School of Mathematics and Statistics, UNSW
MATH2601 Higher Linear Algebra, Semester 2, 2013
Class Test 1 Version A

Calculators are not permitted. Time allowed: 40 minutes.

This sheet must be filled in and stapled to the front of your answers

Student's Surname Initials	Tutor	Student Number

Throughout, as in lectures, $\mathbb{F}$ always denotes a field.

(1) [8 marks] For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

(i) Consider the permutation $\sigma = [2\ 3\ 1]$. Then $\sigma^2 = \text{id}$.

(ii) Let $A \in M_{33}(\mathbb{Z})$ be a matrix with determinant 2. Then the vector $A^{-1}\begin{pmatrix} -8 \\ 10 \\ 4 \end{pmatrix}$ has integer entries.

(iii) The row matrix $(x^2+3x+1 \ \ x^2+2x \ \ x+1) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$ defines a co-ordinate system on $\mathbb{R}[x]_{\leq 2}$.

(iv) Let $V^+ \leq M_{33}(\mathbb{R})$ be the subspace of symmetric matrices and W be the subspace spanned by the matrices

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ -2 & 2 & 4 \end{pmatrix}.$$

Then the sum $V^+ + W$ is direct.

(2) [3 marks] Let $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be orthogonal projection onto $\binom{1}{2}$.

(i) Find the matrix representing P.

(ii) Show that $(\text{id} - P)^2 = \text{id} - P$.

(3) [3 marks] Consider the map $T : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}^3$ defined by

$$Tf = \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix}.$$

(i) Show that T is linear by finding a representing matrix of linear maps or otherwise.

(ii) Show that T is an isomorphism.

(iii) Express the co-ordinate system $T^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$ as a row vector with entries in $\mathbb{R}[x]_{\leq 2}$.

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(4) [3 marks]

Consider the linear map $T : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

with respect to the co-ordinate system $(1 \; x \; x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$ (and the standard co-ordinate system $\mathrm{id} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ on $\mathbb{R}^2$).

(i) Find $T(3 - 2x^2)$.

(ii) Compute a basis for $\ker T$ and hence a co-ordinate system for $\ker T$.

(5) [3 marks] In this question, be sure to argue logically and provide complete proofs. Marks will be deducted for poorly written proofs.

Let $T : V \longrightarrow W$ be a surjective linear map and $B \subseteq V$.

(i) Prove that if B is a spanning set for V, then $T(B)$ is a spanning set for W.

(ii) If B is linearly independent, is it true that $T(B)$ is linearly independent too? Justify your answer fully.

School of Mathematics and Statistics, UNSW
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Class Test 1 Version B

Calculators are not permitted. Time allowed: 40 minutes.

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Student's Surname Initials	Tutor	Student Number

Throughout, as in lectures, $\mathbb{F}$ always denotes a field. Recall also that $\mathrm{ev}_x(f) = f(x)$.

(1) [8 marks] For each of the following, say whether the statement is true or false and give a brief reason. You will get one mark for a correct true/false answer, and if your true/false answer is correct then you will get one mark for a good reason.

(i) The permutation $\sigma = [2\ 1\ 3]$ is even.

(ii) Consider the matrix

$$C = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$

Then the matrix

$$C^{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} C$$

represents reflection about a line in $\mathbb{R}^2$.

(iii) Let W, W' be subspaces of a vector space V and $T : W \oplus W' \longrightarrow V$ be the natural map defined by $T\binom{\mathbf{w}}{\mathbf{w}'} = \mathbf{w} + \mathbf{w}'$. Then $\ker T$ is isomorphic to $W \cap W'$.

(iv) The row matrix $(2x + 4\ \ x - 3\ \ x - 5) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$ defines a co-ordinate system on $\mathbb{R}[x]_{\leq 2}$.

(2) [3 marks] Consider the vector space $V = L(\mathbb{R}[x]_{\leq 1}, \mathbb{R})$ and note that we have the following vectors $\frac{d}{dx}, \mathrm{ev}_1, \mathrm{ev}_2 \in V$. Express $\frac{d}{dx}$ as a linear combination of $\mathrm{ev}_1, \mathrm{ev}_2$. (Recall that $\mathrm{ev}_x(f) = f(x)$).

(3) [3 marks] Consider the subspaces

$$W = \mathbb{R}(1 + x^3) + \mathbb{R}(x + x^3), \quad W' = \mathbb{R}(1 + x + x^2 + x^3) + \mathbb{R}(3 + x^2 + 2x^3).$$

(i) Compute $W \cap W'$.

(ii) Is the sum $W + W'$ direct?

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(4) [3 marks] Let $T : \mathbb{R}[x]_{\leq 1} \longrightarrow \mathbb{R}[x]_{\leq 2}$ be the map defined by

$$(Tf)(x) = (x^2 - 2)\frac{df}{dx} - 3xf(x).$$

(i) Explain briefly why T is linear.

(ii) Find the matrix representing T with respect to the standard co-ordinate systems $(1\ x) : \mathbb{R}^2 \longrightarrow \mathbb{R}[x]_{\leq 1}$ and $(1\ x\ x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$.

(5) [3 marks] In this question, be sure to argue logically and provide complete proofs. Marks will be deducted for poorly written proofs.

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a linearly independent set of vectors in the vector space V and $\mathbf{v} \in V$. Prove that $B \cup \{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \notin \mathrm{Span}(B)$.