MATH2601: Higher Linear Algebra (2013,S2)
Problem Set 6: Invariant subspaces & diagonalisation [1]

This problem set covers material from lectures 19-24. You saw the basic theory of diagonalisation in first year. The theory however was clearly incomplete as not all square matrices were diagonalisable, even if one works over an algebraically closed field. To complete the theory, we use the notion of direct sums and more precisely, invariant direct sums to break up the vector space under consideration into simpler parts. This new point of view allows us to better understand linear endomorphisms in general and in particular, prompts the theory of Jordan canonical forms which gives the "best" possible form for arbitrary complex square matrices.

1. Let $T = \frac{d}{dx} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$. Show that $W = \text{Span}(1, x-2, (x-2)^2)$ is a T-invariant subspace of $\mathbb{R}[x]$. (HARD) Show that there is no vector space complement to W in $\mathbb{R}[x]$ which is T-invariant.

2. Let $\mathbf{u} \in \mathbb{R}^3$ be a unit vector and $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be reflection about the hyperplane $\mathbf{u}^\perp$. Determine geometrically all T-invariant subspaces of $\mathbb{R}^3$. Write $\mathbb{R}^3$ as an invariant direct sum of 1-dimensional subspaces and hence diagonalise T.

3. Show that $W = \text{Span}((1,1,0,1)^T, (1,0,2,0)^T), W' = \text{Span}((1,1,0,0)^T, (1,1,-1,0)^T)$ are T-invariant subspaces of the linear map given by the matrix
$$\begin{pmatrix} 4 & -2 & -1 & -1 \\ 3 & -1 & -1 & -1 \\ -2 & 2 & 2 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$
Hence find a change of co-ordinates matrix $C \in GL_4(\mathbb{R})$ such that $C^{-1}TC$ has block diagonal form $A_1 \oplus A_2$ for 2×2-matrices A_1, A_2. Determine A_1, A_2.

4. We consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$. i) If we view A as a rational matrix defining a $\mathbb{Q}$-linear map $\mathbb{Q}^2 \longrightarrow \mathbb{Q}^2$, show that the only A-invariant subspaces of $\mathbb{Q}^2$ are $\mathbf{0}$ and $\mathbb{Q}^2$ itself. ii) Find all A-invariant subspaces if we view A as a real matrix defining an $\mathbb{R}$-linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

5. Show that the following pairs of matrices are not similar
$$\text{i)} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 1 \\ 2 & 0 & 6 \\ 2 & -3 & 1 \end{pmatrix}, \quad \text{ii)} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}.$$

6. Show that any two real matrices with characteristic polynomial $p(\lambda) = -\lambda^3 + 3\lambda^2 + \lambda - 3$ are similar. Find two non-similar matrices with characteristic polynomial $p(\lambda) = -\lambda^3 + \lambda^2 + \lambda - 1$.

7. Suppose you know one of the e-values of $A \in M_{44}(\mathbb{C})$ is 2 and it has algebraic multiplicity 3. If A has diagonal entries $1, 3, 3, 6$, find the fourth eigenvalue.

8. Show that any non-scalar complex 2×2-matrix with only a single eigenvalue λ must be non-diagonalisable.

9. You are given the following information about the matrix $A \in M_{33}(\mathbb{R})$: $\ker(A - I) = \mathbb{R}(1,0,2)^T + \mathbb{R}(3,1,1)^T, \ker(A - 2I) = \mathbb{R}(4,1,1)^T$. Show that A is diagonalisable and diagonalise A.

[1] by Daniel Chan

10. Note that $V = \mathbb{C}\cos x + \mathbb{C}\sin x$ is a $\frac{d}{dx}$-invariant subspace of $C^\infty(\mathbb{C})$ the $\mathbb{C}$-space of analytic functions on $\mathbb{C}$. Hence we may consider the linear endomorphism $T = \frac{d}{dx} + \mathrm{id} : V \longrightarrow V$. Find the characteristic polynomial, eigenvalues and eigenvectors of T and hence write V as a T-invariant direct sum of 1-dimensional subspaces. Find a co-ordinate system adapted to this direct sum decomposition and the corresponding diagonal matrix representing T.

11. Consider the linear endomorphism $T = (x-1)\frac{d}{dx} - 2\,\mathrm{id} : \mathbb{R}[x]_{\leq 1} \longrightarrow \mathbb{R}[x]_{\leq 1}$. Find the characteristic polynomial, eigenvalues and eigenvectors of T and hence write $\mathbb{R}[x]_{\leq 1}$ as a T-invariant direct sum of 1-dimensional subspaces. Find a co-ordinate system adapted to this direct sum decomposition and the corresponding diagonal matrix representing T.

12. Show (preferably without diagonalising) that the following 3×3-matrix is diagonalisable (over $\mathbb{C}$).

$$\begin{pmatrix} 0 & 0 & -7 \\ 1 & 0 & -9 \\ 0 & 1 & 6 \end{pmatrix}$$

Hint: Check that the characteristic polynomial has no repeated roots.

13. Suppose you are given a real 3×3-matrix A with characteristic polynomial $p(\lambda) = -\lambda^3 + \lambda^2 + \lambda - 1$ such that $(1,1,1)^T, (2,0,1)^T$ are eigenvectors with eigenvalue 1. Show that A is diagonalisable.

14. Consider the linear map $T = (x-3)\frac{d}{dx} : \mathbb{R}[x]_{\leq 1} \longrightarrow \mathbb{R}[x]_{\leq 1}$. Find the matrix representing T with respect to the co-ordinate system $(1\ x) : \mathbb{R}^2 \longrightarrow \mathbb{R}[x]_{\leq 1}$. Hence find the matrix representing T^2 and show that $T^n = T^{n+1}$ for any integer $n \geq 1$.

15. Let $A \in M_{33}(\mathbb{F})$ be a matrix with characteristic polynomial $p(\lambda) = 2 - \lambda - \lambda^3$. Use the Cayley-Hamilton theorem to write A^4 as a linear combination of I, A, A^2.

16. Let $A \in M_{33}(\mathbb{F})$ be a matrix with characteristic polynomial $p(\lambda) = 2 - \lambda + 3\lambda^2 - \lambda^3$. Use the Cayley-Hamilton theorem to write A^{-1} as a polynomial function of A of degree 2. (You may want to check invertibility of A by determining $\det(A)$ first).

17. Let A be a square matrix with characteristic polynomial $p(\lambda) = \lambda^2 - 3\lambda + 2$. Using the Cayley-Hamilton theorem or otherwise, show that $\mathrm{im}\,(A - 2I_2) \subseteq \ker(A - I_2)$.

18. Write out explicitly the following block diagonal matrices: a) $J_1(\sqrt{2}) \oplus J_2(\pi)$, b) $J_2(3) \oplus J_1(i) \oplus J_3(-1)$.

19. Determine all the eigenvalues of $A = J_1(\sqrt{2}) \oplus J_1(\sqrt{2}) \oplus J_2(\pi)$ together with their geometric and algebraic multiplicity.

20. Determine if the following matrices are similar.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}.$$

21. For each of the matrices A below, find a co-ordinate system $C : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ such that $J = C^{-1}AC$ has Jordan canonical form i.e. is a direct sum of Jordan blocks. Write down J as well.

$$\begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 3 & 4 \end{pmatrix}.$$

22. For each of the matrices A below, find $C \in GL_2(\mathbb{C})$ and Jordan canonical form $J \in M_{22}(\mathbb{C})$ such that $A = CJC^{-1}$.

$$\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}.$$

23. For any complex number β, show that the following matrices are similar (by comparing Jordan canonical forms or otherwise).

$$\begin{pmatrix} \beta & 2 \\ 0 & \beta \end{pmatrix}, \quad \begin{pmatrix} \beta & 3 \\ 0 & \beta \end{pmatrix}.$$

Selected partial answers and hints

1. Check T-invariance of W by checking $T(x-2)^i \in W$.

2. The T-invariant subspaces are $\mathbb{R}\mathbf{u}$ and all the subspaces of $\mathbf{u}^\perp$. The diagonal matrix is $(-1) \oplus (1) \oplus (1)$.

3. The columns of C are $(1,1,0,1)^T, (1,0,2,0)^T, (1,1,0,0)^T, (1,1,-1,0)^T$. Then

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

4. Any subspace other than $\mathbf{0}$ or $\mathbb{F}^2$ must be 1-dimensional so has form $\mathbb{F}\mathbf{v}$. This is T-invariant iff $\mathbf{v}$ is an eigenvector.

5. Show the traces or determinants are different.

6. If $p(\lambda) = -\lambda^3 + 3\lambda^2 + \lambda - 3$ then show matrix must be diagonalisable. If $p(\lambda) = -\lambda^3 + \lambda^2 + \lambda - 1$ construct different Jordan canonical forms with this characteristic polynomial.

7. Fourth eigenvalue is 7.

8. Any diagonalisable 2×2-matrix with only e-value λ must have form $C\lambda I_2 C^{-1}$.

9. Sum of geom multiplicities is 3 so diagonalisable. $D = C^{-1}AC$ where

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

10. $\mathrm{cp}_T(\lambda) = \lambda^2 - 2\lambda + 2$. T-invariant direct sum is $V = \mathbb{C}(\cos x + i\sin x) \oplus \mathbb{C}(\cos x - i \sin x)$. Corresponding matrix is

$$\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}.$$

11. $\mathrm{cp}_T(\lambda) = \lambda^2 + 3\lambda + 2$. T-invariant direct sum is $\mathbb{R}[x]_{\leq 1} = \mathbb{R}1 \oplus \mathbb{R}(x-1)$. Adapted co-ord system is $(1 \quad x-1) : \mathbb{R}^2 \longrightarrow \mathbb{R}[x]_{\leq 1}$. Corresponding matrix is

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

13. Show that -1 is also an eigenvalue so the sum of geometric multiplicities is ≥ 3.

14. $\begin{pmatrix} 0 & -3 \\ 0 & 1 \end{pmatrix}$.

15. $A^4 = -A^2 + 2A$.

16. $\det(A) = 2, A^{-1} = I - 3A + A^2$.

18. $J_1(\sqrt{2}) \oplus J_2(\pi) = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \pi & 1 \\ 0 & 0 & \pi \end{pmatrix}$.

19. $\lambda = \sqrt{2}$ has geom & alg mult 2. $\lambda = \pi$ has geom mult 1 but alg mult 2.
20. No.
21. $A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}, \quad J = J_2(3), \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$
22. $A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, \quad J = J_2(3), \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$