School of Mathematics & Statistics University of New South Wales

MATH2601: Higher Linear Algebra (2013,S2)
Problem Set 8: Jordan canonical form [1]

This problem set covers material from lectures 28-31 on the Jordan canonical form and applications to variation of parameters. The Jordan canonical form can be used to completely describe the similarity classes of square matrices over an algebraically closed field. The proof of the Jordan canonical form is quite involved and introduces several new important concepts, such as generalised eigenspaces and Jordan chains/tableaux. You should try to get an understanding of these. For a linear endomorphism $T : V \longrightarrow V$ we will use the notation in lectures $E_\lambda(n) = \ker(T - \lambda\,\mathrm{id}\,)^n$.

1. Consider the matrix
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$
   (a) Find all the subspaces $E_\lambda(n)$ for all eigenvalues λ and $n \in \mathbb{N}$.
   (b) What are all the generalised eigenspaces of A?
   (c) What is the Jordan canonical form for A?

2. Let $A \in M_{44}(\mathbb{C})$ be a matrix with only one eigenvalue λ. Find $E_\lambda(3)$ if you know that $\dim E_\lambda(2) = 3$.

3. Let $T : V \longrightarrow V$ be a linear map such that $T^3 = T^2$. Using primary decomposition or otherwise, show that any vector $\mathbf{v} \in V$ can be written in the form $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$ where $T^2\mathbf{v}' = \mathbf{0}$ and $T\mathbf{v}'' = \mathbf{v}''$.

4. Let $T : V \longrightarrow V$ be an $\mathbb{F}$-linear map with $\dim V < \infty$. Suppose that $p(T) = 0$ for some monic polynomial $p(x) \in \mathbb{F}[x]$ which factorises into distinct (monic) linear factors. Show that T is diagonalisable.

5. Let W be a subspace of V and $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ be linear independent modulo W. Show that $m \leq \dim V - \dim W$.

6. Find two vectors in $\mathbb{R}^4$ which are linearly independent modulo $\mathrm{Span}((1, 1, 2, -1)^T, (2, 1, 2, -1)^T)$.

7. Let A be a matrix with Jordan canonical form $J_1(2) \oplus J_1(2) \oplus J_1(1) \oplus J_4(5) \oplus J_4(2)$. Determine the dimensions of the subspaces $E_\lambda(n)$ for eigenvalues λ and $n \in \mathbb{N}$. For each eigenvalue, write down the corresponding Jordan form tableau.

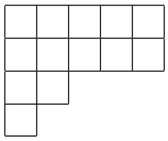
8. Let $A \in M_{55}(\mathbb{C})$ be a matrix whose subspaces $E_\lambda(n)$ satisfy $\dim E_2(1) = 2, \dim E_2(2) = 4, \dim E_2(3) = 5$. Find a Jordan canonical form for A.

9. Let $A \in M_{99}(\mathbb{C})$ be a matrix with minimal polynomial $(x^2 - 1)(x - 1)^2$. Suppose further that the geometric multiplicities of the eigenvalues 1 and -1 are 2 and 3 respectively. Can you determine from this information the Jordan canonical form for A? Hint: Use primary decomposition to help you.

10. Suppose given 8 matrices in $M_{66}(\mathbb{C})$ with cube zero i.e. $A^3 = 0$. Show that two of the matrices must be similar.

[1] by Daniel Chan

11. Let A be a matrix with a single eigenvalue 3. Suppose the corresponding Jordan form tableau is



Determine the Jordan canonical form for A and also the dimensions of the subspaces $E_3(n)$ for $n \in \mathbb{N}$.

12. Let $A \in M_{33}(\mathbb{C})$ be a matrix with Jordan chain

$$(\mathbf{v}_2\ \mathbf{v}_1\ \mathbf{v}_0) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

corresponding to the eigenvalue $\lambda = 2$. Determine A from this information.

13. Each of the matrices A below has 5 as an eigenvalue. Find a Jordan canonical form J for A and a change of co-ordinates matrix C such that $J = C^{-1}AC$.

$$A = \begin{pmatrix} 8 & 6 & 3 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 7 & 5 & 1 \\ 0 & 4 & 1 \\ -1 & -2 & 4 \end{pmatrix}.$$

14. Find a Jordan canonical form J for

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -4 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and a change of co-ordinates matrix C such that $J = C^{-1}AC$.

15. Consider

$$A = \begin{pmatrix} -1 & -2 & 1 & 2 \\ -1 & 2 & 1 & 0 \\ -4 & -3 & 3 & 3 \\ -4 & -2 & 2 & 4 \end{pmatrix}, \quad (\mathbf{v}_2\ \mathbf{v}_1\ \mathbf{v}_0) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 2 & 2 \end{pmatrix}.$$

(a) Show that $(\mathbf{v}_2\ \mathbf{v}_1\ \mathbf{v}_0)$ is a Jordan chain for A and compute the corresponding eigenvalue λ.

(b) Show using the trace or otherwise that A only has one eigenvalue and show it has geometric multiplicity 2.

(c) Complete the following partially filled-in Jordan form tableau

$\mathbf{v}_2$	
$\mathbf{v}_1$	
$\mathbf{v}_0$	

.

(d) Use the filled-in Jordan form tableau above to find a change of co-ordinate matrix C such that $J = C^{-1}AC$ has Jordan canonical form.

16. Solve the initial value problem

$$\frac{d\mathbf{y}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\mathbf{y} + 4e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{y}(0) = \mathbf{0}.$$

Note in this case that $\binom{1}{1}$ is an eigenvector of the above 2×2-matrix.

17. For $t_0 \in \mathbb{R}$, show that the solution to $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t) + \mathbf{v}(t), \mathbf{y}(t_0) = \mathbf{y}_0$ is given by

$$\mathbf{y}(t) = \exp((t - t_0)A)\mathbf{y}_0 + \exp(tA) \int_{t_0}^{t} \exp(-sA)\mathbf{v}(s)ds.$$

18. In this question, we need to work over the field $\mathbb{F} = \mathbb{C}$ (or more generally, any field not of characteristic 2). Show that $J_n(\lambda)^2$ is similar to $J_n(\lambda^2)$ for $\lambda \neq 0$. What happens if $\lambda = 0$?

19. Let $T : V \longrightarrow V$ be a linear map. Consider the linear map $S = \left(\begin{smallmatrix} 0 & T \\ T & 0 \end{smallmatrix}\right) : V \oplus V \longrightarrow V \oplus V$.

    (a) Show that for any T-invariant subspace $W \leq V$ that $\operatorname{im}\left(\left(\begin{smallmatrix} \iota \\ \iota \end{smallmatrix}\right) : W \longrightarrow V \oplus V\right)$ is S-invariant where $\iota : W \longrightarrow V$ is the inclusion map. Similarly show that $\operatorname{im}\left(\begin{smallmatrix} \iota \\ -\iota \end{smallmatrix}\right)$ is also S-invariant.

    (b) With the above notation, show that $\left(\begin{smallmatrix} \iota \\ \iota \end{smallmatrix}\right) \circ T = S \circ \left(\begin{smallmatrix} \iota \\ \iota \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \iota \\ -\iota \end{smallmatrix}\right) \circ (-T) = S \circ \left(\begin{smallmatrix} \iota \\ -\iota \end{smallmatrix}\right)$.

    (c) In the case $W = V$, we let $V^+ = \operatorname{im}\left(\left(\begin{smallmatrix} \mathrm{id} \\ \mathrm{id} \end{smallmatrix}\right) : V \longrightarrow V \oplus V\right), V^- = \operatorname{im}\left(\left(\begin{smallmatrix} \mathrm{id} \\ -\mathrm{id} \end{smallmatrix}\right) : V \longrightarrow V \oplus V\right)$. Show that $V \oplus V = V^+ \oplus V^-$.

    (d) Suppose $T \in M_{nn}(\mathbb{C})$ has Jordan canonical form $J_{n_1}(\lambda_1) \oplus \ldots \oplus J_{n_r}(\lambda_r)$. Find the Jordan canonical form for S.

Selected partial answers and hints

1. There is only 1 e-value $\lambda = 1$. $E_1(1) = \mathbb{C}(0, -1, 1), E_1(2) = \mathbb{C}(0, 1, 0)^T + \mathbb{C}(0, 0, 1)^T$ and the generalised e-space is $\mathbb{C}^3 = E_1(\infty) = E_1(3) = E_1(4) = \ldots$. The Jordan canonical form is $J = J_3(1)$.

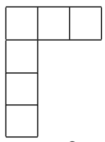
2. $E_\lambda(3) = \mathbb{C}^4$.

3. $V = \ker T^2 \oplus \ker(T - I)$.

4. Use primary decomposition.

5. Observe $W \oplus \mathbb{F}\mathbf{v}_1 \oplus \ldots \oplus \mathbb{F}\mathbf{v}_m$ is a subspace of V.

6. $(0, 1, 0, 0)^T, (0, 0, 1, 0)^T$.

7. For $\lambda = 2$ the Jordan canonical form tableau is 

so $\dim E_2(1) = 3, \dim E_2(2) = 4, \dim E_2(n) = 5$ for $n \geq 3$.

For $\lambda = 5$ the Jordan canonical form tableau is 

so $\dim E_5(1) = 1, \dim E_5(2) = 2, \dim E_5(3) = 3, \dim E_5(n) = 4$ for $n \geq 4$.
Finally, $\dim E_1(1) = \dim E_1(\infty) = 1$.

8. $J_3(2) \oplus J_2(2)$.

9. $J_3(1) \oplus J_3(1) \oplus J_1(-1) \oplus J_1(-1) \oplus J_1(-1)$.

10. There are only 7 Jordan canonical form tableau with 6 boxes and at most 3 rows.

11. $J = J_4(3) \oplus J_3(3) \oplus J_2(3) \oplus J_2(3) \oplus J_2(3)$ and $\dim E_3(1) = 5, \dim E_3(2) = 10, \dim E_3(3) = 12, \dim E_3(n) = 13$ for $n \geq 4$.

12. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$.

13. Respective possible answers are

$$C = \begin{pmatrix} 3 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}, J = J_2(5) \oplus J_1(5)$$

$$C = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}, J = J_3(5).$$

14. One possibility is $C = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, J = J_2(1) \oplus J_1(-1)$.

15. a) $\lambda = 2$ so just check $\mathbf{v}_{i+1} = (A - 2I)\mathbf{v}_i$. c) Fill the empty box with any vector in $\mathbb{C}(1, -1, 1, 0)^T + \mathbb{C}(0, 1, 0, 1)^T$ that is not a scalar multiple of $\mathbf{v}_2$. d) One possibility is

$$J = J_3(2) \oplus J_1(2) \text{ and } C = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & 1 \\ 2 & 2 & 2 & 0 \end{pmatrix}.$$

16. $\mathbf{y}(t) = 2(e^{5t} - e^{3t})\binom{1}{1}$.

17. Verify directly or shift the time variable.

18. Compute the nullity of $(J_n(\lambda)^2 - \lambda^2 I)^k$ for $k \in \mathbb{N}$.