School of Mathematics & Statistics University of New South Wales

MATH2601: Higher Linear Algebra (2013,S2)
Problem Set 2: The determinant [1]

This problem set covers material from lectures 5-6. We will assume you all know how to compute the determinant of a square matrix. The purpose of this chapter of work is to make sure you understand the theory. In particular, you will gain an appreciation of how the permutation group S_n clarifies our understanding of the determinant. Recall that for us, the definition of the determinant uses S_n and if you are asked to prove results from definitions, you will be required to use this defintion as opposed to the one you were given in first year.

1. Let $\sigma = [2143] \in S_4$. Compute the sign of σ by writing σ as a product of 2-cycles and by computing $\sigma.\Delta_4$.

2. Prove from definitions that the determinant is multi-linear in the rows.

3. Compute that $(1, 2)$-minor and cofactor of the matrix
$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

4. Consider the following matrices
$$A = \begin{pmatrix} 1 & 0 & a \\ 2 & 0 & b \\ 1 & 1 & c \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & a+2 \\ 2 & 0 & b-1 \\ 1 & 1 & c+4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2a & 2a \\ 2 & b & b \\ 1 & c+1 & c \end{pmatrix}.$$
If $\det A = 5$, find $\det B, \det C$ using properties of the determinant or otherwise.

5. Prove from definitions that $\det(A^T) = \det(A)$.

6. Prove from definitions that if a column of A is $\mathbf{0}$ then $\det(A) = 0$.

7. Let $A \in M_{nn}(\mathbb{F})$ and B be the matrix obtained from A by adding the first row to the last. Prove from definitions that $\det(B) = \det(A)$.

8. Let $A = (a_{ij})_{ij} \in M_{nn}(\mathbb{F})$ and $\sigma \in S_n$. If $B = (a_{i\sigma(j)})_{ij}$ is the matrix with columns permuted according to σ^{-1}, show that $\det(B) = \operatorname{sgn}(\sigma)\det(A)$.

9. Compute the classical adjoint of
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$
and verify Cramer's rule in this case.

10. (HARD) Let $A \in M_{mm}(\mathbb{F}), B \in M_{mn}(\mathbb{F}), C \in M_{nn}(\mathbb{F})$ and 0 be the $n \times m$ zero matrix. Show that the determinant of the $(m+n) \times (m+n)$-matrix $(d_{ij}) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ is $\det(A)\det(C)$.

11. Let $A \in M_{nn}(\mathbb{Q})$ be such that there is a complex matrix $B \in M_{nn}(\mathbb{C})$ with $AB = I_n$. Use Cramer's rule to give a clean proof that B has rational entries.

[1] by Daniel Chan

12. You are given a matrix $A \in M_{33}(\mathbb{Z})$ with determinant 2. Your friend tells you the solution to $A\mathbf{v} = \begin{pmatrix} 4 \\ -8 \\ 16 \end{pmatrix}$ is $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$. How do you know s/he is lying?

Some answers and hints:

1. $\sigma = (12)(34)$ so $\operatorname{sgn}(\sigma) = (-1)^2 = 1$. Also $\sigma.\Delta_4 = \Delta_4$.

3. $A(1,2) = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ so $C_{12} = (-1)^3 \det A(1,2) = -7$.

4. $\det B = 5 + 3 = 8, \det C = 2 \det A = 10$.

9. $\mathrm{Adj}A = \begin{pmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ and $A(\mathrm{Adj}A) = 2I_3$.

10. This question is for high achieving students who wish to be extended. The key is to show terms $\prod d_{i\sigma(i)}$ are zero unless, unless σ is a permutation which shuffles the elements of $J = \{1, \ldots, m\}$ and $J' = \{m+1, \ldots, m+n\}$ but does not send an element of J to J' or vice versa.

11. See example in lectures.

12. Use Cramer's rule to see that the entries of A^{-1} are all rational with denominator at most 2.