School of Mathematics & Statistics University of New South Wales

MATH2601: Higher Linear Algebra (2013,S2)
Problem Set 9: Inner product spaces [1]

This problem set covers lectures 32-37 and includes material on orthogonal complements and projections, dual vector spaces, adjoints and applications to least squares. The inner product is additional data you can add to a vector space so that concepts such as lengths and angles make sense. This in turn enables one to generalise many geometric notions to arbitrary inner product spaces. In this problem set $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ unless otherwise specified.

1. Prove some of the properties of conjugates and adjoints of matrices listed in lecture 32 such as: i) $\overline{(A+B)} = \overline{A} + \overline{B}$, ii) $(\beta A)^* = \overline{\beta} A^*$, iii) $(AX)^* = X^* A^*$.

2. Let V be the vector space of continuous $\mathbb{R}$-valued functions on $[0,1]$ equipped with the standard inner product. If $f(x) = x, g(x) = e^x$ determine $(f|g)$ and $\|f\|$.

3. Let $V = M_{mn}(\mathbb{F})$. We generalise the standard inner product on $\mathbb{F}^n$ to V as follows: for $A, B \in V$ let $(A|B) = \text{tr}(A^* B)$. Show that this does indeed define an inner product on V.

4. Let V, W be inner product spaces. Show that $V \oplus W$ has the structure of an inner product space with inner product given by $(\binom{\mathbf{v}}{\mathbf{w}}|\binom{\mathbf{v}'}{\mathbf{w}'}) = (\mathbf{v}|\mathbf{v}') + (\mathbf{w}|\mathbf{w}')$. This inner product space is sometimes called the *orthogonal direct sum* of V and W since the canonical copies of V and W in $V \oplus W$ are orthogonal.

5. Prove that the composite of conjugate linear maps is linear.

6. Show that in any inner product space V over $\mathbb{C}$ and $\mathbf{v}, \mathbf{v}' \in V$ we have
$$4(\mathbf{v}|\mathbf{v}') = \|\mathbf{v}+\mathbf{v}'\|^2 - \|\mathbf{v}-\mathbf{v}'\|^2 - i\|\mathbf{v}+i\mathbf{v}'\|^2 + i\|\mathbf{v}-i\mathbf{v}'\|^2.$$
In particular, the inner product can be recovered from the norm.

7. Prove the Cauchy-Schwarz inequality in an arbitrary inner product space V as follows. Let $\mathbf{v}, \mathbf{v}' \in V$ and suppose the polar form of $(\mathbf{v}|\mathbf{v}') = r_0 e^{i\theta}$. For $r \in \mathbb{R}$, show that we have a real non-negative quadratic function of r defined by $q(r) = \|\mathbf{v} + re^{-i\theta}\mathbf{v}'\|^2$. Deduce the Cauchy-Schwarz inequality from non-positivity of the discrimant of $q(r)$.

8. Find a basis for the orthogonal complement of $W = \mathbb{R}(1,1,1,0)^T + \mathbb{R}(2,2,1,1)^T$ in $\mathbb{R}^4$.

9. Let V be the vector space of continuous $\mathbb{R}$-valued functions on $[0,1]$ equipped with the standard inner product. Find the orthogonal complement to $\mathbb{R}1$ in V.

10. Let V be the vector space of continuous $\mathbb{R}$-valued functions on $[0,2\pi]$ equipped with the standard inner product. Show that the sum $W = \mathbb{R}\sin(x) + \mathbb{R}\cos(x)$ is orthogonal. Hence compute $\text{proj}_W x$.

11. Show that the sum $\mathbb{C}(2,1,i)^T + \mathbb{C}(1,-2,0)^T = W$ is orthogonal. Hence or otherwise compute $\text{proj}_W(1,0,i)^T$.

12. Consider the standard inner product $(f|g) = \int_0^1 \overline{f(x)} g(x) dx$ on $V = \mathbb{C}[x]_{\leq 2}$. Find the orthogonal complement to $\mathbb{C}[x]_{\leq 1}$ in V.

[1] by Daniel Chan

13. Let V be a finite dimensional inner product space and $W \leq V$. Let $P : V \longrightarrow V$ be orthogonal projection onto W and $Q : V \longrightarrow V$ be orthogonal projection onto $W^\perp$. Show that $Q = \text{id} - P$. Show that $Q \circ P = 0 = P \circ Q$ in the following 3 ways:

    (a) algebraically, using the fact that $P^2 = P$.

    (b) geometrically, by drawing a picture and using geometric definitions of P, Q.

    (c) algebraically, by using the description of P, Q in terms of linear maps from $W \oplus W^\perp \longrightarrow W \oplus W^\perp$.

14. Let V be an inner product space and $W \leq V$. Show that $W \subseteq (W^\perp)^\perp$.

15. Let

$$S = \{(0,1,1)^T, (3,2,0)^T\}, \quad A = \begin{pmatrix} 0 & 3 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Apply the Gram-Schmidt algorithm to S to find an orthonormal basis for $W = \text{Span}(S)$ and hence QR-factorise A. You should check your answer by verifying $A = QR$. Hence find the matrix representing $\text{proj}_W : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

16. Let

$$S = \{(1,1,1)^T, (-1,1,2)^T, (1,1,-1)^T\}, \quad A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Apply the Gram-Schmidt algorithm to S to find an orthonormal basis for $\mathbb{R}^3$ and hence QR-factorise A. You should check your answer by verifying $A = QR$.

17. Find an orthonormal basis for $V = \mathbb{R}1 \oplus \mathbb{R}e^x \leq \mathcal{C}^0([0,1])$ equipped with the inner product $(f|g) = \int_0^1 \overline{f(t)} g(t) dt$.

18. Consider the linear map $S = \frac{d}{dx} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ and its transpose $S^T : \mathbb{R}[x]^* \longrightarrow \mathbb{R}[x]^*$. Let $l \in \mathbb{R}[x]^*$ be the linear functional defined by $l = \text{ev}_0 \circ \frac{d}{dx}$. Find $(S^T l)(x^3 + x^2)$ and $S^T l$.

19. Consider the vector space $V = M_{mn}(\mathbb{F})$.

    (a) For each pair of integers $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}$ we define the function $l_{ij} : V \longrightarrow \mathbb{F} : (a_{rs})_{rs} \longrightarrow a_{ij}$. Show that $l_{ij} \in V^*$.

    (b) Show that the set $\{l_{ij} |\ i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}\}$ is a basis for V^*.

20. Fix a matrix $B = (b_{ij}) \in M_{nn}(\mathbb{F})$. Recall that the function $S : M_{nn}(\mathbb{F}) \longrightarrow M_{nn} : A \mapsto BA$ is a linear map. Using the notation of the previous question, express $S^T l_{11}$ a linear combination of the basis vectors l_{ij}.

21. Let $V = \mathbb{R}[x]_{\leq 1}$ with inner product $(f|g) = \int_0^1 f(t)g(t)dt$. Find $f \in V$ such that $(f|\cdot) = \text{ev}_1$.

22. Let $V = \mathbb{R}\cos(x) + \mathbb{R}\sin(x)$ with inner product $(f|g) = \int_0^{2\pi} f(t)g(t)dt$. Find $f \in V$ such that $(f|\cdot) = \text{ev}_0$.

23. For a linear endomorphism $S : V \longrightarrow$ on an inner product space V, simplify the expression $((1+i)S \circ S^* + 3S^* \circ S)^*$.

24. Let $V \leq \mathcal{C}^\infty(\mathbb{R})$ be the subspace of infinitely differentiable functions on $\mathbb{R}$ which are periodic with 2π as a period. We equip V with the inner product $(f|g) = \int_0^{2\pi} f(t)g(t)dt$. Consider the linear map $T = \frac{d}{dx} : V \longrightarrow V$. Show that $T^* = -T$.

25. Let $T : V \longrightarrow W$ be a linear map between finite dimensional inner product spaces. Show that $\ker T^* = (\text{im } T)^\perp$.

26. Let $A \in M_{mn}(\mathbb{R})$. The *row space* of A, denoted $\text{row}(A)$ is the span of the rows of A. Show that $\dim \text{row}(A) = \text{rank}(A)$.

27. Let $T : V \longrightarrow W$ be a linear map between finite dimensional inner product spaces. Show that i) $\ker(T^* \circ T)^2 = \ker T$ and that ii) $\text{rank}(T^* \circ T)^2 = \text{rank} T$.

28. Consider the data points $(t, y) =$

$$(0,1),(1,1),(2,2),(4,3).$$

   (a) Find the line of best fit $y = \alpha + \beta t$ to this data.
   (b) Find the curve of the form $y = \alpha + \beta t + \gamma t^2$ which best fits this data.

29. Find the curve of the form $y = \alpha \cos(\pi t) + \beta \sin(\pi t)$ which best fits the data points $(t, y) = (0, 4), (.5, 3), (1, 3)$.

30. A scientist performs an experiment which gives the following data points

$$(t_1, y_1) = (0, 1),\ (t_2, y_2) = (1, e^2),\ (t_3, y_3) = (2, e^3).$$

The scientist knows that the variables y, t are related by some differential equation of the form $\frac{dy}{dt} = \alpha y$ for some constant $\alpha \in \mathbb{R}$, but does not know what the parameter α is. S/he wishes to estimate it from the data points above as follows. Given some set of functions F, we say the function $y \in F$ *log best fits* the data if it minimises

$$\sum_i |\ln y_i - \ln y(t_i)|^2.$$

Find the value of the parameter α so that there is a solution which log best fits the data points.

Selected partial answers and hints

1. Easiest to write $A = (a_{ij})$ etc.
2. $(f|g) = 1, \|f\| = \frac{1}{\sqrt{3}}$.
3. Hardest axiom to verify is that $(A|A) > 0$ for $A \neq 0$. Do this by writing $A = (a_{ij})$.
8. $(-1, 0, 1, 1)^T, (-1, 1, 0, 0)^T$.
9. The orthogonal complement is $\{f \in V | \int_0^1 f(t)dt = 0\}$, the space of continuous functions with zero average value.
10. Note $\int_0^{2\pi} \sin(x)\cos(x)dx = 0$ so sum is orthogonal. $\text{proj}_W x = -2\sin(x)$.
11. $\frac{1}{2}(2, 1, i)^T + \frac{1}{5}(1, -2, 0)^T$.
12. $\mathbb{C}(x^2 - x + \frac{1}{6})$.
15. $Q = \begin{pmatrix} 0 & \frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{11}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{11} \end{pmatrix}.$

The projection matrix is $\begin{pmatrix} \frac{9}{11} & \frac{3}{11} & -\frac{3}{11} \\ \frac{3}{11} & \frac{13}{22} & \frac{9}{22} \\ -\frac{3}{11} & \frac{9}{22} & \frac{13}{22} \end{pmatrix}$.

16. $Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{5}{\sqrt{42}} & -\frac{1}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{42}} & \frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{42}} & -\frac{2}{\sqrt{14}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{42}}{3} & -\frac{8}{\sqrt{42}} \\ 0 & 0 & \frac{4}{\sqrt{14}} \end{pmatrix}.$

17. $\{1, \frac{e^x - e + 1}{\sqrt{3/2 + 2e - e^2/2}}\}$.

18. $S^T l = \mathrm{ev}_0 \circ \frac{d^2}{dx^2}$ so $(S^T l)(x^3 + x^2) = 2$.

19. b) Check linear independence by evaluating any linear combination $\sum_{i,j} a_{ij} l_{ij}$ at the standard basis vector E_{rs}.

20. $S^T l_{11} = \sum_j b_{1j} l_{j1}$.

21. $f(x) = 6x - 2$.

22. $f(x) = \frac{1}{\pi} \cos(x)$.

23. $(1 - i) S \circ S^* + 3 S^* \circ S$.

24. Similar to e.g. lecture 36, slide 3.

25. Similar to prop, lecture 36 slide 6.

26. Just note $\mathrm{row}(A) = \mathrm{col}(A^T)$.

27. Use $\ker S^* \circ S = \ker S$ for $S = T^* \circ T$.

28. a) $y = \frac{4}{5} + \frac{19}{35} t$. b) $y = \frac{49}{55} + \frac{39}{110} t + \frac{1}{22} t^2$.

29. $y = \frac{1}{2} \cos(\pi t) + 3 \sin(\pi t)$.

30. Note that $\ln y$ is a linear combination of $1, t$ and we wish to (usual) best fit $\ln y$ to the data points $(t_i, \ln y_i)$. Hence $\alpha = \frac{3}{2}$.