MATH2601: Higher Linear Algebra (2013,S2)
Problem Set 5: "Revision" of kernels, images and co-ordinates [1]

This problem set covers material from lectures 13-18. This includes the notions of kernel, image, linear independence, span, co-ordinate systems, dimension, rank and nullity. The only really new concepts are perhaps that of change of co-ordinates, which is already implicit in the diagonalisation theory you learnt in first year, and that of vector space complements. The material here is primarily revision plus some optional extension material for MATH1241, which you may have seen briefly either in lectures or in the harder exercises. There is a lot of material covered in this problem set, but hopefully, much of it is reasonably familiar.

1. Does $S = \{1 + 2x + x^2, -1 + 3x + x^2, 1 + x^2\}$ span $\mathbb{C}[x]_{\leq 2}$?
2. Does $S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 9 \end{pmatrix} \right\}$ span $M_{22}(\mathbb{C})$?
3. Find a minimal spanning set for
$$\mathbb{R}\begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}.$$
4. Let V be a vector space and $S_1, S_2 \subseteq V$. Show that $\mathrm{Span}(S_1) + \mathrm{Span}(S_2)$ is spanned by $S_1 \cup S_2$.
5. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be orthogonal projection onto the line $\mathrm{Span}(\mathbf{u})$. Describe geometrically the solutions $\mathbf{v}$ to $T\mathbf{v} = \mathbf{w}$ as $\mathbf{w}$ varies over $\mathbb{R}^3$.
6. Let $T : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ be the linear map defined by $T = \frac{d^2}{dx^2} - 3\frac{d}{dx} + 2\,\mathrm{id}$. Find a co-ordinate system for $\ker T$ and $\mathrm{null}T$.
7. Let $S : U \longrightarrow V, T : V \longrightarrow W$ be linear maps between $\mathbb{F}$-spaces. Show that $\ker S \subseteq \ker T \circ S$ and $\mathrm{im}\ T \circ S \subseteq \mathrm{im}\ T$.
8. Let $S : U \longrightarrow V, T : V \longrightarrow W$ be linear maps between $\mathbb{F}$-spaces such that $\mathrm{im}\ T \circ S = \mathrm{im}\ T$. If T is injective, show that S is surjective. If T is not injective, can you still conclude that S is surjective?
9. Find a basis and hence co-ordinate system for the kernel of the matrix
$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 3 & -1 & 1 \\ 1 & 0 & -2 & -1 \end{pmatrix}.$$
10. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be two non-parallel vectors. Is the map $(\mathbf{v} \quad \mathbf{w}) : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ injective? Explain (preferably in one sentence).
11. Find a co-ordinate system for the $\mathbb{R}$-space $V = M_{22}(\mathbb{R}) \oplus \mathbb{R}[x]_{\leq 1}$. Hence or otherwise find the dimension of V.
12. This question concerns the vector space V. True or false?

[1] by Daniel Chan

(a) Let $B \subset V$ and $\mathbf{v} \in V$ be a vector such that $B \cup \mathbf{v}$ is linearly independent. Then B is never a basis for V.

(b) Let B be a basis for V and $S \subseteq B$ which spans V. Then $S = B$.

13. Find a basis and hence co-ordinate system for Span(S) where $S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \right\}$.

14. Let $T : \mathbb{R}[x]_{\leq 1} \longrightarrow \mathbb{R}[x]_{\leq 2}$ be the linear map defined by the differential operator $T = (x-2)\frac{d}{dx} - \text{id}$. Find the matrix representing T wrt the standard co-ordinate systems $(1\ x) : \mathbb{R}^2 \longrightarrow \mathbb{R}[x]_{\leq 1}$ and $(1\ x\ x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$. Hence determine bases for the kernel and image of T. What are the rank and nullity of T? Verify your answer by doing the question without co-ordinates.

15. Show that the map $T : M_{22}(\mathbb{R}) \longrightarrow M_{22}(\mathbb{R}) : A \mapsto A^T$ is linear and let $S = T - \text{id} : M_{22}(\mathbb{R}) \longrightarrow M_{22}(\mathbb{R})$. Using the ordered basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

to construct a co-ordinate system for $M_{22}(\mathbb{R})$, find the matrix representing S. Hence or otherwise compute bases for the kernel and image of S.

16. Consider the linear map $T : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix}$$

wrt to the co-ordinate system $(1\ x\ x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$ (and the standard co-ordinate system $\text{id} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ on $\mathbb{R}^2$).

(a) Find $T(3 - 2x^2)$.

(b) Compute a basis for $\ker T$ and hence a co-ordinate system for $\ker T$.

17. Use change of co-ordinates to write down the matrix representing rotation about the axis $\mathbb{R}(1,1,0)^T$ about angle $\pi/3$. Use the right hand rule with thumb pointing in the direction of $(1,1,0)^T$. You may leave the matrix in factored form.

18. We work over $\mathbb{F} = \mathbb{R}$. Consider the change of co-ordinates

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Write out the linear map $\binom{y_1}{y_2} \mapsto \binom{y_1}{0}$ in terms of the x-co-ordinates and describe the corresponding transformation of the (x_1, x_2)-plane geometrically.

19. Suppose we have new x and y co-ordinate systems on $\mathbb{R}^2$ given by

$$C_x = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}, C_y = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

What is the change of co-ordinates matrix to go from x to y co-ordinates?

20. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for $\mathbb{C}^2$. Is $\{\mathbf{v}_1 + 2\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$ also a basis?

21. Let $T : V \longrightarrow W$ be a surjective linear map. If $\dim V \leq \dim W$, prove that T is actually an isomorphism.

22. Let $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a linear map of rank m. Show that T is injective.

23. Find a co-ordinate system for a vector space complement to $\mathbb{R}(1, 2, 2)^T + \mathbb{R}(1, 1, 1)^T$ in $\mathbb{R}^3$.

24. Let $A = (2\ 3\ 1\ \ -1) \in M_{14}(\mathbb{R})$. Show that $\mathbf{v} = (2, -1, -1, 0)^T \in \ker A$ and find a basis for a vector space complement to $\mathbb{R}\mathbf{v}$ in $\ker A$.

25. Find a vector space complement to $\mathbb{R}(1 + x + x^3) + \mathbb{R}(x - x^2 - x^3)$ in $\mathbb{R}[x]_{\leq 3}$.

26. Let W be a 3-dim subspace of $M_{22}(\mathbb{R}) \oplus M_{23}(\mathbb{R})$. What is the dimension of any vector space complement to W?

27. Let $A \in M_{mn}(\mathbb{F})$ be a rank l matrix. Find the dimensions of all possible vector space complements to $\ker A$.

28. Let A be a matrix with zero determinant but whose $(1, 1)$-minor $A(1, 1)$ has non-zero determinant. Prove that null $A = 1$.

29. Consider the linear map $T = x^3 : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ which multiplies polynomials by x^3. Show that the map T is injective but not surjective. Find a vector space complement to im T. By using the isomorphism theorem appropriately, prove that all vector space complements to im T have the same dimension. (Note that rank-nullity does not apply here!)

Selected partial answers and hints

1. Yes.
2. Yes.
3. $\left\{\begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\right\}$ since it is lin indep and the third matrix is a lin combination of these.
5. The solutions are empty or planes perpedicular to the line.
6. A co-ord system is $(e^x\ \ e^{2x}) : \mathbb{R}^2 \longrightarrow \mathcal{C}^\infty(\mathbb{R})$ so the nullity is 2.
9. Yes, because the the vectors are linearly indep.
10. Co-ord system is $\begin{pmatrix} 2 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.
11. $\dim V = 4 + 2 = 6$. Co-ord system is

$$\left(\begin{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ 0 \end{matrix} \quad \begin{matrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ 0 \end{matrix} \quad \begin{matrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ 0 \end{matrix} \quad \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ 0 \end{matrix} \quad \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ 1 \end{matrix} \quad \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ x \end{matrix}\right)$$

12. a) T, b) T.
13. Co-ord system is $\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\right)$
14. Representing matrix is $\begin{pmatrix} -1 & -2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. Bases for the kernel and image are $\{x - 2\}$ and $\{1\}$ respectively. Rank and nullity both 1.
15. Representing matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so bases for the kernel and image are resp.

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\} \quad \text{and} \quad \left\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}.$$

16. a) $\binom{-1}{0}$. b) $\{3 + x - 2x^2\}$.
17. CRC^{-1} where

$$R = \begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0\\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{pmatrix}, C = \begin{pmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 1 & 0 & 0\end{pmatrix}.$$

18. $\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ gives a non-orthogonal projection onto $\mathbb{R}\binom{2}{-1}$ by sliding parallel to $\binom{-1}{1}$.
19. $\begin{pmatrix}-2 & 0\\ 5 & 1\end{pmatrix}$.
20. Yes. The corresponding co-ordinate system is given by the product of invertible matrices $(\mathbf{v}_1\ \mathbf{v}_2)\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}$.
21. & 22. Use rank-nullity to show that $\ker T = \mathbf{0}$.
23. One possibility is $(0, 1, 0)^T$.
24. One possibility is $\{(1, 0, 0, 2)^T, (0, 1, 0, 3)^T\}$.
25. One possibility is $\{1, x\}$.
26. 7.
27. l.
28. Note $\ker A \neq \mathbf{0}$ and that the sum $\ker A + \text{Span}(\mathbf{e}_2, \ldots, \mathbf{e}_n)$ is direct where n is the size of the matrix A.
29. Apply the isomorphism theorem to the linear map $S : \mathbb{R}[x] \longrightarrow \mathbb{R}[x] : \sum_i p_i x^i \mapsto p_0 + p_1 x + p_2 x^2$.