Solutions

MATH2601: Class Test 1, Version A

① i) F, $\sigma^2 = [3 \;\; 1 \;\; 2] \neq [1 \;\; 2 \;\; 3]$

ii) T. From Cramer's rule we know

$$A^{-1}\begin{pmatrix}-8\\10\\4\end{pmatrix} = \frac{1}{2}(\operatorname{adj} A)\begin{pmatrix}-8\\10\\4\end{pmatrix} = (\operatorname{adj} A)\begin{pmatrix}-4\\5\\2\end{pmatrix}$$

which has integer entries since $\operatorname{adj} A$ does

iii) F. $\{x^2+3x+1, \; x^2+2x, \; x+1\}$ is not a basis

since $x^2+3x+1 = (x^2+2x)+(x+1)$.

iv) $V^+ \cap W$ contains the non-zero matrix

$$\begin{pmatrix}1 & 2 & -1\\0 & 1 & 1\\0 & 0 & 2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2 & 0 & 0\\4 & 2 & 0\\-2 & 4 & 4\end{pmatrix} = \begin{pmatrix}2 & 2 & -1\\2 & 2 & 1\\-1 & 2 & 4\end{pmatrix}$$

② (i) $P = \underset{\sim}{u}\,\underset{\sim}{u}^T$ where $\underset{\sim}{u} = \dfrac{\binom{1}{2}}{\left\|\binom{1}{2}\right\|} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$

$$\therefore \; P = \frac{1}{5}\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}.$$

(ii) Since P is a projection, $P^2 = \text{id}$ and

$(\text{id}-P)^2 = \text{id} - \text{id}P - P\text{id} + P^2 = \text{id} - 2P + P = \text{id}-$

③. i) $$T = \begin{pmatrix} ev_0 \\ ev_0 \circ \frac{d}{dx} \\ ev_0 \circ \frac{d^2}{dx^2} \end{pmatrix}$$ which is a matrix of linear maps since the entries are composites of the linear maps ev_0 & $\frac{d}{dx}$.

ii) Let $f = a + bx + cx^2 \in \mathbb{R}[x]_{\le 2}$. Then

$$Tf = \begin{pmatrix} a \\ b \\ 2c \end{pmatrix}.$$ Hence, for each $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathbb{R}^3$ there exists a unique solution to $Tf = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ namely,

$$T^{-1}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = f = \alpha + \beta x + \tfrac{1}{2}\gamma x^2.$$

iii) The representing matrix is

$$(T^{-1}\underset{\sim}{e}_1 \quad T^{-1}\underset{\sim}{e}_2 \quad T^{-1}\underset{\sim}{e}_3) = (1 \quad x \quad \tfrac{1}{2}x^2).$$

④ Note $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = T \circ (1 \quad x \quad x^2)$

$$\mathbb{R}^3 \xrightarrow{(1 \ x \ x^2)} \mathbb{R}[x]_{\le 2} \xrightarrow{T} \mathbb{R}^2$$

i) $$T(3-2x^2) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}(1 \quad x \quad x^2)^{-1}(3-2x^2)$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}.$$

ii) $\ker A = \ker \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \mathbb{R}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ which has basis $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\ker T$ has basis $\left\{ (1 \quad x \quad x^2)\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = -1 - x + x^2 \right\}$ & co-ord system $(-1 - x + x^2)$

5. i) Let B be a spanning set for V and $w \in W$. We show w is a linear combination of elements of $T(B)$. Since T is surjective, we can find $v \in V$ with $Tv = w$. Now B spans V so

$$v = \lambda_1 v_1 + \dots + \lambda_n v_n$$ for some scalars λ_i and $v_1, \dots, v_n \in B$.

Hence
$$
\begin{aligned}
w = Tv &= T(\lambda_1 v_1 + \dots + \lambda_n v_n) \\
&= \lambda_1 Tv_1 + \dots + \lambda_n Tv_n \quad \text{as } T \text{ is line} \\
&\in \operatorname{Span} T(B) \quad \text{as each } Tv_i \in T(B).
\end{aligned}
$$

This shows $T(B)$ spans W.

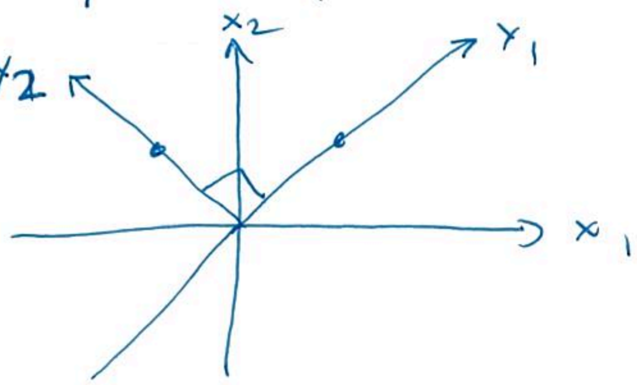
ii) False. Let V be a non-zero vector space with a non-zero vector $v \in V$. Let $T: V \to 0$ be the zero map which is surjective. Then $B = \{v\}$ is lin. indep but $T(B) = 0$ is not.

Solutions

MATH 2601 : Class Test 1: Version B

① i) F. $\sigma = (1\ 2)$ which is odd.

ii) T. $C : \mathbb{R}^2_Y \longrightarrow \mathbb{R}^2_X$ is a co-ord system



obtained by rotating the standard one by $\tan^{-1}\left(\frac{3}{4}\right)$. The matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

represents reflection about the y_2-axis above.

iii) T. $\ker T = \left\{ \begin{pmatrix} v \\ -v \end{pmatrix} \middle| \ v \in W \cap W' \right\}$.

We have an isomorphism $\begin{pmatrix} id \\ -id \end{pmatrix} : W \cap W' \longrightarrow \ker T$

iv) F. $\{2x+4, x-3, x-5\}$ is not a basis as $x^2 \notin \text{Span}(2x+4, x-3, x-5)$.

②. Let $a+bx \in \mathbb{R}[x]_{\leq 1}$. We solve for $\alpha, \beta \in \mathbb{R}$

$$\frac{d}{dx} = \alpha\, ev_1 + \beta\, ev_2.$$

Evaluating both sides at $a+bx$ gives

$$b = \alpha(a+b) + \beta(a+2b)$$

Solving gives $\alpha = -1, \beta = 1$ so $\frac{d}{dx} = ev_2 - ev_1$

③. (i). Any $v \in W \cap W'$ has form

$$v = \alpha(1+x^3) + \beta(x+x^3) = \gamma(1+x+x^2+x^3) + \delta(3+x^2+$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

We solve using 1st yr Gaussian elimination

$$\begin{array}{cccc} \alpha & \beta & -\gamma & -\delta \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & \cancel{1} & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The only condition on γ, δ for a solution is $-\gamma - \delta = 0$ i.e. $\gamma = -\delta$.

Hence

$$W \cap W' = \{\delta(3+x^2+2x^3) - \delta(1+x+x^2+x^3) \mid \delta \in \mathbb{R}\}$$

$$= \mathbb{R}(2 - x + x^3)$$

(ii) $W \cap W' \neq 0$ so the sum is not direct.

④ ⓘ $T = (x^2-2)\frac{d}{dx} - 3x$ is linear being the linear combⁿ of the linear map left multⁿ by x and the composite of linear maps $\frac{d}{dx}$ and left multⁿ by x^2-2.

ii). The representing matrix is

$$A = (1\ x\ x^2)\circ T \circ (1\ x) : \mathbb{R}^2 \xrightarrow{(1\ x)} \mathbb{R}[x]_{\leq 1} \xrightarrow{T} \mathbb{R}[x]_{\leq 2} \xrightarrow{(1\ x\ x^2)^{-1}} \mathbb{R}^3$$

which has first column

$$(1\ x\ x^2)^{-1}\circ T\circ(1\ x)\begin{pmatrix}1\\0\end{pmatrix} = (1\ x\ x^2)^{-1}\, T\, \cancel{3x}\,(1)$$

$$= (1\ x\ x^2)^{-1}(-3x)$$

$$= \begin{pmatrix}0\\-3\\0\end{pmatrix}.$$

and 2nd column

$$(1\ x\ x^2)^{-1}\circ T\circ(1\ x)\begin{pmatrix}0\\1\end{pmatrix} = (1\ x\ x^2)^{-1}\, T(x)$$

$$= (1\ x\ x^2)^{-1}(-2x^2-2)$$

$$= \begin{pmatrix}-2\\0\\-2\end{pmatrix}$$

Hence $A = \begin{pmatrix}0 & -2\\-3 & 0\\0 & -2\end{pmatrix}$

5) ($\Rightarrow$). Suppose first $B \cup \{v\}$ is linearly indep. If $v \in \mathrm{Span}(B)$ then $v = \lambda_1 v_1 + \dots + \lambda_n v_n$ for some scalars $\lambda_1, \dots, \lambda_n$. Hence

$$-v + \lambda_1 v_1 + \dots + \lambda_n v_n = 0$$

is a non-trivial linear comb^{n} giving the zero vector. Hence $B \cup \{v\}$ is linearly dependent a contradiction so $v \notin \mathrm{Span}(B)$.

($\Leftarrow$). Suppose now $v \notin \mathrm{Span}(B)$. Consider scalars $\lambda, \lambda_1, \dots, \lambda_n$ such that

$$\lambda v + \lambda_1 v_1 + \dots + \lambda_n v_n = 0.$$

We need to show all scalars λ, λ_i are 0. If $\lambda = 0$, this follows since B is linearly indep. Suppose now $\lambda \neq 0$ so we can solve for

$$v = -\frac{1}{\lambda}(\lambda_1 v_1 + \dots + \lambda_n v_n) \in \mathrm{Span}(B)$$

a contradiction. Hence $\lambda = 0$ and, as noted above all the other $\lambda_i = 0$ too. Thus $B \cup \{v\}$ is linearly indep.