MATH2601: Higher Linear Algebra (2013,S2)
Problem Set 7: Discrete & continuous time systems [1]

In first year, you saw that discrete and continuous time systems given by certain recurrence or differential equations, could be studied using diagonalisation theory. There were various ways to view this, perhaps easiliest as a change of variables to decouple equations. In general, we could triangularise to form cascaded recurrence or differential equations. However, the approach we focus on here is to use Jordan canonical forms and compute powers and exponential matrices in these.

1. Write down $J_3(2)^2, J_3(2)^3$ and then the general formula for $J_3(2)^n$ when $n \geq 3$ is an integer.

2. Consider the discrete time system $\mathbf{v}(k+1) = A\mathbf{v}(k)$ where $A \in M_{44}(\mathbb{R})$ has Jordan canonical form $C^{-1}AC = J = J_3(3) \oplus J_1(1)$. Solve for $\mathbf{v}(k)$ if
$$C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 2 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{v}(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

3. Consider the discrete time system $\mathbf{v}(k+1) = A\mathbf{v}(k)$ where $A \in M_{44}(\mathbb{R})$ has Jordan canonical form $C^{-1}AC = J = J_2(2) \oplus J_2(-1)$. Solve for $\mathbf{v}(k)$ if
$$C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 2 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{v}(0) = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}.$$

4. In this question we study the asymptotic behaviour of the discrete time system $\mathbf{v}(k+1) = A\mathbf{v}(k)$ where $A \in M_{33}(\mathbb{R})$ has Jordan canonical form $C^{-1}AC = J = J_2(3) \oplus J_1(\frac{1}{2})$ and
$$C = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$
For the various possible initial conditions $\mathbf{v}(0)$, find the limit of $\frac{\mathbf{v}(k)}{\|\mathbf{v}(k)\|}$ as $k \longrightarrow \infty$.

5. Find a closed formula for A^k if $A = \begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix}$ and k is a positive integer. Also compute $\exp(tA)$.

6. Let R_θ be the rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $T = R_{\pi/4} \oplus R_{\pi/6}$. Find the integers n such that $T^n = I_4$.

7. Let N be a nilpotent square matrix. Show that the only eigenvalue of N is 0.

8. Let $L, N \in M_{nn}(\mathbb{F})$ be nilpotent matrices which commute. Show that $L + N$ is also nilpotent. Give an example to show that this is no longer the case if you drop the commutativity assumption.

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9. For $t \in \mathbb{R}$, compute $\exp(A)$ and $\exp(tA)$ if $A = C(J_3(3) \oplus J_1(-1))C^{-1}$ &

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

You may leave your answer in factored form CEC^{-1} for some matrix E.

10. Let $A \in M_{22}(\mathbb{C})$ be a matrix whose eigenvalues all have modulus strictly less than 1. Show that $I - A$ is invertible by finding a suitable power series formula for the inverse. You may assume that the sum $\sum_{k=1}^{\infty} kz^{k-1}$ converges for $|z| < 1$.

11. Solve the inital value problem $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t), \quad \mathbf{y}(0) = (1,1,1,1)^T$ if $A = C(J_3(3) \oplus J_1(-2))C^{-1}$ &

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

12. Solve the inital value problem

$$\frac{d\mathbf{y}}{dt} = \begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix} \mathbf{y}(t), \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

13. Find the general solution to the homogenous ODE

$$\frac{dy_1}{dt} = y_1 - 4y_2$$
$$\frac{dy_2}{dt} = y_1 - 3y_2$$

14. Let $A \in M_{nn}(\mathbb{R})$ and $\mathbf{w} \in \mathbb{R}^n$. Given some fixed $t_0 \in \mathbb{R}$, show that $\mathbf{y}(t) = \exp(tA)\exp(-t_0A)\mathbf{w}$ is a solution to the initial value problem $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t), \quad \mathbf{y}(t_0) = \mathbf{w}$. Hence or otherwise solve the initial value problem $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t), \quad \mathbf{y}(1) = (1,1,1,1)^T$ if $A = C(J_3(3) \oplus J_1(-2))C^{-1}$ &

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Selected partial answers and hints

1. $\begin{pmatrix} 2^n & n2^n & \frac{n(n-1)}{2}2^{n-2} \\ 0 & 2^n & n2^n \\ 0 & 0 & 2^n \end{pmatrix}.$

2. $\mathbf{v}(k) = \begin{pmatrix} \frac{k(k-1)}{2}3^{k-2} + k3^{k-1} \\ \frac{k(k-1)}{2}3^{k-2} - k3^{k-1} + 3^k \\ -\frac{k(k-1)}{2}3^{k-2} + 3^k \\ -\frac{k(k-1)}{2}3^{k-2} \end{pmatrix}.$

3. $\mathbf{v}(k) = \begin{pmatrix} 2^k \\ 2^k + (-1)^k \\ -2^k + (-1)^k \\ -2^k \end{pmatrix}$.

4. If $\mathbf{v}(0) \notin \mathbb{R}(0,1,1)^T$ then the limit is $\frac{1}{\sqrt{6}}(1,2,-1)^T$.

5. $A^k = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2^k & k2^{k-1} \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$, $\exp(tA) = e^{2t} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$.

6. n is a multiple of 24.

7. If $N^n = 0$, and $\mathbf{v}$ is an e-vector, compute $N^n\mathbf{v}$ in two ways.

8. Use the binomial formula.

9. $E = \begin{pmatrix} e^{3t} & te^{3t} & \frac{t^2}{2}e^{3t} & 0 \\ 0 & e^{3t} & te^{3t} & 0 \\ 0 & 0 & e^{3t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}$.

10. $(I - A)^{-1} = I + A + A^2 + A^3 + \dots$.

11. $\mathbf{y}(t) = \begin{pmatrix} te^{3t} + e^{-2t} \\ e^{3t} \\ (t+1)e^{3t} \\ (t+1)e^{3t} \end{pmatrix}$.

12. $\mathbf{y}(t) = e^{2t} \begin{pmatrix} -2t+1 \\ -t \end{pmatrix}$.

13. The general solution is $\mathbf{y}(t) = e^{-t}\left(\alpha\binom{2t+1}{t} + \beta\binom{-4t}{1-2t}\right)$ for arbitrary real constants α, β.

14. Use either the product rule to verify the solution directly or shift the time variable.