Composite of linear maps

Aim lecture: Linear maps can be combined via composition. We introduce this concept here.

Prop

Let $S, T : V \longrightarrow V', S', T' : V' \longrightarrow V''$ be $\mathbb{F}$-lin maps.

1. The composite fn $T' \circ T : V \longrightarrow V''$ defined by $(T' \circ T)\mathbf{v} = T'(T\mathbf{v})$ is $\mathbb{F}$-linear.
2. $(S' + T') \circ (S + T) = S' \circ S + S' \circ T + T' \circ S + T' \circ T$ as maps from $V \longrightarrow V''$. (Distributive law)
3. For $\beta \in \mathbb{F}$ we have $(\beta T') \circ T = \beta(T' \circ T) = T' \circ (\beta T)$.

Proof. 1) Follows from checking axioms whilst 2) & 3) are computations. Note similarity with matrix arithmetic. We prove 2). For any $\mathbf{v} \in V$ we have
$[(S' + T') \circ (S + T)]\mathbf{v} = (S' + T')[(S + T)\mathbf{v}] = (S' + T')(S\mathbf{v} + T\mathbf{v}) =$
$S'(S\mathbf{v} + T\mathbf{v}) + T'(S\mathbf{v} + T\mathbf{v}) = S'(S\mathbf{v}) + S'(T\mathbf{v}) + T'(S\mathbf{v}) + T'(T\mathbf{v}) =$
$(S' \circ S)\mathbf{v} + (S' \circ T)\mathbf{v} + (T' \circ S)\mathbf{v} + (T' \circ T)\mathbf{v} = [S' \circ S + S' \circ T + T' \circ S + T' \circ T]\mathbf{v}$
so 2) follows.

Example of composite linear maps

E.g. Show that $T : C^{\infty}(\mathbb{R}) \longrightarrow C^{\infty}(\mathbb{R}) : f(x) \mapsto \frac{d^2 f}{dx^2} + 3f(x)$ is $\mathbb{R}$-linear.

A $\frac{d^2}{dx^2} = \frac{d}{dx} \circ \frac{d}{dx}$ is lin $\because$

Hence T is lin being the

Geometric example

E.g Let $P_{\mathbf{u}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be projn onto $\mathbf{u} = \frac{1}{\sqrt{2}}(1,1,0)^T$. Show $P_{\mathbf{u}}^2 \stackrel{\text{def}}{=} P_{\mathbf{u}} \circ P_{\mathbf{u}} = P_{\mathbf{u}}$ geometrically & algebraically. Hence show algebraically the reflection $T = \text{id} - 2P_{\mathbf{u}}$ satisfies $T^2 = \text{id}$.

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Geometrically, for any $\mathbf{w} \in \mathbb{R}\,\mathbf{u}$ we have $P_{\mathbf{u}}\mathbf{w} = \mathbf{w}$. But for any $\mathbf{v} \in \mathbb{R}^3$ we have $P_{\mathbf{u}}\mathbf{v} \in \mathbb{R}\,\mathbf{u}$ so

$P_{\mathbf{u}}^2\mathbf{v} =$

Composites & matrix product

Prop

Let $A \in M_{lm}(\mathbb{F}), B \in M_{mn}(\mathbb{F})$ so their associated linear maps are $T_A : \mathbb{F}^m \longrightarrow \mathbb{F}^l$ & $T_B : \mathbb{F}^n \longrightarrow \mathbb{F}^m$. Then

$$T_A \circ T_B = T_{AB}.$$

In other words, composites correspond to matrix multn.

Proof.

Rem Actually, the unusual definition of matrix multn was designed precisely to make this formula work.

Rotations

Rotation anti-clockwise through angle θ about $\binom{0}{0} \in \mathbb{R}^2$ is given by the 2×2-matrix

Reminder on bijectivity

Prop-Defn

Recall that a function $f : X \longrightarrow Y$ is

1. *surjective* or *onto* if for any $y \in Y$ the eqn $f(x) = y$ always has a soln $x \in X$.
2. *injective* or *1-1* if for any $y \in Y$ the eqn $f(x) = y$ has at most one soln $x \in X$.
3. *bijective* if for any $y \in Y$, there is a unique soln $x \in X$ to $f(x) = y$ i.e. f is surjective & injective. This occurs iff f is invertible, in which case $f^{-1}(y) = x$.

Recall also that $f \circ f^{-1} = \text{id}_Y, f^{-1} \circ f = \text{id}_X$ and indeed, these equations can be used to define invertibility & the inverse function.

Isomorphisms of vector spaces

Prop-Defn

An *isomorphism* of vector spaces is a bijective linear map $T : V \longrightarrow V'$. Given such an isomorphism, the inverse $T^{-1} : V' \longrightarrow V$ is also linear. We say V & V' are *isomorphic* and write $V \simeq V'$.

Proof. Good ex.

E.g. Let $A \in M_{nn}$ be an invertible matrix. The assoc lin map $T_A : \mathbb{F}^n \longrightarrow F^n$ is invertible with inverse $T_{A^{-1}}$.
Indeed for any $\mathbf{w} \in \mathbb{F}^n$, the unique soln to $\mathbf{w} = T_A\mathbf{v} = A\mathbf{v}$ is $\mathbf{v} = A^{-1}\mathbf{w} = T_{A^{-1}}\mathbf{w}$.

Defn

A *co-ordinate system* on an $\mathbb{F}$-space V is an isomorphism of the form $C : \mathbb{F}^n \longrightarrow V$.

Philosophy of isomorphisms & co-ordinate systems

In the geometric examples above, we often confused the vector space V of geometric vectors in 3-dim space with $\mathbb{R}^3$ it is important to fully understand precisely what permits us to make this identification & the subtleties involved.
Given a triple of non-coplanar vectors in V we can put a co-ordinate system on V i.e. find a co-ord system $C : \mathbb{R}^3 \longrightarrow V$. Bijectivity of C means that every geometric vector corresponds (via C) to a triple in $\mathbb{R}^3$.
To *calculate* with geom vectors, we can use C^{-1} to pass to the corresponding co-ordinates in $\mathbb{R}^3$, then calculate using these co-ord, then use C to pass back to geom vectors.
If calculations only involve addn & scalar multn then linearity of C, C^{-1} ensures we are fine. If the calculations involve other operations e.g. dot product, then we may need our co-ord system to be special for this to work.
One subtlety is that there are usually many co-ord systems and a theme of linear algebra is *they are not all the same*! Some are better than others.
Isomorphisms allow one more generally, to identify two vector spaces in the same way we identify V with $\mathbb{R}^3$.