MATH2601: Higher Linear Algebra (2013,S2)
Problem Set 1: Permutations, groups & fields [1]

This deals with material from lectures 1-4.

1. Determine which of the following are permutations in S_4: i) [4322], ii) [2134], iii) [2341].
2. For the permutations σ in the previous question, compute σ^{-1} & σ^2.
3. Compute the product permutation $\sigma\tau$ if $\sigma = [231], \tau = [213]$.
4. Write [2314] as a product of 2-cycles in two different ways.
5. Consider the permutation $\sigma = [2431]$ & matrix $A = (a_{ij}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$. What is the matrix $(a_{i\sigma(j)})_{ij}$?
6. Let $\sigma = (1\ 2)$. For $(a_{ij}) \in M_{22}(\mathbb{F})$, write out the 2×2-matrix $(a_{\sigma(i)\sigma(j)})_{ij}$ explicitly.
7. Let σ be a permutation in S_n for some $n \in \mathbb{Z}^+$. Show there is some positive integer m such that $\sigma^m = \mathrm{id}$.
8. Make sure you know the group identities for the groups $\mathbb{Z}, \mathbb{R}, \mathbb{R}^\times, S_n, GL_n(\mathbb{C})$ and can verify that they are indeed identities for the group.
9. Let $(G, *)$ be a group. For $g, h \in G$, simplify the expression $g^{-1} * h * g^2 * g^{-3} * g^2 * h$. Can you further simplify the expression if the group is known to be commutative?
10. Note that for any $m, n \in \mathbb{N}$, matrix addition $+$ is a binary operation on $M_{mn}(\mathbb{R})$. Show that $(M_{mn}(\mathbb{R}), +)$ is an abelian group.
11. Let $(G, *)$ be a group. Prove the left cancellation law, that is, if $h * g = h * g'$ then $g = g'$ for all $g, g', h \in G$.
12. Consider the set of positive reals $\mathbb{R}^+$. Show that the usual addition $+$ is an associative, commutative binary operation on $\mathbb{R}^+$. Is $(\mathbb{R}^+, +)$ a group? Given $\alpha \in \mathbb{R}^+$ is the map $+\alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ : \beta \mapsto \beta + \alpha$ bijective?
13. Consider the set of real polynomials $\mathbb{R}[x]$ equipped with the binary operation $*$ = usual multiplication of polynomials. Is $(\mathbb{R}[x], *)$ a group? Does the cancellation law hold?
14. Let $(G, *)$ be a group. We consider a new binary operation $\dagger$ on G defined by $g \dagger h = h * g$ for all $g, h \in G$. Show that $(G, \dagger)$ is also a group and compute some examples of $g \dagger h$ for $G = \mathbb{R}, \mathbb{C}^\times, GL_n(\mathbb{R})$.
15. For $\sigma = (2\ 3)$ and the real polynomial $f(x_1, x_2, x_3) = x_1^2 x_3 - 5x_2 x_3^4$, find $(\sigma.f)(x_1, x_2, x_3)$.
16. Which of the following polynomials in x_1, x_2, x_3 are not symmetric: i) $x_1x_2 + x_2x_3 + x_3x_1$, ii) $x_1 + x_2$. Prove from definitions that the non-symmetric polynomial is indeed not symmetric.
17. Show that the following function defines a group action of the group $G = GL_n(\mathbb{C})$ on the set $X = M_{nn}(\mathbb{C})$: for $U \in GL_n(\mathbb{C}), A \in M_{nn}(\mathbb{C})$ we define $U.A = UAU^{-1}$. Later, we will say that two matrices $A, B \in M_{nn}(\mathbb{C})$ are *similar* if there is a matrix $U \in GL_n(\mathbb{C})$ such that $B = U.A$.

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18. Let G be a group acting on a set X. For $x, x' \in X$, we will write $x \sim x'$ iff there exists some $g \in G$ such that $g.x = x'$. Show that for all $x, x', x'' \in X$ we have

    (a) $x \sim x$.

    (b) if $x \sim x'$ then $x' \sim x$.

    (c) if $x \sim x'$ and $x' \sim x''$ then $x \sim x''$.

    If you have done MATH1081, this says that the relation $\sim$ is an equivalence relation on X.

19. Let $\mathbb{Q}[i]$ be the set of complex numbers with rational real and imaginary part. Show the usual addition and multiplication of complex numbers also defines a binary operation on $\mathbb{Q}[i]$. $\mathbb{Q}[i]$ is a field. Verify some of the field axioms.

20. Let $\mathbb{F}$ be the field $\mathbb{R}(x)$. Show that $\frac{x^3y - xy^3 + 7y^3}{x^2} \in \mathbb{F}[y]$ by expressing it as a polynomial in y with co-efficients in $\mathbb{F}$.

Some answers and hints:
1. ii) and iii).
2. If $\sigma = [2341]$ then $\sigma^{-1} = [4123], \sigma^2 = [3412]$.
3. $\sigma\tau = [321]$.
4. $[2314] = (23)(13) = (23)(13)(12)(12)$.
5. $(a_{i\sigma(j)})_{ij} = \begin{pmatrix} 2 & 4 & 3 & 1 \\ 6 & 8 & 7 & 5 \end{pmatrix}$.
6. $\begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}$
7. First show there are distinct $i, j \in \mathbb{Z}$ such that $\sigma^i = \sigma^j$.
8. $e_{\mathbb{Z}} = 0, e_{\mathbb{R}} = 0, e_{\mathbb{R}^\times} = 1, e_{S_n} = \mathrm{id}, e_{GL_n} = I_n$.
9. $g^{-1} * h * g * h$. If commutative this is h^2.
12. $(\mathbb{R}^+, +)$ is not a group as it has no group identity.
13. $(\mathbb{R}[x], *)$ is not a group as group inverses do not always exist. However, the cancellation law does hold.
15. $(\sigma.f)(x_1, x_2, x_3) = x_1^2 x_2 - 5x_3 x_2^4$
16. ii) is not symmetric (though it is a symmetric function of x_1, x_2).