MATH2601: Higher Linear Algebra (2013,S2)
Problem Set 4: Direct sums [1]

This problem set looks at material from lectures 10-12 concerning material on direct sums of vector spaces and linear maps between direct sums. This new notion is the key concept which allows us to better understand the linear algebra you learnt in first year. With this improved understanding, it will be easier to further develop linear algebra. The basic idea is simple enough, the direct sum allows you to build a bigger vector space from smaller ones. One can easily understand the direct sum if you understand the summands. The key point later on is that given a "big" vector space that you don't understand, we seek to write it nicely as a direct sum of smaller vector spaces that you can understand.

1. In this question we consider the $\mathbb{R}$-space $V = C^\infty(\mathbb{R}) \oplus \mathbb{R}$.
   (a) Is $\binom{\cos(x+\pi/4)}{0} \in V$ a linear combination of $\binom{\cos x}{1}, \binom{\sin x}{1}$?
   (b) Show that the map $T : V \longrightarrow C^\infty(\mathbb{R}) : \binom{f(x)}{\beta} \mapsto \beta + \int_0^x f(t)dt$ is linear.
   (c) For linear maps $\frac{d}{dx} : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}), \mathrm{ev}_1 : C^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$ compute the matrix-vector product
   $$\begin{pmatrix} \frac{d}{dx} & 0 \\ \mathrm{ev}_1 & -2\,\mathrm{id} \end{pmatrix} \begin{pmatrix} \sin x \\ 1 \end{pmatrix}.$$
   (d) Let $T : C^\infty(\mathbb{R}) \longrightarrow V$ be the linear map given by the matrix $\binom{\frac{d^2}{dx^2} - 4\,\mathrm{id}}{\mathrm{ev}_1}$. Re-write the equation $Ty = \mathbf{0}$ as an initial value problem (with y a function of x).

2. Let $W = \mathbb{C}(1,0,0)^T, W' = \mathbb{C}(1,1,2i)^T + \mathbb{C}(0,i,-1)^T \leq \mathbb{C}^3$. Show that the sum $W + W'$ is direct and that in fact $W + W' = \mathbb{C}^3$. Consider the natural isomorphism $\Phi : W \oplus W' \longrightarrow W + W' : \binom{\mathbf{w}}{\mathbf{w}'} \mapsto \mathbf{w} + \mathbf{w}'$. Find $\Phi^{-1}(0,0,1)^T$.

3. Let $W = \mathbb{R}(1 + x^3) + \mathbb{R}(x + x^3)$. Is the sum $\mathbb{R}[x]_{\leq 2} + W$ direct?

4. For $j = 1, 2$, let $W_j \subseteq M_{m2}(\mathbb{R})$ be the set of matrices with all non-zero entries in the j-th column. Show that W_j is a subspace of $M_{m2}(\mathbb{R})$ and that we have an internal direct sum $M_{m2}(\mathbb{R}) = W_1 \oplus W_2$.

5. In this question, we look at linear maps from $C^\infty(\mathbb{R})^2 \longrightarrow C^\infty(\mathbb{R})^2$ by considering 2×2-matrices with entries in $L(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}))$. Compute the product
$$\begin{pmatrix} \frac{d}{dx} & 3\,\mathrm{id} \\ \mathrm{id} & \frac{d}{dx} \end{pmatrix} \begin{pmatrix} 0 & \frac{d^2}{dx^2} \\ \mathrm{id} & \frac{d}{dx} \end{pmatrix}.$$

6. Show that the row matrix $(1 \quad x+x^2 \quad x^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$ defines a co-ordinate system on $\mathbb{R}[x]_{\leq 2}$. Recall that here we identify $L(\mathbb{R}, \mathbb{R}[x]_{\leq 2}) = \mathbb{R}[x]_{\leq 2}$.

7. Write down the block diagonal 3×3-matrix A corresponding to the direct sum $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \oplus (3)$. Find all $X \in M_{33}(\mathbb{R})$ such that $AX = XA$.

8. Consider the following linear maps of $\mathbb{F}$-spaces $T_1 : V_1 \longrightarrow W_1, T_2 : V_2 \longrightarrow W_2$. Show that the composite of $T_1 \oplus 0 : V_1 \oplus V_2 \longrightarrow W_1 \oplus V_2$ with $0 \oplus T_2 : W_1 \oplus V_2 \longrightarrow W_1 \oplus W_2$ is zero. Note that the two 0s above refer to different zero maps!

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9. Show that the map $T : \mathbb{R}[x] \oplus \mathbb{R}[x] \longrightarrow \mathbb{R} \oplus \mathbb{R}$ below is linear by showing it is represented by some matrix of linear maps.
$$T\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} f(1) + g(3) \\ f(2) + g'(2) \end{pmatrix}.$$

10. Show that the following map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]$ is linear by finding a row matrix in $\mathbb{R}[x]^3$ representing it.
$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a - b + (b + 2c)x + cx^3.$$

11. Show that the following map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is linear by finding a row matrix in $(\mathbb{R}^3)^3$ (i.e. a row matrix of 3-tuples) representing it.
$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b + c \\ a - c \\ 2a - b \end{pmatrix}.$$

12. The following map $T : \mathbb{R}^4 \longrightarrow M_{22}(\mathbb{R})$ is linear. Find a row matrix (of matrices!) which represents it.
$$T\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} \beta_1 - 2\beta_4 & \beta_3 + \beta_4 \\ \beta_2 & 5\beta_4 - \beta_3 \end{pmatrix}.$$

Some answers and hints:

1. a) Yes, $\binom{\cos(x+\pi/4)}{0} = \sqrt{2}\binom{\cos x}{1} - \sqrt{2}\binom{\sin x}{1}$. c) $\binom{\cos x}{\sin(1)-2}$ d) IVP $\frac{d^2y}{dx^2} - 4y = 0, y(1) = 0$.

3. No, $1 - x \in \mathbb{R}[x]_{\leq 2} \cap W$.

5. $\begin{pmatrix} 3\,\mathrm{id} & \frac{d^3}{dx^3} + 3\frac{d}{dx} \\ \frac{d}{dx} & 2\frac{d^2}{dx^2} \end{pmatrix}$

9. $\begin{pmatrix} \mathrm{ev}_1 & \mathrm{ev}_3 \\ \mathrm{ev}_2 & \mathrm{ev}_2 \circ \frac{d}{dx} \end{pmatrix}$

10. $\begin{pmatrix} 1 & -1 + x & 2x + x^3 \end{pmatrix}$

11. $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$

12.
$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}\right)$$