School of Mathematics & Statistics University of New South Wales

MATH2601: Higher Linear Algebra (2013,S2)
Problem Set 3: Linearity [1]

This problem set looks at material from lectures 7-9 on vector spaces, linear maps, and the algebra of linear maps. This is an in depth look at material that you would have looked at in MATH1241/1251. Old material includes the notions of linear combinations, subspaces, rotations and projections. New material includes the notions of sums/intersections of subpsaces, the vector space $L(V, W)$, reflections, composites of linear maps and isomorphisms.

1. For a set X, verify some of the vector space axioms for $\mathrm{Fun}(X, \mathbb{F})$.

2. Show that for any integer d, the set $\mathbb{R}[x]_d$ of polynomials of degree d is not a subspace of $\mathbb{R}[x]$.

3. Compute the following subspaces of $\mathbb{C}^3$:

$$\text{i)} \left(\mathbb{C}\begin{pmatrix}1\\2\\3\end{pmatrix} + \mathbb{C}\begin{pmatrix}2\\-1\\-1\end{pmatrix}\right) \cap \mathbb{C}\begin{pmatrix}3\\1\\2\end{pmatrix}, \quad \text{ii)} \left(\mathbb{C}\begin{pmatrix}1\\2\\3\end{pmatrix} + \mathbb{C}\begin{pmatrix}2\\-1\\-1\end{pmatrix}\right) \cap \mathbb{C}\begin{pmatrix}3\\1\\1\end{pmatrix}$$

.

4. Show that $\mathbb{R}(1 + x) + \mathbb{R}(2 - x) = \mathbb{R}[x]_{\leq 1}$.

5. Compute the following subspace of $\mathbb{C}[x]$: $\mathbb{C}[x]_{\leq 1} \cap (\mathbb{C}(2 - 3x + x^3) + \mathbb{C}(x + x^3))$.

6. Let $V', V'', W \leq V$. Prove that $(V' \cap W) + (V'' \cap W) \subseteq (V' + V'') \cap W$. Does the reverse inclusion always hold?

7. Let $V', V'' \leq V$. Show that if $V' \cup V''$ is a subspace of V then either $V' \subseteq V''$ or $V'' \subseteq V'$. This shows that in general, unions of subspaces are not subspaces.

8. Let $W, W' \leq V$. Show that if W'' is any subspace of V containing W and W' then also $W'' \supseteq W + W'$.

9. (For those who have done MATH1081). Fix an $\mathbb{F}$-space V and let $\mathcal{S}$ be the poset of subspaces of V ordered by inclusion. The previous question shows that for any $W, W' \in \mathcal{S}$, there is a least upper bound for $\{W, W'\}$ and it is $W + W'$. Show that $W \cap W'$ is a greatest lower bound. (HARD) If you are game, show that least upper bounds and greatest lower bounds exist for arbitrary subsets of $\mathcal{S}$.

10. Let X be a set and $V = \mathrm{Fun}(X, \mathbb{F})$. For any $x \in X$, show that the evaluation at x function $\mathrm{ev}_x : V \longrightarrow \mathbb{F} : f \mapsto f(x)$ is $\mathbb{F}$-linear.

11. Consider the vector space $V = L(\mathbb{R}[x]_{\leq 1}, \mathbb{R}[x]_{\leq 1})$. Show that the function $T : \mathbb{R}[x]_{\leq 1} \longrightarrow \mathbb{R}[x]_{\leq 1}$ defined by $T(a + bx) = ax$ for $a, b \in \mathbb{R}$ is linear and hence defines a vector in V. Show that T is not a linear combination of $\frac{d}{dx}, \mathrm{id} \in V$.

12. Consider the vector space $V = L(\mathbb{R}[x]_{\leq 1}, \mathbb{R})$ and note that we have the following vectors $\frac{d}{dx}, \mathrm{ev}_0 \in V$ (make sure you know why!). Show that the function $T : \mathbb{R}[x]_{\leq 1} \longrightarrow \mathbb{R} : a + bx \mapsto 3a - 4b$ is linear by showing it is a linear combination of $\frac{d}{dx}$ and ev_0.

[1] by Daniel Chan

13. Let $P_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be orthogonal projection onto $\binom{1}{1}$ and $P_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be orthogonal projection onto $\binom{1}{-1}$. Find matrices representing P_1, P_2. Determine the sum $P_1 + P_2$ by: i) finding the matrix representing it and, ii) analysing an appropriate geometric picture.

14. Find the matrix $A \in M_{33}(\mathbb{R})$ representing reflection about the plane $2x - y + 3z = 0$. Explain geometrically why $A^2 = I_3$ and verify this by matrix multiplication.

15. Let $T_1, T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be reflections about the x-axis and the line $x - y = 0$. Show that the composite $T_1 \circ T_2$ is a rotation and determine which one it is.

16. Let $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be anti-clockwise rotation about $\binom{0}{0}$ about angle θ and $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be reflection about the x-axis. Describe geometrically the composite $R_\theta^{-1} \circ T \circ R_\theta$.

17. In this question, we consider linear maps from $C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$. Simplify the linear map

$$\left(4\frac{d}{dx} + 5\,\mathrm{id}\right) \circ \left(\frac{d}{dx} - 2\,\mathrm{id}\right) + 2\frac{d}{dx}.$$

18. Show that the transpose map $(\cdot)^T : M_{mn}(\mathbb{F}) \longrightarrow M_{nm}(\mathbb{F}) : A \mapsto A^T$ is an isomorphism of $\mathbb{F}$-spaces.

19. Let $C : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear map associated to the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Show that C defines a co-ordinate system on $\mathbb{R}^2$.

20. Consider the linear map $T : \mathbb{R}[x]_{\leq d} \longrightarrow \mathbb{R}^{d+1}$ defined by $Tp = (p(0), p'(0), \dots, p^{(d)}(0))^T$ (linearity will be clear from lecture 11 material on the linearity of the matrix-vector product). Show that T is an isomorphism and compute the induced co-ordinate system $T^{-1} : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}[x]_{\leq d}$.

21. Fix distinct reals t_1, t_2, t_3 and consider the linear map $T : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}^3 : p \mapsto (p(t_1), p(t_2), p(t_3))^T$. Show that T is an isomorphism. Can you describe the co-ordinate system T^{-1}. You may wish to try this question again after lecture 12.

22. Let $T : V \longrightarrow W, S : U \longrightarrow V$ be isomorphisms of $\mathbb{F}$-spaces. Show that $T \circ S$ is also an isomorphism.

23. Consider a linear map $N : V \longrightarrow V$ such that $N^n = 0$ for some positive integer n (such linear maps are said to be nilpotent). Show that $\mathrm{id} + N$ is an isomorphism. Hint: use the sum of a geometric progression formula.

Some answers and hints:

2. $\mathbb{R}[x]_d$ is not closed under addition and does not contain 0.

3. i) $\mathbb{C}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, ii) 0.

4. Show that given any $a, b \in \mathbb{R}$, you can always solve $\alpha(1 + x) + \beta(2 - x) = a + bx$.

5. Solve $\alpha(2 - 3x + x^3) + \beta(x + x^3) = a + bx$ for $\alpha, \beta, a, b \in \mathbb{C}$ to obtain the intersection as $\mathbb{C}(2 - 4x) = \mathbb{C}(1 - 2x)$.

6. Reverse inclusion fails on setting subspaces to be non-parallel lines in $\mathbb{R}^2$.

11. To show that T is not a linear combination, show that you cannot solve $T = \alpha\frac{d}{dx} + \beta\,\mathrm{id}$ for $\alpha, \mathrm{id} \in \mathbb{R}$ by comparing co-efficients of $T(a + bx) = (\alpha\frac{d}{dx} + \beta\,\mathrm{id})(a + bx)$

12. In fact $T = 3\mathrm{ev}_0 - 4\frac{d}{dx}$.

13. P_1, P_2 are represented by matrices

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

i) Adding matrices see $P_1 + P_2$ represented by I_2. ii) $P_1 + P_2 = \mathrm{id}_{\mathbb{R}^2}$ expresses the fact that any vector can be expressed as the sum of a "component" parallel to $\binom{1}{1}$ and an orthogonal "component" parallel to $\binom{1}{-1}$.

14. Normal is $(2, -1, 3)^T$ so $A = \frac{1}{7}\begin{pmatrix} 3 & 2 & -6 \\ 2 & 6 & 3 \\ -6 & 3 & -2 \end{pmatrix}$

15. T_1, T_2 represented by matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence $T_1 \circ T_2$ represented by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so must be clock-wise rotation by $\frac{\pi}{2}$. You can also do this question geometrically.

16. $R_\theta^{-1} \circ T \circ R_\theta$ is reflection about the line L obtained by rotating the x-axis clockwise about $\binom{0}{0}$ through angle θ.

17. $4\frac{d^2}{dx^2} - \frac{d}{dx} - 10\,\mathrm{id}$.

18. Check the transpose is linear and bijectivity follows from the fact that transpose is its own inverse.

19. Since C is linear, you need only check bijectivity or equivalently invertibility e.g. by examining the determinant.

20. $T^{-1}(\alpha_0, \ldots, \alpha_d)^T = \alpha_0 + \alpha_1 x + \frac{\alpha_2}{2}x^2 + \ldots + \frac{\alpha_d}{d!}x^d$.

21. If you did MATH1151, this is a reformulation of the Lagrange interpolation question.

$$T^{-1}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = y_1\frac{(t - t_2)(t - t_3)}{(t_1 - t_2)(t_1 - t_3)} + y_2\frac{(t - t_1)(t - t_3)}{(t_2 - t_1)(t_2 - t_3)} + y_3\frac{(t - t_2)(t - t_1)}{(t_3 - t_2)(t_3 - t_2)}$$