = \mathcal{O}_E(3q_1 + 3q_2 + q_3' + q_3'') \otimes_E \pi^* \mathcal{O}_{\mathbb{P}^1}(-2) \simeq \mathcal{O}_E(-q_1 - q_2 + q_3' + q_3'').$$

The Riemann-Hurwitz formula ensures that as a sheaf on E we have $\omega_{\mathbb{Y}} \simeq \mathcal{O}_E$. The isomorphism is not G-equivariant and in fact, looking locally at any ramification point, we see that $\omega_{\mathbb{Y}} \simeq \mathcal{O}_E \otimes \chi$ for some character χ which generates the character group $G^\vee$. Hence for any $\mathcal{L} \in \mathsf{coh}(\mathbb{Y})$,

$$\mathcal{L} \otimes kG^\vee := \bigoplus_{i=0}^{3} \mathcal{L} \otimes \chi^i \in \mathsf{coh}(\mathbb{Y})$$

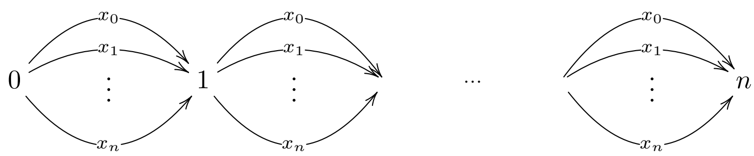
is Serre stable. Furthermore, if $\mathcal{M} \in \mathsf{coh}(\mathbb{Y})$ then

$$\mathbf{R}\mathrm{Hom}_{\mathbb{Y}}(\mathcal{M}, \mathcal{L} \otimes kG^\vee) = \mathbf{R}\mathrm{Hom}_E(\mathcal{M}, \mathcal{L} \otimes kG^\vee)^G = \mathbf{R}\mathrm{Hom}_E(\mathcal{M}, \mathcal{L}).$$

Hence $\mathrm{Hom}_{\mathbb{Y}}(\mathcal{T}, \mathcal{L} \otimes kG^\vee)$ has dimension vector $\vec{d}'$ whenever $\mathcal{L}$ is a G-equivariant line bundle whose underlying sheaf on E is degree 4 and not isomorphic to $\pi^* \mathcal{O}_{\mathbb{P}^1}(1)$. An example of such a sheaf is $\mathcal{L} = \mathcal{O}_E(3q_1 + q_2)$.

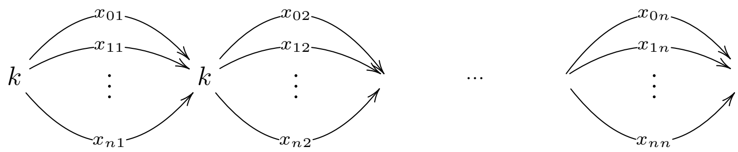
6. Beilinson algebra

In this section, we examine the Beilinson algebra B_n, which is given by the following quiver



with commutative relations $x_i x_j = x_j x_i$ for all i, j. We will compare our approach using the Serre stable moduli stack with the traditional quiver GIT approach for the case $\vec{d} = \mathbf{1}, s = n$.

In the quiver GIT approach, one has to pick a stability condition, and here the usual one used is the linear map $\rho \colon K_0(B_n) \longrightarrow \mathbb{Z}$ defined by $\rho([M]) = \dim Me_n - \dim Me_0$ where $e_i \in B_n$ denotes the primitive idempotent corresponding to the vertex i. One considers only ρ-*stable* modules with dimension vector **1**, which by definition means only those M such that for any proper submodule N of M, we have $\rho([N]) > 0$. One readily sees that these are precisely the representations of the form



(3)

where $\vec{x}_i := (x_{0i}, \dots, x_{ni}) \neq \vec{0}$ for all i. This is an open condition. In this case, the commutativitiy relations show that all $\vec{x}_i$ are proportional and define a common point of $\mathbb{P}^n$. If $\mathbb{X}_{\mathbf{1}}^{\rho}$ denotes the open substack of $\mathbb{X}_{\mathbf{1}}$ consisting of ρ-stable modules, then quiver GIT in this case gives $\mathbb{X}_{\mathbf{1}}^{\rho} = \mathbb{P}^n$ (this will also be clear from our calculation of the Serre stable moduli stack below).

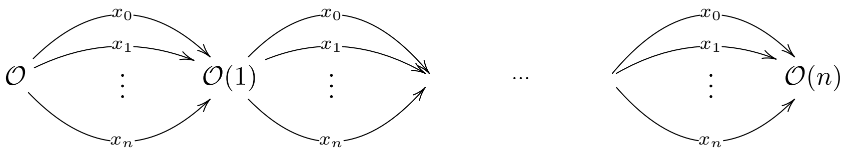
We now turn to the Serre stable moduli stack.

Proposition 6.1. *Let M be a Serre stable B_n-module with dimension vector* **1**. *Then M is ρ-stable.*

Proof. First note that if S_i denotes the simple B_n-module at vertex i, then a simple downward induction shows $\operatorname{pd} S_i = n - i$. Suppose now that M is not ρ-stable, so in the notation of (3) $\vec{x}_i = \vec{0}$ say. Then there exists a direct sum decomposition $M = M_{\leq i} \oplus M_{>i}$ of B_n-modules such that $\operatorname{pd} M_{>i} < n$. It follows that $\nu_n M_{>i}$ is not a module, so M cannot be Serre stable. □

Theorem 6.2. *We have $\mathbb{X}^S_{\mathbf{1},n} = \mathbb{P}^n$. Furthermore, if $\mathcal{U}$ is the universal representation, then $\nu_n \mathcal{U} \simeq \omega_{\mathbb{P}^n} \otimes_{\mathbb{P}^n} \mathcal{U}$.*

Proof. Consider the following family $\mathcal{U}$ of B_n-modules over $\mathbb{P}^n_{x_0:\dots:x_n}$:



Recall that $\mathcal{U}$ is the universal representation on $\mathbb{X}_{\mathbf{1}}^{\rho} \simeq \mathbb{P}^n$ and that its dual $\mathcal{T} = \mathcal{U}^{\vee} = \mathcal{O} \oplus \dots \oplus \mathcal{O}(-n)$ is a tilting bundle on $\mathbb{P}^n$ [Be] with endomorphism ring isomorphic to B_n.

We first show that $\mathcal{U}$ is Serre stable, so picking any isomorphism $\theta \colon \mathcal{U} \simeq \omega_{\mathbb{P}^n}^{-1} \otimes_{\mathbb{P}^n} \nu_n \mathcal{U}$ determines a morphism $\mathbb{P}^n \longrightarrow \mathbb{X}^S$. Let P_i denote the indecomposable projective B_n-module at vertex i and I_i denote the injective at i. We need the following

Proposition 6.3. *We have the following exact sequences of B_n-modules over $\mathbb{P}^n$.*

$$0 \longrightarrow \Omega^n(n) \otimes P_n \longrightarrow \Omega^{n-1}(n-1) \otimes P_{n-1} \longrightarrow \dots \longrightarrow \Omega^1(1) \otimes P_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \otimes P_0 \longrightarrow \mathcal{U} \longrightarrow 0 \quad (4)$$

$$0 \longrightarrow \omega_{\mathbb{P}^n} \otimes_{\mathbb{P}^n} \mathcal{U} \longrightarrow \Omega^n(n) \otimes I_n \longrightarrow \Omega^{n-1}(n-1) \otimes I_{n-1} \longrightarrow \dots \longrightarrow \Omega^1(1) \otimes I_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \otimes I_0 \longrightarrow 0 \quad (5)$$

Furthermore, the second sequence is, up to twisting by a line bundle, the dual of the left module version of the first sequence.

Proof. We have from [Be], the following resolution of the diagonal $\Delta \colon \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$:

$$0 \longrightarrow p^*\Omega^n(n) \otimes q^*\mathcal{O}(-n) \longrightarrow \dots \longrightarrow p^*\Omega^1(1) \otimes q^*\mathcal{O}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \longrightarrow \mathcal{O}_{\Delta} \longrightarrow 0$$

where

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}^n & \xrightarrow{p} & \mathbb{P}^n \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ \mathbb{P}^n & \xrightarrow{u} & \mathrm{pt} \end{array}$$

p is the projection onto the first factor and q is the projection onto the second. We now apply the functor $p_* \mathcal{H}om(q^* \mathcal{T}, -)$ to this exact sequence. Since

$$\begin{aligned} R^j p_* \mathcal{H}om(q^* \mathcal{T}, p^*\Omega^i(i) \otimes q^*\mathcal{O}(-i)) &= R^j p_*(p^*\Omega^i(i) \otimes q^* \mathcal{H}om(\mathcal{T}, \mathcal{O}(-i))) \\ &= \Omega^i(i) \otimes_{\mathbb{P}^n} R^j p_* q^* \mathcal{H}om(\mathcal{T}, \mathcal{O}(-i)) \quad \text{[Har, Ch. III, Ex. 8.2]} \\ &= \Omega^i(i) \otimes_{\mathbb{P}^n} f^* R^j u_* \mathcal{H}om(\mathcal{T}, \mathcal{O}(-i)) \quad \text{[Har, Ch. III, Proposition 9.3]} \\ &= \Omega^i(i) \otimes \operatorname{Ext}^j_{\mathbb{P}^n}(\mathcal{T}, \mathcal{O}(-i)) \\ &= \begin{cases} \Omega^i(i) \otimes P_i & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

we see that the functor preserves the exactness of the Beilinson resolution and furthermore, transforms it to the exact sequence (4) since $p_* \mathcal{H}om(q^* \mathcal{T}, \mathcal{O}_\Delta) \simeq \mathcal{H}om(\mathcal{T}, \mathcal{O}) = \mathcal{U}$.

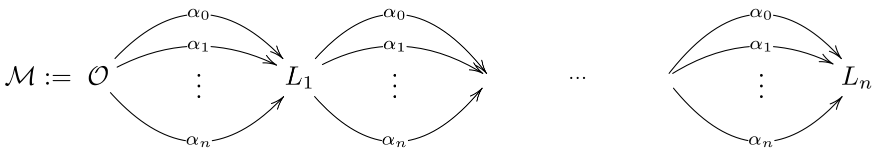
Since (4) is an exact complex of locally free sheaves, applying $\mathcal{H}om(-, \mathcal{O}(-1))$ to it gives the following exact sequence of left B_n-modules over $\mathbb{P}^n$

$$0 \longrightarrow \omega_{\mathbb{P}^n} \otimes_{\mathbb{P}^n} \mathcal{U} \longrightarrow I'_0 \otimes_k \Omega^n(n) \longrightarrow I'_1 \otimes_k \Omega^{n-1}(n-1) \longrightarrow \dots \longrightarrow I'_{n-1} \otimes_k \Omega^1(1) \longrightarrow I'_n \otimes_k \mathcal{O} \longrightarrow 0$$

where I'_i is the injective left B_n-module at vertex i. Note that $B_n \simeq B_n^{op}$ and that the isomorphism switches the idempotent at vertex i with that at vertex $n-i$. Using this isomorphism gives (5). $\square$

Serre stability of $\mathcal{U}$ is now easily verified. Applying $- \otimes_{B_n} DB_n$ to the resolution of $\mathcal{U}$ in (4) gives the coresolution of $\omega_{\mathbb{P}^n} \otimes_{\mathbb{P}^n} \mathcal{U}$ in (5). Hence $\nu_n \mathcal{U} \simeq \omega_{\mathbb{P}^n} \otimes_{\mathbb{P}^n} \mathcal{U}$ as desired.

We wish to show that any isomorphism $\mathcal{U} \xrightarrow{\sim} \omega_{\mathbb{P}^n}^{-1} \otimes_{\mathbb{P}^n} \nu_n \mathcal{U}$ defines the universal family on $\mathbb{X}^S$. To this end, consider a test scheme T and let $(\mathcal{M}, \theta) \in \mathbb{X}^S(T)$ be given by the family of B_n-modules



and the isomorphism $\theta\colon \mathcal{M} \xrightarrow{\sim} L \otimes_T \nu_n \mathcal{M}$ where L, L_i are invertible sheaves on T. Now $\mathcal{U}$ is the universal family on $\mathbb{X}^\rho_{\mathbf{1}}$ and $\mathcal{M}$ is ρ-stable by Proposition 6.1, so there is a unique map $f\colon Y \to \mathbb{P}^n$ such that $\mathcal{M} = f^*\mathcal{U}$. Hence it suffices to show that the T-point $(\mathcal{M}, \theta)$ is independent of θ. Suppose we vary θ by pre-composing with an automorphism $\psi = (\psi_0, \dots, \psi_n)$ where $\psi_i \in \operatorname{Aut} L_i = H^0(\mathcal{O}_T^\times)$. Now ρ-stability implies that the sections $\alpha_1, \dots, \alpha_n$ generate L_1, so we see in fact that we must have all the ψ_i are equal and an automorphism ψ of $\mathcal{M}$ is just given by an element of $H^0(\mathcal{O}_T^\times)$. Since we are working in the rigidified moduli stack, varying by ψ does not alter the T-point. $\square$

This example of the Beilinson algebra exhibits several phenomena arising in the theory of Serre stable moduli stacks that are worth emphasising. The first is that unlike in the theory of quiver GIT, we do not need to pick a separate stability condition, Serre stability already gives ρ-stability. Furthermore, we have

Proposition 6.4. $\vec{d} = \mathbf{1}$ *is the unique minimal Coxeter stable dimension vector.*

Proof. As in Proposition 5.1, one can readily check this from first principles. The easiest way to see the result is to use Beilinson's derived equivalence or more precisely, the ensuing isomorphism $K_0(B_n) \simeq K_0(\mathbb{P}^n)$. If $H^i \subset \mathbb{P}^n$ denotes a linear subspace of codimension i, then

$$[\mathcal{O}], [\mathcal{O}_H], [\mathcal{O}_{H^2}], \dots, [\mathcal{O}_{H^n}]$$

is a basis for $K_0(\mathbb{P}^n)$ and the (shifted) Coxeter transformation Φ sends $\mathcal{O}_{H^i}$ to

$$[\omega_{\mathbb{P}^n} \otimes_{\mathbb{P}^n} \mathcal{O}_{H^i}] = [\mathcal{O}_{H^i}] - (n+1)[\mathcal{O}_{H^{i+1}}].$$

Hence Φ, with respect to the basis above is a single Jordan block with eigenvalue 1. $\square$

The proposition shows that the choice of dimension vector $\mathbf{1}$ is also the only natural one for Serre stable moduli stacks. Hence, there is much less choice in the theory here.

The universal sheaf $\mathcal{U}$ on $\mathbb{X}^S$ is the same as for $\mathbb{X}^\rho_{\mathbf{1}}$, so its dual is a tilting bundle inducing Beilinson's equivalence $\mathsf{D}^b(\mathsf{coh}(\mathbb{P}^n)) \simeq \mathsf{D}^b(\mathsf{mod}\, B_n)$ [Be]. In particular, we see that the Serre stable moduli stack is useful even for algebras which are not derived equivalent to a weighted projective curve, although in this case, the shift parameter may be different from 1.

7. Serre functor for cyclic quotient stacks and orders

The Serre stable moduli stack can be defined in other settings too as long as there is a suitable notion of a Serre functor in families. For stacks and orders, this can be done using the notion of the canonical (bi)-module. In this section, we collect some basic facts about canonical sheaves and bimodules that we need for the rest of this paper. The Serre stability condition is most illuminating in the case of cyclic quotient stacks, so we examine it in this context.

Fix a cyclic group $G = \langle\sigma\rangle$ of order p. Let Y be a smooth (quasi-projective) variety. Consider a G-*cover* of Y, which for us will mean a cover of the form $\tilde{Y} = \underline{\mathrm{Spec}}_Y \tilde{\mathcal{O}}$ where

$$\tilde{\mathcal{O}} = \bigoplus_{\chi \in G^\vee} \mathcal{L}_\chi$$

and G acts on the line bundle $\mathcal{L}_\chi$ by the character χ of G (see [KM, pages 63-65] for details). We will say that $\tilde{Y}$ is *1-generated* if $\tilde{\mathcal{O}}$ is generated as an $\mathcal{O}_Y$-algebra by a single eigensheaf, say $\mathcal{L}_\chi$. In this case, we will also say $\tilde{Y}$ is χ-*generated*. The cyclic covering trick allows us to construct such a G-cover. Given any effective divisor D on Y and line bundle $\mathcal{L}_\chi$ with an isomorphism $m\colon \mathcal{L}_\chi^p \xrightarrow{\sim} \mathcal{O}_Y(-D)$, the algebra $\bigoplus_{i=0}^{p-1} \mathcal{L}_\chi^i$ defines a χ-generated G-cover if we use m to define the multiplication (see [KM, Definition 2.50]). This G-cover is étale on the complement of D. If D is smooth, then so is $\tilde{Y}$.

If $\mathbb{Y}$ is an s-dimensional smooth Deligne-Mumford stack of finite type, then it has a canonical sheaf $\omega_{\mathbb{Y}}$ and $\nu_s := \omega_{\mathbb{Y}} \otimes_{\mathbb{Y}} -$ is an auto-equivalence of $\mathsf{coh}(\mathbb{Y})$ which serves as our shifted Serre functor. To understand this functor, we specialise to the case where $\tilde{Y}$ is a smooth G-cover of a smooth variety Y. By [V, Example 7.21], the quasi-coherent sheaves on the stack $\mathbb{Y} = [\tilde{Y}/G]$ can be viewed as G-equivariant quasi-coherent sheaves on Y. These in turn are given by left modules over the skew group ring $\mathcal{A} := \mathcal{O}_{\tilde{Y}} \# G$, which is an *order* on Y, that is, a torsion-free coherent sheaf of algebras with $\mathcal{A} \otimes_{k(Y)} k(Y)$ a central simple $k(Y)$-algebra (see [AdJ] for further information). Indeed, the data required to give an $\mathcal{A}$-module structure is both an $\mathcal{O}_{\tilde{Y}}$-module together with a G-action, and compatiblity of these two corresponds to the skew relations in the skew group ring. We hence obtain an equivalence of categories $\mathsf{coh}(\mathbb{Y}) \simeq \mathcal{A} - \mathsf{mod}$. The canonical sheaf then corresponds to the G-equivariant sheaf $\omega_{\tilde{Y}}$ (see [V, page 667] for further details).

We wish to understand ν_s in this special context by considering the induced auto-equivalence of $\mathcal{A} - \mathsf{mod}$ which we shall also denote by ν_s. Let $\mathcal{O} = \mathcal{O}_Y, \tilde{\mathcal{O}} = \mathcal{O}_{\tilde{Y}}, \omega := \omega_Y$ and $\tilde{\omega} := \omega_{\tilde{Y}}$. Recall that the adjunction formula gives

$$\tilde{\omega} = \mathcal{H}om_Y(\tilde{\mathcal{O}}, \omega) \simeq \mathcal{H}om_Y(\tilde{\mathcal{O}}, \mathcal{O}) \otimes_Y \omega\,.$$

Consider the trace map $\mathrm{tr}\colon k(\tilde{Y}) \longrightarrow k(Y)$ where $k(Y), k(\tilde{Y})$ are the function fields of Y and $\tilde{Y}$. The trace pairing $k(\tilde{Y}) \times k(\tilde{Y}) \longrightarrow k(Y)\colon (a,b) \mapsto \mathrm{tr(ab)}$ is non-degenerate, so we may use it to identify the dual sheaf $\mathcal{H}om_Y(\tilde{\mathcal{O}}, \mathcal{O})$ with a subsheaf of $k(\tilde{Y})$ and hence $\tilde{\omega}$ with a subsheaf of $k(\tilde{Y}) \otimes_Y \omega$. Now $\mathcal{A}$ is an order in the matrix $k(Y)$-algebra $\mathcal{A} \otimes_Y k(Y)$. Following the custom in non-commutative algebraic geometry, we mimic the adjunction formula and define the *canonical* $\mathcal{A}$-*bimodule* to be

$$\omega_{\mathcal{A}} = \mathcal{H}om_Y(\mathcal{A}, \omega) \simeq \mathcal{H}om_Y(\mathcal{A}, \mathcal{O}) \otimes_Y \omega\,.$$

Then $\nu_s = \omega_{\mathcal{A}} \otimes_{\mathcal{A}} -\colon \mathcal{A} - \mathsf{mod} \longrightarrow \mathcal{A} - \mathsf{mod}$. This is a well-known fact which follows easily from Lemma 7.1(ii) below. As for $\tilde{\omega}$, we may use the (reduced) trace map $\mathrm{tr}\colon \mathcal{A} \otimes_Y k(Y) \to k(Y)$ (see [CR, Section 7D] for details) to identify $\mathrm{Hom}_Y(\mathcal{A}, \mathcal{O}_Y)$ with a sub-bimodule of $\mathcal{A} \otimes_Y k(Y)$. Now $\mathcal{A} \otimes_Y k(Y)$ naturally contains $k(\tilde{Y})$, so $\omega_{\mathcal{A}}$ naturally contains $\tilde{\omega}$ too.

Lemma 7.1. *Let $\tilde{Y}$ be a χ-generated G-cover of Y.*

(i) *Let $\tilde{\omega}_\chi$ be the eigensheaf on Y of $\tilde{\omega}$ corresponding to the character χ. Then multiplication induces an isomorphism $\tilde{\omega} \simeq \tilde{\omega}_\chi \otimes_Y \tilde{\mathcal{O}}$.*

(ii) *The bimodule multiplication map induces isomorphisms $\omega_{\mathcal{A}} \simeq \tilde{\omega} \otimes_{\tilde{Y}} \mathcal{A} \simeq \mathcal{A} \otimes_{\tilde{Y}} \tilde{\omega}$.*

Proof. We may work locally over Y and so assume $\tilde{\mathcal{O}} = \mathcal{O}[y]/(y^p - f) = \mathcal{O} \oplus \mathcal{O}y \cdots \oplus \mathcal{O}y^{p-1}$ where G acts on y via the character χ. It follows that

$$\mathcal{H}om_{\mathcal{O}}(\tilde{\mathcal{O}}, \mathcal{O}) = \mathcal{O} \oplus \mathcal{O}y^{-1} \oplus \cdots \oplus \mathcal{O}y^{1-p} = y^{1-p}\tilde{\mathcal{O}}.$$

Hence $\tilde{\omega} = y^{1-p}\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \omega$ and (i) follows since $\tilde{\omega}_\chi = y^{1-p}\,\omega$.

It remains only to show that $\omega_{\mathcal{A}} = y^{1-p}\,\mathcal{A} \otimes_{\mathcal{O}} \omega$. To verify this, we need a matrix embedding of $\mathcal{A}$ which can easily be obtained from a Peirce decomposition as follows. Recall $kG = \prod_{\mu \in G^\vee} k\varepsilon_\mu$ where ε_μ is the primitive idempotent corresponding to the character μ. Then $\mathcal{A} = \bigoplus_{\mu,\lambda} \varepsilon_\mu \mathcal{A} \varepsilon_\lambda$ and ordering the elements of $G^\vee$ appropriately, we obtain the following algebra homomorphism $\iota\colon \mathcal{A} \hookrightarrow M_p(\mathcal{O})$ which is compatible with the Peirce decomposition:

$$\mathcal{O} \ni t \mapsto \begin{pmatrix} t & 0 & \dots & 0 \\ 0 & t & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & t \end{pmatrix} \qquad \sigma \mapsto \begin{pmatrix} \zeta & 0 & \dots & 0 \\ 0 & \zeta^2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \zeta^p \end{pmatrix} \qquad y \mapsto \begin{pmatrix} 0 & 0 & \dots & \dots & f \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

The image of ι is easily seen to be

$$\begin{pmatrix} \mathcal{O} & (f) & \dots & (f) \\ \mathcal{O} & \mathcal{O} & & \vdots \\ \vdots & \vdots & \ddots & (f) \\ \mathcal{O} & \dots & \dots & \mathcal{O} \end{pmatrix} \tag{6}$$

The trace pairing is now easily computed, allowing us to identify

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}}(\mathcal{A}, \mathcal{O}) &= \{g \in M_p(k(\mathcal{O})) \mid \operatorname{tr}(g\,\mathcal{A}) = 0\} \\ &= \begin{pmatrix} \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ (f^{-1}) & \mathcal{O} & & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ (f^{-1}) & (f^{-1}) & \dots & \mathcal{O} \end{pmatrix} = \quad \mathcal{A} \begin{pmatrix} 0 & \dots & 0 & 1 \\ f^{-1} & 0 & \dots & 0 \\ 0 & f^{-1} & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \dots & f^{-1} & 0 \end{pmatrix} \\ &= \mathcal{A}\, y f^{-1} = \mathcal{A}\, y^{1-p} \end{aligned}$$

from which (ii) follows. ☐

Lemma 7.2. *Let $\tilde{Y}$ be a χ-generated G-cover of Y and $\mathbb{Y} = [\tilde{Y}/G]$ the corresponding quotient stack. Then for any $N \geq |G|$ the sheaf $\mathcal{E} = \oplus_{i=0}^{N-1} \omega_{\mathbb{Y}}^{\otimes i}$ is a generating sheaf for $\mathbb{Y}$. In particular, Y is quasi-projective.*

Proof. We use the pointwise criterion for generation given in [OS, Theorem 5.2]. On the scheme locus of $\mathbb{Y}$, $\mathcal{E}$ generates as it is just the locally free sheaf $\oplus \omega_Y^{\otimes i}$. Hence we may work locally around a stacky point and assume that $\tilde{\mathcal{O}} = \mathcal{O}[y]/(y^p - f)$ where $\mathcal{O}$ is a local ring with maximal ideal $\mathfrak{m} \ni f$. Let $\kappa = \tilde{\mathcal{O}}/(y, \mathfrak{m})$ denote the residue field at the unique point of $\operatorname{Spec} \tilde{\mathcal{O}}$ lying over the closed point of $\operatorname{Spec} \mathcal{O}$. Now by [OS, Theorem 5.2], we need only check that the G-module $\mathcal{E} \otimes_{\tilde{\mathcal{O}}} \kappa$ contains every irreducible representation of G. Now from the local computations in the proof of Lemma 7.1 we see that

$$\mathcal{E} \otimes_{\tilde{\mathcal{O}}} \kappa \simeq \kappa \oplus \kappa y \oplus \ldots \oplus \kappa y^{N-1}$$

where G acts trivially on κ and by the character χ on y. Since χ generates the character group of G, we are done. ☐

Recall that if Y is a smooth variety, then any skyscraper sheaf $k(q)$ is Serre stable in the sense that $\omega_Y \otimes_Y k(q) \simeq k(q)$. This in part motivated Bondal and Orlov's definition of point objects. For stacks, simple sheaves (that is, those with exactly two subsheaves) are not necessarily Serre stable.

Example 7.3 (Serre unstable simple sheaves on a smooth Deligne-Mumford stack.)**.** Let $Y = \operatorname{Spec}(\mathcal{O})$ be an affine curve and suppose $q \in Y$ is a closed point defined by the local equation $f = 0$ where $f \in \mathcal{O}$. We consider the cover $\tilde{Y} = \operatorname{Spec}(\tilde{\mathcal{O}})$ where $\tilde{\mathcal{O}} = \mathcal{O}[y]/(y^p - f)$ which is totally ramified over q and unramified elsewhere. Hence $\mathbb{Y} = [\tilde{Y}/G]$ is a smooth Deligne-Mumford stack with a single stacky point above q with inertia group $G = \mu_p$ and elsewhere is isomorphic to $Y - q$.

Let $\mu \in G^\vee$ and $\varepsilon_\mu \in kG$ be the corresponding primitive idempotent. Then $P_\mu = \mathcal{A}\varepsilon_\mu$ is an indecomposable $\mathcal{A}$-module corresponding to a "column" of the matrix form of $\mathcal{A}$ in (6) above. The P_μ are pairwise non-isomorphic, being the projective covers of non-isomorphic simple modules $S_\mu = P_\mu / y P_\mu$. Now P_μ is an $(\mathcal{A}, \mathcal{O})$-bimodule which is free as a right $\mathcal{O}$-module. We may thus view P_μ as a flat family of $\mathcal{A}$-modules over Y. Furthermore, away from q, the corresponding family of sheaves on $\mathbb{Y}$ is just the structure sheaf $\mathcal{O}_{\mathbb{Y}}$ on $\mathbb{Y}$. Hence, the P_μ give p flat families of $\mathcal{A}$-modules over Y which are all isomorphic away from q, but different at q. Informally speaking, this shows that the rigidified moduli stack $\mathbb{X}$ is non-separated above q and does not look like $\mathbb{Y}$.

We now fix the shift parameter $s = 1$ and see how these families are 1-Serre unstable. From our local computations in the proof of Lemma 7.1, we see

$$\nu_s P_\mu = \omega_{\mathcal{A}} \otimes_{\mathcal{A}} P_\mu \simeq y P_\mu.$$

Thus ν_s permutes the P_μ and S_μ cyclically. The instability of the P_μ is caused by the Serre unstable $\mathcal{A}$-module $P_\mu \otimes_{\mathcal{O}} k(q)$ which is a non-split extension of all the S_μ. As can be expected and will be seen later, in the Serre stable moduli stack, stable reduction will replace this unstable fibre with the Serre stable module $\bigoplus_{\mu \in G^\vee} S_\mu$. This stable reduction will require passing to the ramified G-cover, $\tilde{Y}$ of Y. This completes the example.

Notation 7.4. Let $\mathbb{Y}$ be a weighted projective curve with coarse moduli scheme Y and $q \in Y$ be a closed point. Locally in a neighbourhood of q, we may write $\mathbb{Y} = [\tilde{Y}/G]$ in the notation of Example 7.3. We let $k_{\mathbb{Y}}(q)$ denote the Serre stable sheaf

$$k_{\mathbb{Y}}(q) = \bigoplus_{\mu \in G^\vee} S_\mu$$

on $\mathbb{Y}$. It is the direct sum of p simple sheaves where p is the weight of $\mathbb{Y}$ at q.

8. The dual of the universal bundle is tilting

In this section, we apply the tilting theory of [BKR], to give a module+moduli-theoretic criterion for the dual of the universal sheaf on the Serre stable moduli stack to be tilting. This allows us to recover Geigle-Lenzing's derived equivalence [GL] for canonical algebras.

Let A be a finite dimensional algebra which is basic and *connected* in the sense that its quiver is connected. We fix the shift parameter $s = 1$ and a minimal Coxeter stable dimension vector $\vec{d}$ and form the corresponding Serre stable moduli stack $\mathbb{X}^S$ and universal sheaf $\mathcal{U}$. We assume that gl.dim $A < \infty$ so that A has a chance of being concealed-canonical.

We will unfortunately need to assume that we know $\mathbb{X}^S$ is a weighted projective curve. To check this abstractly, it suffices to show it is a smooth proper irreducible Deligne-Mumford stack of finite type over k. We saw in Theorem 5.4 that this moduli-theoretic condition holds for a canonical algebra A. For concealed-canonical algebras, this also should follow from general moduli principles, though we do not have the required stack technology at present to prove this. In this case, there is an obvious choice for $\vec{d}$, in the language of Lenzing- de la Peña [LdP], it is the unique minimal Coxeter stable dimension vector of rank zero. Given any k-point $M \xrightarrow{\sim} \nu_1 M$ of $\mathbb{X}^S$, we know $\mathrm{pd}\, M = 1$ so $\mathrm{Ext}^2_A(M, M) = 0$ (for example by the proof of Theorem 8.2 below). Hence at least the corresponding point of $\mathbb{X}$ is smooth. Generically M is regular simple by [LdP] so $\langle \vec{d}, \vec{d} \rangle = 0$ yields $\mathrm{Ext}^1_A(M, M) = k$. We see thus that $\mathbb{X}$ is 1-dimensional. Presumably, a closer analysis will show that the same results hold true for $\mathbb{X}^S$. What seems hard to prove is that some form of stable reduction holds and hence that $\mathbb{X}^S$ is proper. One of the goals of ongoing research is to replace our moduli-theoretic assumption here with a module-theoretic one, something which usually occurs in the tilting theory of [BKR].

Let C be the coarse moduli scheme of $\mathbb{X}^S$ and using Notation 7.4 we let

$$\mathcal{S} = \{k_{\mathbb{X}^S}(q) \mid \text{closed } q \in C\}.$$

Let $\mathcal{T} = \mathcal{U}^\vee$ which we can view as an $(A, \mathcal{O}_{\mathbb{X}^S})$-bimodule. We wish of course to show that the functor

$$F := \mathbf{R}\mathrm{Hom}_{\mathbb{X}^S}(\mathcal{T}, -) = \mathbf{R}\Gamma(- \otimes_{\mathbb{X}^S} \mathcal{U}) \colon \mathsf{D}^b(\mathbb{X}^S) \longrightarrow \mathsf{D}^b(A)$$

is an equivalence. If $\mathcal{M}$ is a coherent sheaf on $\mathbb{X}^S$ that is supported on a finite set, then we will abuse notation and write $F\mathcal{M} = \mathcal{M} \otimes_{\mathbb{X}^S} \mathcal{U}$.

It is instructive to describe explicitly the modules $Fk_{\mathbb{X}^S}(q)$. We may compute $Fk_{\mathbb{X}^S}(q)$ locally in a neighbourhood of q and so assume that $\mathbb{X}^S$ is the cyclic group quotient $[\tilde{Y}/G]$ in the notation of Example 7.3. In this language, $\mathcal{U}$ is a G-equivariant sheaf on $\tilde{Y}$. Let $\tilde{q}$ be the point of $\tilde{Y}$ lying above q which we assume to be fixed by G, as in Example 7.3. Then $k_{\mathbb{X}^S}(q) = \bigoplus_{\mu \in G^\vee} k(\tilde{q})_\mu$ where $k(\tilde{q})_\mu$ is the skyscraper sheaf at $\tilde{q}$ with G-action given by the character μ, and

$$Fk_{\mathbb{X}^S}(q) = \Big(\bigoplus_{\mu \in G^\vee} k(\tilde{q})_\mu \otimes_{\tilde{Y}} \mathcal{U} \Big)^G = k(\tilde{q}) \otimes_{\tilde{Y}} \mathcal{U}$$

where the superscript G denotes G-invariants. In particular we see that $Fk_{\mathbb{X}^S}(q)$ is a Serre stable A-module with dimension vector $\vec{d}$.

We wish to apply Bridegeland-King-Reid's general criterion for an exact functor to be an equivalence. In our case, the version we need is the following.

Lemma 8.1. *Suppose that A is a basic connected finite dimensional algebra and $\vec{d}$ a dimension vector such that $\mathbb{X}^S$ is a weighted projective curve. Then F is an equivalence of categories provided it induces isomorphisms*

$$F\colon \operatorname{Ext}^i_{\mathbb{X}^S}(k_{\mathbb{X}^S}(q), k_{\mathbb{X}^S}(q')) \longrightarrow \operatorname{Ext}^i_A(Fk_{\mathbb{X}^S}(q), Fk_{\mathbb{X}^S}(q'))$$

for all i and $q, q' \in C$.

Proof. This is just a special case of [BKR, Theorem 2.4] and we need only check that the hypotheses there hold. First note that $\mathcal{S} = \{k_{\mathbb{X}^S}(q) \mid \text{closed } q \in C\}$ is a spanning class for $\mathsf{D}^b(\mathbb{X}^S)$ as is easily seen by repeating the proof of [Br, Example 2.2]. Connectedness of A ensures indecomposability of $\mathsf{D}^b(A)$. Finally, recall from Notation 7.4 that the $k_{\mathbb{X}^S}(q)$ are Serre stable and so are the $Fk_{\mathbb{X}^S}(q)$ as we have just observed. Hence $F(\nu k_{\mathbb{X}^S}(q)) \simeq \nu(Fk_{\mathbb{X}^S}(q))$ where ν denotes the Serre functors on $\mathsf{D}^b(\mathbb{X}^S)$ and $\mathsf{D}^b(A)$. □

Theorem 8.2. *Let A be a basic connected finite dimensional algebra of finite global dimension and $\vec{d} \in K_0(A)$ be a minimal Coxeter stable dimension vector. Suppose that*

(i) *the Serre stable moduli stack $\mathbb{X}^S$ is a weighted projective curve, and*

(ii) *any Serre stable module M of dimension vector $\vec{d}$ is regular semisimple.*

Then $\mathbf{R}\mathrm{Hom}_{\mathbb{X}^S}(\mathcal{T}, -)\colon \mathsf{D}^b(\mathbb{X}^S) \longrightarrow \mathsf{D}^b(A)$ is an equivalence of categories, where $\mathcal{T}$ is the dual of the universal representation on $\mathbb{X}^S$.

Proof. We need only check the hypothesis of Lemma 8.1 concerning isomorphisms of Ext groups. First note that $\mathsf{coh}(\mathbb{X}^S)$ is hereditary and together with Notation 7.4 we find

$$\operatorname{Hom}_{\mathbb{X}^S}(k_{\mathbb{X}^S}(q), k_{\mathbb{X}^S}(q')) = D\operatorname{Ext}^1_{\mathbb{X}^S}(k_{\mathbb{X}^S}(q'), k_{\mathbb{X}^S}(q)) = \begin{cases} k^{p_q} & \text{if } q = q' \\ 0 & \text{else} \end{cases}$$

where p_q is the weight of $\mathbb{X}^S$ at q. Let $M = Fk_{\mathbb{X}^S}(q)$ and $M' = Fk_{\mathbb{X}^S}(q')$. By Serre duality and stability we know that $\operatorname{Ext}^i_A(M, A) = 0$ if $i \neq 1$. As gl.dim $A < \infty$ we see that the projective dimension of M is 1. In particular $\operatorname{Ext}^i_A(M, M') = 0$ if $i \geq 2$. If $q \neq q'$ then $M \not\simeq M'$ by Proposition 4.1 and hence $\operatorname{Hom}_A(M, M') = 0$ by Proposition 4.4 and our hypothesis (ii). Serre duality and stability then show $\operatorname{Ext}^1_A(M, M') = 0$. Next, we consider the case $q = q'$. We know from Proposition 4.1 that M is a direct sum of p_q regular simple modules and that these form a single ν_1-orbit, so say $M = \bigoplus_{i=0}^{p_q-1} \nu_1^i N$ for some regular simple N. In particular, $\operatorname{End}_A M$ is a p_q-dimensional vector space as is $\operatorname{Ext}^1_A(M, M)$ by Serre duality. Hence F certainly induces an isomorphism

$$F\colon \operatorname{Hom}_{\mathbb{X}^S}(k_{\mathbb{X}^S}(q), k_{\mathbb{X}^S}(q)) \longrightarrow \operatorname{Hom}_A(Fk_{\mathbb{X}^S}(q), Fk_{\mathbb{X}^S}(q))$$

We may work locally near q and so assume $\mathbb{X}^S = [\tilde{Y}/G]$ in the notation of Example 7.3 where G is cyclic of order p_q and $k_{\mathbb{X}^S}(q) = \bigoplus_{i=0}^{p_q-1} \nu_1^i k(\tilde{q})$ for $\tilde{q} \in \tilde{Y}$ the point lying over q. It remains only to show that F induces a non-zero map

$$F\colon \operatorname{Ext}^1_{\mathbb{X}^S}(\nu_1^{i+1}k(\tilde{q}), \nu_1^i k(\tilde{q})) \longrightarrow \operatorname{Ext}^1_A(F\nu_1^{i+1}k(\tilde{q}), F\nu_1^i k(\tilde{q}))$$

We compute this "Kodaira-Spencer" map explicitly by constructing $\mathcal{O}_{2\tilde{q}} \otimes_{\tilde{Y}} \mathcal{U}$ as follows. Serre duality gives a unique non-split extension

$$0 \longrightarrow \nu_1 N \longrightarrow E \longrightarrow N \longrightarrow 0.$$

Applying powers of ν_1 to this exact sequence and taking direct sums gives an extension

$$0 \longrightarrow M \longrightarrow \bar{\mathcal{U}} \longrightarrow M \longrightarrow 0 \tag{7}$$

where $\bar{\mathcal{U}} = \bigoplus_{i=0}^{p_q-1} \nu_1^i E$. This module is a Serre stable self-extension of M, so gives a flat family of Serre stable modules over the ring of dual numbers $k[\varepsilon]$. Furthermore, G-torsors of $\operatorname{Spec}(k[\varepsilon])$ are trivial so this $k[\varepsilon]$-point of $\mathbb{X}^S$ is given by a morphism $\phi\colon \operatorname{Spec}(k[\varepsilon]) \longrightarrow \tilde{Y}$. Now $\bar{\mathcal{U}}$ is non-split, so ϕ is not constant and so must give an isomorphism of $k[\varepsilon]$ with $\mathcal{O}_{2\tilde{q}}$. Hence we may assume that $\mathcal{O}_{2\tilde{q}} \otimes_{\tilde{Y}} \mathcal{U} \simeq \bar{\mathcal{U}}$. Consider now a non-split extension

$$0 \longrightarrow \nu_1^{i+1}k(\tilde{q}) \longrightarrow \mathcal{M}_i \longrightarrow \nu_1^i k(\tilde{q}) \longrightarrow 0$$

which represents a non-zero element of $\operatorname{Ext}^1_{\mathbb{X}^S}(\nu_1^{i+1}k(\tilde{q}), \nu_1^i k(\tilde{q}))$. Its image under F is the extension

$$0 \longrightarrow \nu_1^{i+1}k(\tilde{q}) \otimes_{\mathbb{X}^S} \bar{\mathcal{U}} \longrightarrow \mathcal{M}_i \otimes_{\mathbb{X}^S} \bar{\mathcal{U}} \longrightarrow \nu_1^i k(\tilde{q}) \otimes_{\mathbb{X}^S} \bar{\mathcal{U}} \longrightarrow 0$$

which is non-split since it must be one of the p_q direct summands of (7). □

We immediately arrive at an independent proof of a slight refinement of Geigle-Lenzing's derived equivalence [GL].

Corollary 8.3. *Let A be a canonical algebra, $\mathbb{X}^S_{\mathbf{1},1}$ be the Serre stable moduli stack and $\mathcal{U}$ be the universal sheaf. Then $\mathcal{T} = \mathcal{U}^\vee$ is tilting and induces a derived equivalence between the weighted projective line $\mathbb{X}^S_{\mathbf{1},1}$ and A.*

9. The tautological moduli problem

In this section, we introduce *smoothly weighted varieties* which generalise weighted projective curves. Such stacks can be recovered as tautological Serre stable moduli stacks of sheaves. This explains why the Serre stable moduli stack is useful, whenever we have an abelian category derived equivalent to such a stack.

Definition 9.1. A stack $\mathbb{Y}$ is called a *smoothly weighted (projective) variety* if there is a morphism $\pi\colon \mathbb{Y} \longrightarrow Y$ to a (projective) variety Y such that every point $q \in Y$ has an open neighbourhood U and the restricted map $\pi_{\pi^{-1}(U)}\colon \pi^{-1}(U) \longrightarrow U$ has the form $[\tilde{U}/G] \longrightarrow U$ for some 1-generated cyclic cover $\tilde{U} \to U$ (defined in Section 7) with Galois group G and $\tilde{U}$ smooth.

Note smoothness of $\tilde{U}$ implies smoothness of the ramification locus. Also, π is an isomorphism away from the ramification loci of the covers $\tilde{U} \to U$. The coarse moduli scheme of $\mathbb{Y}$ is Y. By construction, weighted projective curves are examples of smoothly weighted projective varieties. The simplest examples of Geigle-Lenzing projective spaces [HIMO] as described below are another.

Example 9.2 (Geigle-Lenzing projective space)**.** Consider the polynomial algebra $R = k[x_0, \dots, x_n]$ graded by $\deg x_0 = 1$ but $\deg x_i = p$ for $i > 0$ and some integer $p > 1$. Let $\mathbb{Y}$ be the projective stack associated to R which can be described as follows. The $\mathbb{Z}$-grading on R amounts to an action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$ and we define the *GL-projective space weighted on a hyperplane with weight p* to be $\mathbb{Y} = [(\mathbb{A}^n - 0)/\mathbb{G}_m]$. There is a map to the scheme-theoretic $\mathrm{Proj}(R) \simeq \mathrm{Proj}(k[t_0 = x_0^p, x_1, \dots, x_n]) = \mathbb{P}^n$. As in the appendix, we have a local description of this map $\mathbb{Y} \longrightarrow \mathbb{P}^n$. Above the affine patch $t_0 \neq 0$, the stack $\mathbb{Y}$ is just $\mathrm{Spec}(R/(x_0 - 1)) \simeq \mathbb{A}^n$. Above the affine patch $x_i \neq 0, i > 0$, it is $[\mathrm{Spec}\,(R/(x_i - 1))\,/\mu_p]$, so the inertia groups along $t_0 = 0$ are μ_p and $\mathbb{Y}$ is a smoothly weighted projective variety.

When studying moduli of A-modules, we fixed a discrete invariant which was an element of $K_0(A)$. In the theory of moduli of sheaves, $K_0(\mathbb{Y})$ may no longer be discrete. If $\mathbb{Y}$ is a projective scheme, then we have a well-defined Euler pairing $\chi\colon K_0(\mathbb{Y}) \times K_0(\mathbb{Y}) \longrightarrow \mathbb{Z}$ and it is more natural in this context to associate to a coherent sheaf $\mathcal{M}$ the invariant $\chi(-, \mathcal{M}) \in \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathbb{Y}), \mathbb{Z})$. In fact, one usually can get away with considering the Hilbert polynomial which is just the restriction of $\chi(-, \mathcal{M})$ to $\{\mathcal{O}(n) \mid n \in \mathbb{Z}\}$ for some choice of polarisation $\mathcal{O}(1)$.

To cater for not necessarily projective stacks, we proceed as follows. Note first that $\mathsf{coh}(\mathbb{Y}) \simeq \mathcal{A} - \mathsf{mod}$ for some order $\mathcal{A}$ on Y. There is quite some choice in the order $\mathcal{A}$ but in our case, there is a natural one. Recall from Lemma 7.2, that $\mathcal{E} = \oplus_{i=0}^N \omega_{\mathbb{Y}}^{\otimes i}$ is a generating sheaf for N sufficiently large so $\mathbb{Y}$ is quasi-projective. Thus if $c\colon \mathbb{Y} \longrightarrow Y$ is the map to the coarse moduli scheme, we may take $\mathcal{A} = c_* \,\mathcal{E}nd_{\mathbb{Y}}\, \mathcal{E}$. Recall that we defined the support of a coherent sheaf $\mathcal{M}$ on $\mathbb{Y}$ to be the support of $c_* \,\mathcal{H}om_{\mathbb{Y}}(\mathcal{E}, \mathcal{M})$ in Y.

Definition 9.3. The *moduli stack $\tilde{\mathbb{W}}$ of skyscraper sheaves* is defined as follows. Given a test scheme T, the category of T-points $\tilde{\mathbb{W}}(T)$ consists of sheaves $\mathcal{M}$ on $\mathbb{Y} \times T$ which are flat over T and furthermore satisfy i) Supp $\mathcal{M}$ is finite over T and ii) $c_* \,\mathcal{H}om_{\mathbb{Y}}(\omega_{\mathbb{Y}}^{\otimes i}, \mathcal{M})$ is locally free of rank 1 over T for all $i \in \mathbb{Z}$. As in Subsection 2.3, we will rigidify this stack to obtain the rigidified moduli stack $\mathbb{W}$.

It is instructive to look on an open patch of the form $\mathbb{U} = [\tilde{U}/G] \subseteq \mathbb{Y}$ and see what condition ii) above corresponds to. In this case, we may identify $\mathsf{coh}(\mathbb{U})$ with $\mathcal{A} - \mathsf{mod}$ where now $\mathcal{A} = \mathcal{O}_{\tilde{U}} \# G$ instead. Let $\mathcal{M} \in \tilde{\mathbb{W}}(\mathrm{Spec}\, k)$ so its support as a sheaf on $\tilde{U}$ is some finite closed G-invariant subscheme $Z \subset \tilde{U}$ by condition i). Local computations such as in the proof of Lemma 7.1 show that $\omega_{\tilde{U}}|_Z \simeq (\mathcal{O}_{\tilde{U}} \otimes \chi)|_Z$ where χ is an appropriate generator of the character group of G. Hence condition ii) requires

$$c_* \,\mathcal{H}om_{\mathbb{Y}}(\omega_{\mathbb{Y}}^{\otimes i}, \mathcal{M}) \simeq (\mathcal{M} \otimes \chi^{-i})^G$$

to be 1-dimensional. This module is of course just the χ^i isotypic component of the G-module $\mathcal{M}$. Working more generally over arbitrary T, we see that condition ii) means that $\varepsilon_\mu \mathcal{M}$ is locally free of rank 1 over T, for all $\mu \in G^\vee$ where $\varepsilon_\mu \in kG$ is the primitive idempotent corresponding to μ.

We fix the shift parameter to be $s = \dim \mathbb{Y}$ which in our case is just $\dim Y$. Recall from Section 7 that we have a (shifted) Serre functor $\nu_s := \omega_{\mathbb{Y}} \otimes_{\mathbb{Y}} -$. This definition naturally generalises to families $\mathcal{M} \in \mathbb{W}(R)$: $\nu_s \mathcal{M} = \omega_{\mathbb{Y}} \otimes_{\mathbb{Y}} \mathcal{M}$ defines a flat family of skyscraper sheaves on $\mathbb{Y}$. Since $\mathbb{Y}$ is smooth, ν_s induces an automorphism $\nu_s \colon \mathbb{W} \xrightarrow{\sim} \mathbb{W}$ and we do not need to worry about its domain of definition as occurs for moduli of A-modules. We can now define Serre stable sheaves as before, as well as the Serre stable moduli stack $\mathbb{W}^S$. More precisely, $\mathbb{W}^S$ is the stackification of the pre-stack $\mathbb{W}^{S,pre}$ whose objects over a test scheme T are flat families $\mathcal{M}$ of skyscraper sheaves on $\mathbb{Y}$ over T, together with an isomorphism $\nu_s \mathcal{M} \xrightarrow{\sim} \mathcal{M} \otimes_T \mathcal{N}$ such that $\mathcal{M}, \mathcal{N}$ are line bundles over T.

Before proving the next result, it will be useful to keep in mind that if $\mathbb{Y} = Y$ is actually a variety, then $Y \simeq \mathbb{W}$ and the isomorphism is given by the universal skyscraper sheaf $\mathcal{O}_\Delta \in \mathsf{coh}(Y \times Y)$ where $\Delta \subset Y \times Y$ is the diagonal copy of Y. If $Y = \operatorname{Spec}(\mathcal{O})$ is affine, then the inverse map Ψ is easily defined as follows. Consider a flat family $\mathcal{M} \in \mathbb{W}(R)$ over R which we view as an $(\mathcal{O}, R)$-bimodule which is locally free of rank 1 over R. Left multiplication induces a ring homomorphism $\mathcal{O} \longrightarrow \operatorname{End}_R \mathcal{M} = R$ and hence an R-point $\operatorname{Spec}(R) \longrightarrow Y$ which we define to be $\Psi(\mathcal{M})$. The proof below generalises this argument.

Lemma 9.4. *Let $\tilde{U} \to U$ be a χ-generated G-cover of a smooth affine variety ramified over a smooth divisor and $\mathbb{Y} = [\tilde{U}/G]$ be the corresponding quotient stack. Then we have an isomorphism $\mathbb{W}^S \simeq \mathbb{Y}$.*

Proof. We fix the usual notation: $U = \operatorname{Spec}(\mathcal{O}), \tilde{U} = \operatorname{Spec}(\tilde{\mathcal{O}})$ and $\mathcal{A} = \tilde{\mathcal{O}} \# G$ is the skew group ring, so there is a category equivalence $\mathsf{coh}(\mathbb{Y}) \simeq \mathcal{A} - \mathsf{mod}$. We will view a flat family of $\mathcal{A}$-modules over a commutative ring R as an $(\mathcal{A}, R)$-bimodule to help us keep track of scalars. In particular, if $\varepsilon_\mu \in kG$ is the primitive idempotent corresponding to $\mu \in G^\vee = \langle \chi \rangle$ and $\mathcal{M} \in \mathbb{W}(R)$, then $\mathcal{M}$ is a left $\mathcal{A}$-module such that $\varepsilon_\mu \mathcal{M}$ is a locally free right R-module of rank 1.

We define a morphism of stacks $\Phi \colon \mathbb{Y} \to \mathbb{W}^S$ by defining the universal family. First note that $\mathcal{U} =_{\mathcal{A}} \mathcal{A}_{\tilde{\mathcal{O}}}$ is a family of left $\mathcal{A}$-modules over $\tilde{\mathcal{O}}$ and furthermore, $\varepsilon_\mu \mathcal{U} \simeq \tilde{\mathcal{O}}$ as a right $\tilde{\mathcal{O}}$-module. This flat family is also G-equivariant since $\mathcal{U}$ is an $(\mathcal{A}, \tilde{\mathcal{O}} \# G)$-bimodule. Hence $\mathcal{U}$ defines a G-equivariant element of $\mathbb{W}(\tilde{\mathcal{O}})$, that is a flat family of $\mathcal{A}$-modules over $\mathbb{Y}$. It corresponds to $\mathcal{O}_\Delta$ above. Note $\mathcal{U}$ is also Serre stable. Indeed we see from Lemma 7.1 that there is a natural isomorphism of $(\mathcal{A}, \tilde{\mathcal{O}})$-bimodules

$$\omega_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{U} = \omega_{\mathcal{A}} \simeq \mathcal{A} \otimes_{\tilde{\mathcal{O}}} \tilde{\omega} = \mathcal{U} \otimes_{\tilde{\mathcal{O}}} \tilde{\omega} \tag{8}$$

We have thus defined an element of $\mathbb{W}^S(\tilde{\mathcal{O}})$. Moreover, all data are G-equivariant so we do indeed have a morphism $\mathbb{Y} = [\operatorname{Spec}(\tilde{\mathcal{O}})/G] \to \mathbb{W}^S$.

We now construct the inverse functor Ψ to Φ and by stackification, it suffices to define $\Psi \colon \mathbb{W}^{S,pre} \to \mathbb{Y}$. Consider an object of $\mathbb{W}^{S,pre}(R)$ given by an $(\mathcal{A}, R)$-bimodule ${}_{\mathcal{A}} \mathcal{M}_R$ and $(\mathcal{A}, R)$-bimodule isomorphism $\theta \colon \omega_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{M} \xrightarrow{\sim} \mathcal{M} \otimes_R \mathcal{N}$ where $\mathcal{N}$ is an invertible R-module. Since $\mathcal{A} = \tilde{\mathcal{O}} \# G$, we may view the $\mathcal{A}$-module structure of $\mathcal{M}$ as a $G^\vee$-grading

$$\mathcal{M} = \bigoplus_{\lambda \in G^\vee} \varepsilon_\lambda \mathcal{M}$$

together with a compatible action of $\tilde{\mathcal{O}}$, that is, a $G^\vee$-graded algebra homomorphism $\tilde{\mathcal{O}} \longrightarrow \operatorname{End}_R \mathcal{M}$. This follows from the more precise formula in $\tilde{\mathcal{O}} \# G$: given a degree λ element $t \in \tilde{\mathcal{O}}$ and $\mu \in G^\vee$ we have $t\varepsilon_\mu = \varepsilon_{\mu+\lambda} t$.

The existence of the isomorphism θ imposes the following structure on $\mathcal{M}$.

Sublemma 9.5. (i) *There is a homomorphism $\rho \colon \mathcal{O} \longrightarrow R$ such that left multiplication by $t \in \mathcal{O}$ on $\mathcal{M}$ is right multiplication by $\rho(t)$.*

(ii) *There is an invertible R-module $\mathcal{L}$ such that θ induces an isomorphism $\theta_\mu \colon \varepsilon_{\mu+\chi} \mathcal{M} \xrightarrow{\sim} \varepsilon_\mu \mathcal{M} \otimes_R \mathcal{L}$.*

(iii) *θ also induces an isomorphism $\mathcal{L}^p \simeq R$ where $p = |G|$.*

Proof. Let $\tilde{\omega}_\chi$ be the eigensheaf of $\tilde{\omega}$ corresponding to the character χ so that by Lemma 7.1, θ can be viewed as an isomorphism $\theta \colon \tilde{\omega}_\chi \otimes_{\mathcal{O}} \mathcal{M} \xrightarrow{\sim} \mathcal{M} \otimes_R \mathcal{N}$. Multiplying by ε_μ and re-arranging gives an isomorphism

$$\varepsilon_{\mu+\chi} \mathcal{M} \simeq \tilde{\omega}_\chi^{-1} \otimes_{\mathcal{O}} \varepsilon_\mu \mathcal{M} \otimes_R \mathcal{N}\,. \tag{9}$$

For each $\mu \in G^\vee$, left multiplication induces a ring homomorphism $\rho_\mu \colon \mathcal{O} \longrightarrow \operatorname{End}_R \varepsilon_\mu \mathcal{M} = R$. Since (9) is an $(\mathcal{O}, R)$-bimodule isomorphism, we see ρ_μ is independent of the choice of μ and part i) follows. We may thus rewrite (9) as

$$\varepsilon_{\mu+\chi} \mathcal{M} \simeq \varepsilon_\mu \mathcal{M} \otimes_R \mathcal{N} \otimes_R \mathcal{P}.$$

where $\mathcal{P} = \tilde{\omega}_\chi^{-1} \otimes_{\mathcal{O}} R$. Now χ has order p, so the invertible module $\mathcal{N} \otimes_R \mathcal{P}$ is p-torsion. Setting $\mathcal{L} = \mathcal{N} \otimes_R \mathcal{P}$ finishes the proof of the sublemma. □

We wish now to define $\Psi(\mathcal{M})$ which consists of an étale G-cover $\tilde{R}$ of R and a G-equivariant homomorphism $\tilde{\mathcal{O}} \longrightarrow \tilde{R}$. As one might suspect, $\tilde{R}$ will be given by the p-torsion line bundle $\mathcal{L}$ in the sublemma, but to make this functorial, we proceed as follows. Note first that by Sublemma 9.5i), any right R-linear endomorphism of $\mathcal{M}$ is automatically left $\mathcal{O}$-linear too. We let $\tilde{R}$ be the R-subalgebra of $\operatorname{End}_R \mathcal{M}$ consisting of endomorphisms ϕ compatible with θ, that is, $(\phi \otimes \mathrm{id}_{\mathcal{N}})\theta = \theta(\mathrm{id}_{\tilde{\omega}_\chi} \otimes \phi)$ as morphisms $\tilde{\omega}_\chi \otimes_{\mathcal{O}} \mathcal{M} \longrightarrow \mathcal{M} \otimes_R \mathcal{N}$. Note first that the construction of $\tilde{R}$ does not depend on the choice of isomorphism θ in its equivalence class. Now $\tilde{\omega}$ is an $\tilde{\mathcal{O}}$-central bimodule containing $\tilde{\omega}_\chi$, so left multiplication by elements of $\tilde{\mathcal{O}}$ is certainly compatible with θ. We thus obtain a $G^\vee$-graded homomorphism $\tilde{\mathcal{O}} \longrightarrow \tilde{R}$ or equivalently, a G-equivariant homomorphism.

We now show that $\tilde{R}$ is isomorphic to the commutative $G^\vee$-graded R-algebra $\bigoplus_{i=0}^{p-1} \mathcal{L}^i$ and so is indeed an étale G-cover of R, thus completing the definition of Ψ. First note that a degree λ endomorphism $\phi \in \operatorname{End}_R \mathcal{M}$ is compatible with θ if the following diagram commutes for all $\mu \in G^\vee$

$$\begin{array}{ccc} \varepsilon_{\mu+\chi}\,\mathcal{M} & \xrightarrow{\theta_\mu} & \varepsilon_\mu\,\mathcal{M} \otimes_R \mathcal{L} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi\otimes\mathrm{id}} \\ \varepsilon_{\mu+\lambda+\chi}\,\mathcal{M} & \xrightarrow{\theta_{\mu+\lambda}} & \varepsilon_{\mu+\lambda}\,\mathcal{M} \otimes_R \mathcal{L} \end{array}$$

We now easily see that multiplication by elements of $\mathcal{L}$ via θ_μ^{-1} are endomorphisms of $\mathcal{M}$ compatible with θ. There is hence a $G^\vee$-graded R-algebra homomorphism $\bigoplus_{i=0}^{p-1} \mathcal{L}^i \longrightarrow \tilde{R}$. This homomorphism is clearly injective and is also surjective since the commutative diagram above shows that any homogeneous $r \in \tilde{R}$ is completely determined by its restriction to $\varepsilon_0\,\mathcal{M}$ which is an element of $\operatorname{Hom}_R(\varepsilon_0\,\mathcal{M}, \varepsilon_{i\chi}\,\mathcal{M}) \simeq \mathcal{L}^i$ where $\deg r = i\chi$. This completes the definition of $\Psi(\mathcal{M})$ which is clearly seen to be functorial in $\mathcal{M}$.

The construction of $\Psi(\mathcal{M})$ is completely reversible, so Ψ does give an equivalence of stacks. We will illustrate this for the "tautological point" $\pi \colon \tilde{U} \longrightarrow \mathbb{Y}$ and so see that the inverse of Ψ is indeed the morphism $\Phi \colon \mathbb{Y} \longrightarrow \mathbb{W}^S$ induced by the universal family $\mathcal{U} =_{\mathcal{A}} \mathcal{A}_{\tilde{\mathcal{O}}}$ described above.

To this end, first note that the tautological point is given by the trivial G-torsor $pr \colon G \times \tilde{U} \longrightarrow \tilde{U}$ and G-action $\alpha \colon G \times \tilde{U} \longrightarrow \tilde{U}$. We determine $\mathcal{M} := \Psi^{-1}(\pi)$. Now the trivial G-torsor corresponds to the torsion line bundle $\mathcal{L} = \tilde{\mathcal{O}}$, so we have $\mathcal{N} \otimes_{\mathcal{O}} \tilde{\omega}_\chi^{-1} = \tilde{\mathcal{O}}$ or equivalently by Lemma 7.1, $\mathcal{N} = \tilde{\omega}_\chi \otimes_{\mathcal{O}} \tilde{\mathcal{O}} = \tilde{\omega}$. Since we are working in the rigidified moduli stack $\mathbb{W}$, we may assume that $\varepsilon_0\,\mathcal{M} = \tilde{\mathcal{O}}$. Reversing equation (9), we define $\varepsilon_{i\chi}\,\mathcal{M} = \tilde{\omega}_\chi^{-i} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \otimes_{\tilde{\mathcal{O}}} \tilde{\omega}^i \simeq \tilde{\mathcal{O}}$. Setting $\mathcal{M} = \bigoplus_i \varepsilon_{i\chi}\,\mathcal{M} = \bigoplus_{\mu \in \mathrm{G}^\vee} \varepsilon_\mu\,\tilde{\mathcal{O}}$, we may sum the isomorphisms in (9) to get an isomorphism $\theta \colon \tilde{\omega}_\chi \otimes_{\mathcal{O}} \mathcal{M} \simeq \mathcal{M} \otimes_{\tilde{\mathcal{O}}} \tilde{\omega}$. Note that $\mathcal{M}$ is defined as a $G^\vee$-graded right $\tilde{\mathcal{O}}$-module or equivalently, a $(kG, \tilde{\mathcal{O}})$-bimodule which is naturally isomorphic to $\mathcal{U} = \tilde{\mathcal{O}} \# G$. To complete the left $\mathcal{A}$-module structure, we need a compatible left $\tilde{\mathcal{O}}$-module structure which is provided by the $G^\vee$-graded homomorphism $\alpha^* \colon \tilde{\mathcal{O}} \longrightarrow \mathcal{O}_{G \times \tilde{U}} = \tilde{\mathcal{O}}\,G^\vee$ where $\tilde{\mathcal{O}}\,G^\vee = \bigoplus_{\mu \in G^\vee} \tilde{\mathcal{O}}\,u_\mu$ is the usual group ring on the dual group $G^\vee$ (so $u_\mu u_\lambda = u_{\mu+\lambda}$). The endomorphisms of $\mathcal{M}$ which are compatible with θ can be identified with $\tilde{\mathcal{O}}\,G^\vee$ and under this identification, u_μ permutes the graded components by $\varepsilon_\lambda \mapsto \varepsilon_{\mu+\lambda}$. We view $\tilde{\mathcal{O}}$ as a kG-module, so for $f \in \tilde{\mathcal{O}}$ and writing "." for scalar multiplication to avoid confusion, we have $\varepsilon_\mu.f$ is the projection of f onto the μ-eigensheaf of $\tilde{\mathcal{O}}$. In this language, $\alpha^* f = \sum_{\mu \in G^\vee} (\varepsilon_\mu.f) u_\mu$. It follows that if f is homogeneous of degree μ, then $\alpha^* f = f u_\mu$ so left multiplication by f induces the map $\varepsilon_\lambda\,\tilde{\mathcal{O}} \longrightarrow \varepsilon_{\mu+\lambda}\,\mathcal{M} \colon \varepsilon_\lambda g \mapsto \varepsilon_{\mu+\lambda} g f$. Hence $\mathcal{M} \simeq \mathcal{U}$ as $(\mathcal{A}, \tilde{\mathcal{O}})$-bimodules. This completes the proof that $\Psi^{-1} = \Phi$. □

Theorem 9.6. *Let $\mathbb{Y}$ be a smoothly weighted variety. Then there is a natural isomorphism $\mathbb{W}^S \simeq \mathbb{Y}$.*

Proof. Let $\Phi \colon \mathbb{Y} \longrightarrow \mathbb{W}^S$ be the morphism defined by the universal family $\theta \colon \omega_{\mathbb{Y}} \otimes_{\mathbb{Y}} \mathcal{O}_\Delta \xrightarrow{\sim} \mathcal{O}_\Delta \otimes_{\mathbb{Y}} \omega_{\mathbb{Y}}$. We show that this restricts to the isomorphism in Lemma 9.4 on any open substack $[\tilde{U}/G]$ of $\mathbb{Y}$. Pulling back via the natural map $f \colon \tilde{U} \times \tilde{U} \longrightarrow \mathbb{Y} \times \mathbb{Y}$, we find that $f^*\theta$ is an isomorphism of $G \times G$-equivariant sheaves. This is precisely the universal family in (8) so we are done. □

Suppose Y is the coarse moduli scheme of the smoothly weighted variety $\mathbb{Y}$. If $\mathbb{Y}$ is weighted along some divisor $D \subset Y$ with weight $p > 1$, then every point $y \in D$ corresponds to a k-point of $\mathbb{W}^S$ of the form $\mathcal{F} \oplus \nu_s \mathcal{F} \oplus \ldots \oplus \nu_s^{p-1} \mathcal{F}$ for some simple sheaf $\mathcal{F}$. Viewing this as a point of $\mathbb{W}$, we see its rigidified automorphism group in $\mathbb{W}$ is $\mathbb{G}_m^{p-1}$, so there is no chance that $\mathbb{W}$ recovers the stack $\mathbb{Y}$. Similarly, if in our definition of smoothly weighted varieties, we had allowed weighting along intersecting divisors, one would find that the rigidified automorphism groups in $\mathbb{W}^S$ are sometimes infinite, so $\mathbb{W}^S$ will not recover $\mathbb{Y}$ in these more general cases. We suspect that there are "higher" versions of the Serre stable moduli stack which will work, perhaps involving the cotangent bundle instead of just the canonical bundle.

10. The dual of the tilting bundle is universal

Let $\mathbb{Y}$ be a smoothly weighted projective variety of dimension s and $\mathcal{T} = \bigoplus_{v\in Q_0} \mathcal{T}_v$ be a tilting bundle on $\mathbb{Y}$ where the $\mathcal{T}_v$ are the indecomposable summands. We will assume $\mathcal{T}$ is *basic* whereby we mean that the summands $\mathcal{T}_v$ are pairwise non-isomorphic so the endomorphism algebra $A = \mathrm{End}_{\mathbb{Y}} \mathcal{T}$ is also basic and the quiver corresponding to A has vertex set Q_0. We consider the dimension vector $\vec{d}\colon Q_0 \longrightarrow \mathbb{N}, v \mapsto \text{rank}\, \mathcal{T}_v$ so that the generic skyscraper sheaf on $\mathbb{Y}$ corresponds to an A-module of dimension vector $\vec{d}$. We may construct the rigidified moduli stack $\mathbb{X}$ of A-modules with dimension vector $\vec{d}$. We fix the shift parameter to be s. The aim of this section is to prove that, at least in the (anti-)Fano case defined below, the Serre stable moduli stack $\mathbb{X}^S$ is isomorphic to $\mathbb{Y}$ and that the universal module is given by $\mathcal{T}^\vee$.

We first recall from [K2, Section 5] how to calculate the derived tensor product $\mathcal{F}\otimes^{\mathbf{L}}_A M$ of an $(\mathcal{O}_{\mathbb{Y}}, A)$-bimodule $\mathcal{F}$ with a left A-module M. We can take a projective A-bimodule resolution $P_\bullet$ of A where each P_j is a finite direct sum of bimodules of the form $Ae_v \otimes_k e_w A$ for various $v, w \in Q_0$. Then

$$\mathcal{F} \otimes^{\mathbf{L}}_A M = \mathcal{F} \otimes_A P_\bullet \otimes_A M. \tag{10}$$

Indeed, $\mathcal{F} \otimes_A Ae_v \otimes_k e_w A = \mathcal{F}_v \otimes_k e_w A$, so $\mathcal{F} \otimes_A P_\bullet$ can be taken to be a projective resolution of $\mathcal{F}$ as an A-module. Of course, a similar statement can be made about $P_\bullet \otimes_A M$. We let $\mathsf{D}^+(\mathbb{Y}, A)$ denote the bounded below derived category of quasi-coherent $(\mathcal{O}_{\mathbb{Y}}, A)$-bimodules. Below, we use the derived global sections functor $\mathbf{R}\Gamma\colon \mathsf{D}^+(\mathbb{Y}) \longrightarrow \mathsf{D}^+(k)$ on $\mathbb{Y}$. We need a preliminary lemma.

Lemma 10.1. *Let $\mathcal{F} \in \mathsf{D}^+(\mathbb{Y}, A)$ and $M \in \mathsf{D}^b_{fg}(A^{op})$. Then*

$$\mathbf{R}\Gamma(\mathcal{F}) \otimes^{\mathbf{L}}_A M \simeq \mathbf{R}\Gamma(\mathcal{F} \otimes^{\mathbf{L}}_A M).$$

Proof. We may replace $\mathcal{F}$ with a bounded below complex of injective $(\mathcal{O}_{\mathbb{Y}}, A)$-bimodules and M with a bounded complex of finite projective left A-modules. Since taking global sections of an $(\mathcal{O}_{\mathbb{Y}}, A)$-bimodule commutes with $-\otimes_A Ae_v$, the lemma follows. □

We now show that $\mathcal{T}, \mathcal{T}^\vee$ are Serre stable left (respectively right) A-modules.

Proposition 10.2. *There are natural isomorphisms of bimodules*

$$DA[-1] \otimes^{\mathbf{L}}_A \mathcal{T} \simeq \mathcal{T} \otimes_{\mathbb{Y}} \omega_{\mathbb{Y}}, \quad \mathcal{T}^\vee \otimes^{\mathbf{L}}_A DA[-1] \simeq \omega_{\mathbb{Y}} \otimes_{\mathbb{Y}} \mathcal{T}^\vee.$$

In particular, $\mathcal{T}^\vee$ is Serre stable.

Proof. Note that $-\otimes^{\mathbf{L}}_A \mathcal{T}\colon \mathsf{D}^b(A) \to \mathsf{D}^b(\mathbb{Y})$ is an equivalence so commutes with the Serre functor. Thus

$$-\otimes^{\mathbf{L}}_A DA[-1] \otimes^{\mathbf{L}}_A \mathcal{T} \simeq -\otimes^{\mathbf{L}}_A \mathcal{T} \otimes_{\mathbb{Y}} \omega_{\mathbb{Y}}$$

from which the first isomorphism follows. Similarly, $\mathbf{R}\mathrm{Hom}_{\mathbb{Y}}(\mathcal{T}, -)$ commutes with the Serre functor and, using Lemma 10.1 we see there are natural isomorphisms

$$\mathbf{R}\Gamma(-\otimes^{\mathbf{L}}_{\mathbb{Y}} \omega_{\mathbb{Y}} \otimes^{\mathbf{L}}_{\mathbb{Y}} \mathcal{T}^\vee) \simeq \mathbf{R}\Gamma(-\otimes^{\mathbf{L}}_{\mathbb{Y}} \mathcal{T}^\vee) \otimes^{\mathbf{L}}_A DA[-1] \simeq \mathbf{R}\Gamma(-\otimes^{\mathbf{L}}_{\mathbb{Y}} \mathcal{T}^\vee \otimes^{\mathbf{L}}_A DA[-1])$$

Applying this functor to $\mathcal{O}_U$ where $U \subset \mathbb{Y}$ varies over open subsets of $\mathbb{Y}$ shows that there is an isomorphism of $(\mathcal{O}_{\mathbb{Y}}, A)$-bimodules $\omega_{\mathbb{Y}} \otimes^{\mathbf{L}}_{\mathbb{Y}} \mathcal{T}^\vee \simeq \mathcal{T}^\vee \otimes^{\mathbf{L}}_A DA[-1]$. □

We wish to show that the derived equivalence $-\otimes^{\mathbf{L}}_A \mathcal{T}$ sends flat families of Serre stable A-modules of dimension vector $\vec{d}$ to flat families of skyscraper sheaves on $\mathbb{Y}$. This will allow us to use Theorem 9.6. Our first task is to show that $M \otimes^{\mathbf{L}}_A \mathcal{T}$ is a sheaf for any k-point M of $\mathbb{X}^S$. As in [BO1], the theory of point objects is most useful when $\mathbb{Y}$ is *(anti-)Fano* whereby we mean that $\omega_{\mathbb{Y}}$ (respectively $\omega_{\mathbb{Y}}^{-1}$) is ample in the sense of [AZ, (4.2.1)]. Recall that $\omega_{\mathbb{Y}}$ is *ample* if any coherent sheaf $\mathcal{F}$ on $\mathbb{Y}$ satisfies i) $\mathcal{F}$ is the epimorphic image of a sheaf of the form $\oplus_{l=1}^r \omega_{\mathbb{Y}}^{\otimes -i_l}$ for some integers $i_1, \dots, i_r$ and ii), $H^j(\mathbb{Y}, \mathcal{F} \otimes_{\mathbb{Y}} \omega^i_{\mathbb{Y}}) = 0$ for all $i \gg 0, j > 0$.

Below, we let $c\colon \mathbb{Y} \longrightarrow Y$ be natural map to the coarse moduli space. Recall that since $\mathbb{Y}$ is smoothly weighted, we defined the support of a coherent sheaf $\mathcal{F}$ on $\mathbb{Y}$ to be $\cup_{i\in\mathbb{Z}} Z_i$ where Z_i is the support of $c_*(\omega_{\mathbb{Y}}^{\otimes -i} \otimes_{\mathbb{Y}} \mathcal{F})$ in Y.

Lemma 10.3. *Suppose that $\mathbb{Y}$ is Fano or anti-Fano. Then $M \otimes^{\mathbf{L}}_A \mathcal{T}$ is a finite length sheaf for any k-point M of $\mathbb{X}^S$.*

Proof. Note that A has finite global dimension, so Serre stability of M means that $\operatorname{Ext}^i_A(M, A) = D\operatorname{Ext}^{s-i}_A(A, M)$ and hence $\operatorname{pd}_A M = s$. Choosing a length s projective resolution for M, we may assume that $F^\bullet = M \otimes^{\mathbf{L}}_A \mathcal{T}$ is a length s complex

$$0 \longrightarrow F^{-s} \xrightarrow{\delta^{-s+1}} F^{-s+1} \xrightarrow{\delta^{-s+2}} \dots \xrightarrow{\delta^0} F^0 \longrightarrow 0$$

of locally free sheaves on $\mathbb{Y}$. From Proposition 10.2, its cohomologies h^{-j} are Serre stable and hence, as shown in the proof of [BO1, Proposition 2.2], have finite support since we are assuming that $\mathbb{Y}$ is either Fano or anti-Fano. We will show by downward induction on j that $h^{-j} = 0$ for $j > 0$. Since the cohomologies are finite length sheaves, we may work locally on Y, in which case the sheaves F^{-j} can be viewed as $\mathcal{A}$-modules where $\mathcal{A}$ is an $\mathcal{O}_Y$-order as in Section 7 and $\mathcal{O}_Y$ is a local ring. Assuming cohomologies vanish in degrees $< -j$, then the $\mathcal{O}_Y$-module $C^{-j} = \operatorname{coker}\left(\delta^{-j} \colon F^{-j-1} \longrightarrow F^{-j}\right)$ has projective dimension $\leq s - j$. By the Auslander-Buchsbaum formula, $\operatorname{depth}_{\mathcal{O}_Y} C^{-j} \geq j$, so C^{-j} has no non-zero finite length submodules unless $j = 0$. □

Lemma 10.4. *Suppose that $\mathbb{Y}$ is a weighted projective line and $\vec{d}$ is minimal Coxeter stable. Then $M \otimes^{\mathbf{L}}_A \mathcal{T}$ is a finite length sheaf for any k-point M of $\mathbb{X}^S$.*

Proof. In this case, $\operatorname{pd} M = 1$ and $\mathsf{coh}(\mathbb{Y})$ is hereditary, so $M \otimes^{\mathbf{L}}_A \mathcal{T} \simeq F_1[1] \oplus F_0$ for Serre stable sheaves F_1, F_0. Hence M is the direct sum of the Serre stable modules $\operatorname{Hom}_{\mathbb{Y}}(\mathcal{T}, F_0)$ and $\operatorname{Hom}_{\mathbb{Y}}(\mathcal{T}, F_1[1])$. Minimality of $\vec{d}$ then ensures that either $F_0 = 0$ or $F_1 = 0$. Now rank considerations show that for $F_1[1] \oplus F_0$ to have the same class in $K_0(\mathbb{Y})$ as a skyscraper sheaf, we must have $F_1 = 0$ and F_0 is a finite length sheaf. □

Proposition 10.5. *Suppose that $M \otimes^{\mathbf{L}}_A \mathcal{T}$ is a finite length sheaf for any k-point M of $\mathbb{X}^S$. Let $\mathcal{M}$ be a flat family of Serre stable A-modules over a noetherian ring R with dimension vector $\vec{d}$. Then $\mathcal{M} \otimes^{\mathbf{L}}_A \mathcal{T}$ is a flat family of $\mathcal{O}_{\mathbb{Y}}$-modules over R.*

Proof. By definition of Serre stability, there is an isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{L} \otimes_R \nu_s \mathcal{M}$ for some line bundle $\mathcal{L}$ on $\operatorname{Spec}(R)$. We may assume that R is a local ring with residue field κ. We let $K_\bullet = \mathcal{M} \otimes^{\mathbf{L}}_A \mathcal{T}$ and use the hypertor spectral sequence for $\kappa \otimes^{\mathbf{L}}_R K_\bullet$. Note that since $\mathcal{M}$ is locally free over R we have $\kappa \otimes^{\mathbf{L}}_R K_\bullet = \kappa \otimes_R K_\bullet$ and by assumption $h_i(\kappa \otimes_R K_\bullet) = 0$ if $i \neq 0$. The homology spectral sequence thus becomes

$$E^2_{i,j} = \operatorname{Tor}^R_i(\kappa, h_j(K_\bullet)) \Rightarrow h_{i+j}(\kappa \otimes_R K_\bullet)$$

with the E^∞ term on the right hand side vanishing when $i + j \neq 0$. Let $Z \subset Y_R$ be the support of $\mathcal{M} \otimes_A \mathcal{T}$. The composite $Z \hookrightarrow Y_R \longrightarrow \operatorname{Spec}(R)$ is both quasi-finite and projective, so by Zariski's main theorem, the morphism is finite. Hence $\mathcal{M} \otimes_A \mathcal{T} = h_0(K_\bullet)$ is a finitely generated R-module. From the spectral sequence we see that $\operatorname{Tor}^R_1(\kappa, h_0(K_\bullet)) = 0$ and so from the local criterion for flatness we deduce that $h_0(K_\bullet) = \mathcal{M} \otimes_A \mathcal{T}$ is flat over R. Hence $\operatorname{Tor}^R_i(\kappa, h_0(K_\bullet)) = 0$ for all $i > 0$. Now, the spectral sequence shows that $\operatorname{Tor}^R_0(\kappa, h_1(K_\bullet)) = \kappa \otimes_R h_1(K_\bullet) = 0$. However, just as in the h_0 case $h_1(K_\bullet)$ is a finitely generated R-module and so $h_1(K_\bullet) = 0$ and in particular $E^2_{i,1} = 0$ for all i. Continuing by induction we see that $h_i(K_\bullet) = 0$ for all $i \neq 0$.

□

Theorem 10.6. *Suppose $\mathbb{Y}$ is either i) Fano or anti-Fano or, ii) that it is a weighted projective line and $\vec{d}$ is minimal Coxeter stable. Then the flat family of Serre stable A-modules $\mathcal{T}^\vee \simeq \omega_{\mathbb{Y}}^{-1} \otimes_{\mathbb{Y}} \nu_s \mathcal{T}^\vee$ defines an isomorphism $\mathbb{Y} \to \mathbb{X}^S$ of stacks.*

Remark Together with Theorem 8.2 and [LdP, Section 2, (S6)(iii)], part (i) gives the characterisation of non-tubular concealed-canonical algebras stated in Theorem 1.3.

Proof. We must show that $\mathcal{T}^\vee \simeq \omega_{\mathbb{Y}}^{-1} \otimes_{\mathbb{Y}} \nu_s \mathcal{T}^\vee$ gives the universal family. We consider a flat family of A-modules $\mathcal{M}$ over R with dimension vector $\vec{d}$ and an isomorphism $\theta \colon \mathcal{M} \to \mathcal{L} \otimes_R \nu_s \mathcal{M}$. From Proposition 10.5, we know that $\mathcal{M} \otimes^{\mathbf{L}}_A \mathcal{T} = \mathcal{M} \otimes_A \mathcal{T}$ is a flat family of $\mathcal{O}_{\mathbb{Y}}$-modules. Furthermore, θ and the natural isomorphism of Proposition 10.2 give isomorphisms

$$\mathcal{M} \otimes_A \mathcal{T} \simeq \mathcal{L} \otimes_R \mathcal{M} \otimes^{\mathbf{L}}_A DA[-1] \otimes^{\mathbf{L}}_A \mathcal{T} \simeq \mathcal{L} \otimes_R \mathcal{M} \otimes^{\mathbf{L}}_A \mathcal{T} \otimes_{\mathbb{Y}} \omega_{\mathbb{Y}}. \tag{11}$$

In particular $\mathcal{M} \otimes_A \mathcal{T}$ is Serre stable and Theorem 9.6 shows that there is a morphism $f\colon \operatorname{Spec}(R) \longrightarrow \mathbb{Y}$ such that $\mathcal{M} \otimes_A \mathcal{T} \simeq f^* \mathcal{O}_\Delta$ where $\Delta \subset \mathbb{Y} \times \mathbb{Y}$ is the diagonal. In fact the isomorphism above is obtained from pulling back $\mathcal{O}_\Delta \simeq \omega_{\mathbb{Y}}^{-1} \otimes_{\mathbb{Y}} \mathcal{O}_\Delta \otimes_A \omega_{\mathbb{Y}}$ via f where here, we view $\mathcal{O}_\Delta$ as a $(\mathcal{O}_{\mathbb{Y}}, \mathcal{O}_{\mathbb{Y}})$-bimodule and the pullback when viewed as a sheaf on $\mathbb{Y} \times \mathbb{Y}$ is via $f \times 1$. Since $\mathcal{T}$ is a tilting bundle we have

$$\begin{aligned} \mathcal{M} &\simeq \mathcal{M} \otimes_A^{\mathbf{L}} \mathbf{R}\Gamma(\mathcal{T} \otimes_{\mathbb{Y}} \mathcal{T}^\vee) \\ &\simeq \mathbf{R}\Gamma(\mathcal{M} \otimes_A \mathcal{T} \otimes_{\mathbb{Y}} \mathcal{T}^\vee) \\ &\simeq \mathbf{R}\Gamma(f^* \mathcal{O}_\Delta \otimes_{\mathbb{Y}} \mathcal{T}^\vee) \\ &\simeq \mathbf{R}\Gamma(f^* \mathcal{T}^\vee) \\ &\simeq f^* \mathcal{T}^\vee \end{aligned}$$

where the second isomorphism follows from Lemma 10.1. This calculation can be used to show that applying $\mathbf{R}\Gamma(- \otimes_{\mathbb{Y}} \mathcal{T}^\vee)$ to the isomorphism in (11) recovers θ and that $\mathcal{T}^\vee \simeq \omega^{-1} \otimes_{\mathbb{Y}} \nu_1 \mathcal{T}^\vee$ is indeed universal. □

Recall from Proposition 5.1 that the theorem applies in the case where $\mathbb{Y}$ is a weighted projective line and $\mathcal{T}$ is the canonical tilting bundle whose endomorphism ring is the canonical algebra. It also applies to the n-canonical algebras (see [HIMO, Section 6]) for a GL-projective space weighted on a hyperplane as defined in Example 9.2. We suspect that the hypotheses of the theorem can be weakened significantly since i) and ii) above are quite independent. In particular, we hope the theorem holds true for all concealed-canonical algebras.

11. Appendix: Classical approach to weighted projective lines

In this appendix, we clarify the relationship between the Geigle-Lenzing description of weighted projective lines in [GL] and ours. The material is implicit in [GL].

Let G be a commutative reductive algebraic group so $G \simeq \mathbb{G}_m^r \times H$ for some finite abelian group H. Let $\Gamma = G^\vee$ be the dual (or character) group, which is a finitely generated abelian group of rank r. We first recall that to give a rational action of G on an affine scheme $\operatorname{Spec}(R)$ amounts to imposing a Γ-grading on R. Given $\gamma \in \Gamma$, G acts on the γ-graded component R_γ via the character γ. Suppose we are indeed given such a grading $R = \bigoplus_\gamma R_\gamma$. In this language, a G-equivariant quasi-coherent sheaf on $\operatorname{Spec}(R)$ corresponds to a Γ-graded R-module. More precisely, we have a category equivalence $\mathsf{qcoh}[\operatorname{Spec}(R)/G] \simeq \mathsf{Gr}_\Gamma R$.

We now restrict to the case where Γ has rank 1, though the ideas here apply in general. In this case, there are precisely two surjective group homomorphisms $\nu\colon \Gamma \longrightarrow \mathbb{Z}$. We will assume that the Γ-graded k-algebra R is *connected* in the sense that $R_0 = k$ and we can choose ν so that for any $0 \neq \gamma \in \Gamma$ with $R_\gamma \neq 0$, we have $\nu(\gamma) > 0$. Connectivity ensures that $\mathfrak{m} := \bigoplus_{\gamma \neq 0} R_\gamma$ is a Γ-graded ideal in R and we let $\mathrm{pt} \in \operatorname{Spec}(R)$ denote that corresponding closed point. Since pt is fixed by G, G acts on the open set $U = \operatorname{Spec}(R) - \mathrm{pt}$, and we may consider the stacky projective scheme $\operatorname{StProj}(R) = [U/G]$. By [V, Example 7.21], we know that there is a category equivalence $\mathsf{qcoh}\operatorname{StProj}(R) \simeq (\mathsf{Gr}_\Gamma R)/\mathrm{tors}$ where tors is the Serre subcategory of $\mathfrak{m}$-torsion modules. Note that the category of quasi-coherent sheaves on a weighted projective line as described in [GL] has the form $(\mathsf{Gr}_\Gamma R)/\mathrm{tors}$ for an appropriate choice of 2-dimensional graded ring R. Hence we may identify the weighted projective lines of Geigle-Lenzing with the associated stack $\operatorname{StProj}(R)$.

We now analyse the stack $\mathbb{X} = \operatorname{StProj}(R)$ in analogy with the standard construction of projective schemes by patching affine open sets. This will connect the Geigle-Lenzing approach with the one given in Subsection 2.2. For each non-zero homogeneous element $t \in R_\gamma, \gamma \neq 0$, the set $U_t = \operatorname{Spec}(R[t^{-1}])$ is a G-invariant open subset of U and hence $[U_t/G]$ is an open substack of $\mathbb{X}$. These cover $\mathbb{X}$ and $[U_t/G]$ has a coarse moduli scheme $\operatorname{Spec}(R[t^{-1}])^G = \operatorname{Spec}(R[t^{-1}]_0)$. Hence $\mathbb{X}$ has a coarse moduli scheme which is the usual scheme-theoretic $\operatorname{Proj}(R)$.

The open substack $[U_t/G]$ has a simpler description which makes it obvious that it is a Deligne-Mumford stack and what the inertia groups are. We change the presentation for the stack by replacing U_t with the closed subscheme $\bar{U}_t = \operatorname{Spec}(R[t^{-1}]/(t-1)) = \operatorname{Spec}(R/(t-1))$. First consider the exact sequence defining $\bar{\Gamma}$ below

$$0 \longrightarrow \mathbb{Z}\gamma \longrightarrow \Gamma \longrightarrow \bar{\Gamma} \longrightarrow 0$$

and the corresponding dual exact sequence

$$1 \longrightarrow G' \longrightarrow G \xrightarrow{\gamma} \mathbb{G}_m \longrightarrow 1$$

where $G' \leq G$ is the subgroup isomorphic to $\bar{\Gamma}^\vee$ and we have identified $(\mathbb{Z}\gamma)^\vee$ with $\mathbb{G}_m$. This sequence shows that G' fixes $t-1$ so acts on $\bar{U}_t$. We leave the reader to verify that $[U_t/G] \simeq [\bar{U}_t/G']$, contenting ourselves with heuristics (which form a proof when U_t is a variety) and a proof that they have equivalent categories of quasi-coherent sheaves (Proposition 11.1 below). Note that G' is finite, so $\mathbb{X}$ is indeed Deligne-Mumford by [BCEFFK, Example 5.17]. We first show that $\bar{U}_t$ meets every G-orbit. Indeed, if $x \in U_t$, then $t(x) = \beta \in k$ is non-zero and there is some $g \in G$ such that $t(g.x) = 1$, just pick g so that $\gamma(g) = \beta^{-1}$. Let now $x, y \in \bar{U}_t$ lie in the same G-orbit so say $y = g.x$. We need to show that $g \in G'$. If this is not the case then $\gamma(g) \neq 1$ so $t(y) = \gamma(g)t(x) \neq 1$ so $y \notin \bar{U}_t$, a contradiction. This completes the heuristics.

Finally, we elucidate the induced category equivalence $\mathsf{qcoh}[U_t/G] \simeq \mathsf{qcoh}[\bar{U}_t/G']$ which amounts to a category equivalence $\mathsf{Gr}_\Gamma R[t^{-1}] \simeq \mathsf{Gr}_{\bar{\Gamma}} R/(t-1)$ where $R[t^{-1}]$ is Γ-graded in the obvious way and $R/(t-1)$ is $\bar{\Gamma}$-graded as in [Sm, p.104]. We now generalise a theorem of Smith-Zhang which can be found in [Sm, Proposition 2.4]. Note that there is a surjective ring homomorphism $\phi\colon R[t^{-1}] \longrightarrow R/(t-1)$.

Proposition 11.1. *The natural morphism* $\iota\colon [\bar{U}_t/G'] \longrightarrow [U_t/G]$ *induces the category equivalence* $\mathsf{Gr}_\Gamma R[t^{-1}] \simeq \mathsf{Gr}_{\bar{\Gamma}} R/(t-1)$ *given by* $\iota^* = R/(t-1) \otimes_{R[t^{-1}]} (-)$.

Proof. The induced functor ι^* is given by $R/(t-1) \otimes_{R[t^{-1}]} (-)$ since it comes from the closed imbedding $\bar{U}_t \longrightarrow U_t$. We wish to define the inverse equivalence Φ so consider $M \in \mathsf{Gr}_{\bar{\Gamma}} R/(t-1)$. We need to convert the additive notation of Γ to multiplicative, so introduce the "placeholder" notation $u^\delta, \delta \in \Gamma$ with the understanding that $u^\delta u^\varepsilon = u^{\delta+\varepsilon}, \delta, \varepsilon \in \Gamma$. We define the "unwrap" module $M[u] \in \mathsf{Gr}_\Gamma R[t^{-1}]$ by

$$M[u]_\delta = M_{\bar{\delta}} u^\delta$$

where $\bar{\delta}$ denote the image of δ in $\bar{\Gamma}$. Given homogeneous elements $a \in R[t^{-1}]_\varepsilon, mu^\delta \in M[u]_\delta$, we define $amu^\delta = \phi(a)mu^{\varepsilon+\delta}$. This construction is clearly functorial and gives the sought for inverse equivalence Φ to ι^*. □

Example 11.2. Let $\Gamma = (\mathbb{Z}\gamma_0 + \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2)/(p_0\gamma_0 = p_1\gamma_1 = p_2\gamma_2)$ and $R = k[x_0, x_1, x_2]/(x_2^{p_2} + \lambda x_0^{p_0} - x_1^{p_1})$ be the Γ-graded algebra with $\deg x_i = \gamma_i$. According to [GL], this gives the weighted projective line, weighted at the 3 points $0, \infty, \lambda$ with weights p_0, p_1, p_2. Now $\mathbb{X} = [(\mathrm{Spec}(R) - \mathrm{pt})\,/G]$ is covered by the open patches $x_0 \neq 0, x_1 \neq 0$. Write $t_0 = x_0^{p_0}, t_1 = x_1^{p_1}$. Note that we have $R[x_0^{-1}]_0 = k[t_1/t_0]$ and $R[x_1^{-1}]_0 = k[t_0/t_1]$ so patching the affine lines together gives $\mathbb{P}^1$ as the coarse moduli space. Let us examine the patch $x_0 \neq 0$ more closely. $R/(x_0 - 1) \simeq k[t_1, x_1, x_2]/(x_1^{p_1} - t_1, x_2^{p_2} - (t_1 - \lambda))$ which is the $\mu_{p_1} \times \mu_{p_2}$-cover of $\mathbb{A}_{t_1}$ ramified at $t_1 = 0, t_1 = \lambda$ with ramification indices p_1, p_2 respectively. Of course, $\mu_{p_1} \times \mu_{p_2} = (\bar{\Gamma})^\vee = (\Gamma/\mathbb{Z}\gamma_0)^\vee$, so the open substack we get here is $\left[\mathrm{Spec}\left(\frac{R}{(x_0-1)}\right)/\mu_{p_1} \times \mu_{p_2}\right]$. This is just the orbifold stack on $\mathbb{A}^1_{t_1}$ with stacky points at $t_1 = 0, \lambda$ and inertia groups μ_{p_1}, μ_{p_2} respectively.

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