School of Mathematics & Statistics University of New South Wales

MATH2601: Higher Linear Algebra (2013,S2)
Problem Set 10: Spectral theory [1]

This problem set covers lectures 38-42 on spectral theory for endomorphisms of inner product spaces. For linear maps which are particularly nice with respect to the inner product, namely, the normal operators, diagonalisation theory works extremely well, and in particular, you can always unitarily diagonalise normal matrices. This has numerous geometric applications. In this problem set $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ unless otherwise specified.

1. Show that the linear map $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ associated to the matrix
$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$
preserves inner products.

2. Consider the standard inner product $(f|g) = \int_0^1 f(x)g(x)dx$. Find an orthonormal co-ordinate system for $\mathbb{R}1 + \mathbb{R}x$.

3. Let V be an inner product space and $T, S : V \longrightarrow V$ be (not necessarily linear) isometries. Show that $T \circ S$ is also an isometry.

4. Consider matrices A, B, C such that A, B are unitarily similar as are B, C. Show that A, C are unitarily similar. If you have done MATH1081, show that being unitarily similar is an equivalence relation.

5. Find all upper triangular orthogonal matrices.

6. Show that the following matrices are similar but not orthogonally similar.
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

7. Let $A \in M_{33}(\mathbb{C})$ be a matrix with $\text{tr}A = 5$. Show that A is not unitary.

8. Show that the eigenvalues of any unitary hermitian matrix A must be ± 1. Find an example of such a matrix.

9. Let $A \in M_{33}(\mathbb{C})$ be an hermitian matrix with eigenvalues $2, 3$. If $E_2 = \mathbb{C}(1, 0, 1)^T$ find A.

10. Unitarily diagonalise the following hermitian matrices.
$$A = \begin{pmatrix} -1 & -3i \\ 3i & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

11. Consider the quadratic form $Q(\mathbf{x}) = x_1^2 - x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_2x_3$. Find a real symmetric matrix $A \in M_{33}(\mathbb{R})$ such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

12. For the following quadratic forms, find their ranks and signatures, and state whether they are degenerate or not.

[1] by Daniel Chan

(a) $Q(\mathbf{x}) = x_1^2 - 2x_1x_2 + x_2^2$.

(b) $Q(\mathbf{x}) = 9x_1^2 - 8x_1x_2 + 7x_2^2$.

13. Determine the nature of the following conics (i.e. ellipse or hyperbola) and find their principal axes. Determine the points which are closest and furthest from $(0,0)$ and the distance to $(0,0)$. Give a rough sketch.

    (a) $2x_1^2 + 5x_2^2 - 4x_1x_2 = 54$.

    (b) $x_1^2 + x_2^2 + 10x_1x_2 = 150$.

14. Determine the nature of the following quadrics (i.e. ellipsoid or hyperboloid of 1 or 2 sheets) and find their principal axes. Determine the points which are closest and furthest from $(0,0)$ and the distance to $(0,0)$. Give a rough sketch.

    (a) $8x_1^2 + 7x_2^2 + 8x_3^2 + 2x_1x_2 + 2x_2x_3 = 144$.

    (b) $-6x_1^2 - x_2^2 + 2x_3^2 + 4x_1x_2 - 2x_1x_3 + 4x_2x_3 = 12$.

15. Let $T : V \longrightarrow V$ be a semisimple operator and $W \leq V$ be T-invariant. Show that W has a T-invariant vector space complement (in V).

16. In this question we compute that eigenvectors and eigenvalues of matrices of the following form

$$A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}, \quad \text{for some } a, b, c \in \mathbb{C}.$$

Find the eigenvectors of A by showing A commutes with

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{for some } a, b, c \in \mathbb{C}.$$

Hence find the eigenvalues of A. Can you generalise this result to $n \times n$-matrices?

17. Which of the following matrices are normal and unitarily diagonalise those.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

18. Find all rotation matrices whose first two columns are given by

$$\frac{1}{25} \begin{pmatrix} 20 & 0 \\ -9 & 20 \\ 12 & 15 \end{pmatrix}.$$

19. Show that the matrix

$$A = \frac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

defines a reflection and determine the line of reflection.

20. Which of the following 3 matrices is a rotation matrix. For the rotation matrices, compute the axis of rotation and the angle of rotation.

$$A = \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \quad B = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}, \quad C = \frac{1}{3} \begin{pmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

21. Let $A \in M_{22}(\mathbb{R})$ be a matrix with eigenvector $\binom{1}{-i}$ of eigenvalue $1+2i$. Determine A.

22. Let $\zeta = e^{2\pi i/3}$. Suppose $A \in M_{33}(\mathbb{R})$ is a rotation matrix such that ζ is an eigenvalue with eigenspace $E_\zeta = \mathbb{C}(1, \zeta, \zeta^2)^T$. Determine the angle of rotation and the axis of rotation.

23. Find two singular value decompositions for

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{pmatrix}.$$

Selected partial answers and hints

1. Check $U^*U = I_2$.
2. $\{1, 2\sqrt{3}(x - \frac{1}{2})\}$.
5. They are the diagonal matrices with ± 1 on the diagonal.
6. Diagonalise B to see matrices are similar. Not orthogonally similar as A is symmetric but B is not.
7. Use triangle inequality and the fact that the trace is the sum of the e-values.
8. Examples are any matrix unitarily similar to a diagonal matrix with ± 1 on the diagonal.
9. $A = \begin{pmatrix} \frac{5}{2} & 0 & -\frac{1}{2} \\ 0 & 3 & 0 \\ -\frac{1}{2} & 0 & \frac{5}{2} \end{pmatrix}$.
10. $A = U^*DU$ where $D = (2) \oplus (-4)$ and $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$.

$(0) \oplus (3) \oplus (3) = U^*BU$ where $U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$

11. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & -2 & 3 \end{pmatrix}$.
12. a) Degenerate, rank 1, signature 1. b) non-degenerate, rank 2 signature 2.
13. a) Ellipse with principal axes in direction $\frac{1}{\sqrt{5}}\binom{2}{1}$, $\frac{1}{\sqrt{5}}\binom{-1}{2}$. Closest points are $\pm\frac{3}{\sqrt{5}}\binom{-1}{2}$ which are 3 units from $(0,0)$ and the furthest points are $\pm\sqrt{\frac{6}{5}}\binom{6}{3}$ which are $3\sqrt{6}$ units from $(0,0)$.

b) Hyperbola with principal axes in direction $\frac{1}{\sqrt{2}}\binom{1}{1}$, $\frac{1}{\sqrt{2}}\binom{-1}{1}$. Closest points are $\pm\frac{5}{\sqrt{2}}\binom{1}{1}$ which are 5 units from $(0,0)$. No furthest points.

14. a) Elliposoid with principal axes in direction $\frac{1}{\sqrt{3}}(1,1,1)^T$, $\frac{1}{\sqrt{2}}(-1,0,1)^T$, $\frac{1}{\sqrt{6}}(1,-2,1)^T$. Closest points are $\pm\frac{4}{\sqrt{3}}(1,1,1)^T$ which are 4 units from $(0,0,0)^T$ and the furthest points are $\pm(2,-4,2)^T$ which are $2\sqrt{6}$ units from $(0,0,0)^T$.

b) Hyperboloid of 2 sheets with principal axes in direction $\frac{1}{\sqrt{30}}(5,-2,1)^T$, $\frac{1}{\sqrt{6}}(-1,-2,1)^T$, $\frac{1}{\sqrt{5}}(0,1,2)^T$. Closest points are $\pm\frac{2}{\sqrt{5}}(0,1,2)^T$ which are 2 units from $(0,0,0)^T$.

15. Use the fact that $T|_W$ is semisimple so W is a sum of eigenspaces.
16. The eigenvectors of A are the eigenvectors of B, namely, the non-zero scalar multiples of

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ e^{2\pi i/3} \\ e^{-2\pi i/3} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ e^{-2\pi i/3} \\ e^{2\pi i/3} \end{pmatrix}.$$

The corresponding e-values are $a+b+c, a+e^{2\pi i/3}b+e^{-2\pi i/3}c, a+e^{-2\pi i/3}b+e^{2\pi i/3}c$. You can also do this question by noting A is a polynomial function of B.

17. B is normal and $(1+i)\oplus(1-i) = U^*BU$ where $U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$.

18.
$$\frac{1}{25}\begin{pmatrix}20 & 0 & -15\\ -9 & 20 & -12\\ 12 & 15 & 16\end{pmatrix}.$$

19. Check $A^TA = I, \det(A) - -1$. Line of reflection is $y = \frac{1}{2}x$.

20. A not orthog because first two columns not orthog. B not rotation because $\det(B) = -1$. C is a rotation matrix because $C^TC = I, \det(C) = 1$. The axis of rotation is in direction $(0,1,1)^T$. Angle of rotation is $\cos^{-1}(-\frac{1}{3})$.

21. $A = \begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$.

22. Axis of rotation is in direction $(1,1,1)^T$ and the angle of rotation is $\frac{2\pi}{3}$.

23. $A = U_lDU_r^T$ where

$$U_l = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -1\end{pmatrix}, D = \begin{pmatrix}\sqrt{10} & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}, U_r = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}.$$

We may swap the second and third columns of U_l and still have an SVD. We can also negate the second column of U_r and/or the second and third columns of U_l. There are other possibilities.