Defn of linear maps

Prop-Defn

Let $V, V' = \mathbb{F}$-spaces. An $\mathbb{F}$-*linear map* or $\mathbb{F}$-*linear transformation* is a function $T : V \longrightarrow V'$ which satisfies any of the following equiv conditions.

1. For any $\beta \in \mathbb{F}, \mathbf{v}, \mathbf{v}' \in V$ we have a) $T(\mathbf{v} + \mathbf{v}') = T\mathbf{v} + T\mathbf{v}'$ & b) $T(\beta\mathbf{v}) = \beta T\mathbf{v}$ i.e. T preserves sums & scalar multiples.
2. For any $\beta \in \mathbb{F}, \mathbf{v}, \mathbf{v}' \in V$ we have $T(\beta\mathbf{v} + \mathbf{v}') = \beta T\mathbf{v} + T\mathbf{v}'$.
3. For any $\beta_1, \ldots, \beta_r \in \mathbb{F}, \mathbf{v}_1, \ldots, \mathbf{v}_r \in V$ we have $T(\sum_i \beta_i \mathbf{v}_i) = \sum_i \beta_i T\mathbf{v}_i$ i.e. T preserves lin combns.

Proof. Easy ex you should know how to do. E.g. Given 1) we have
$T(\beta\mathbf{v} + \mathbf{v}') = T(\beta\mathbf{v}) + T\mathbf{v}' = \beta T\mathbf{v} + T\mathbf{v}'$
so 2) follows.

E.g. Given any $A \in M_{mn}(\mathbb{F})$ we have an associated lin map $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ defined by $T_A(\mathbf{v}) = A\mathbf{v}$.
CHECK: With above notn,
$T(\beta\mathbf{v} + \mathbf{v}') = A(\beta\mathbf{v} + \mathbf{v}') = A(\beta\mathbf{v}) + A\mathbf{v}' = \beta A\mathbf{v} + A\mathbf{v}' = \beta T\mathbf{v} + T\mathbf{v}'$.

Restricting linear maps

Fact-Defn

Let $T : V \longrightarrow V'$ be a lin map & $W \leq V$. The *restriction* of T to W is fn $T|_W : W \longrightarrow V' : \mathbf{w} \mapsto T\mathbf{w}$. It is also linear. Consider the *image* of W under T, i.e. $T(W) = \{T\mathbf{w} | \mathbf{w} \in W\}$. If there's a subspace $W' \leq V'$ with $T(W) \subseteq W'$, then we may consider the function $T|_W : W \longrightarrow W' : \mathbf{w} \mapsto T\mathbf{w}$ which is also linear. Sometimes we write T instead of $T|_W$ if no confusion is likely.

Why?

E.g. 1 $T = \frac{d}{dx} : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ is $\mathbb{R}$-linear since $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}, \frac{d}{dx}(\beta f) = \beta\frac{df}{dx}$ for $f, g \in C^\infty(\mathbb{R}), \beta \in \mathbb{R}$.

E.g. 2 Recall that $\mathbb{R}[x] \leq C^\infty(\mathbb{R})$ (after identifying real poly with poly fns) so we obtain by restriction new linear maps $\frac{d}{dx} : \mathbb{R}[x] \longrightarrow C^\infty(\mathbb{R})$ & also $\frac{d}{dx} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ since derivatives of polys are polys.

E.g. 3 Given any $\mathbb{F}$-space V, the *identity map* $\text{id} = \text{id}_V : V \longrightarrow V : \mathbf{v} \mapsto \mathbf{v}$ is $\mathbb{F}$-lin. For any subspace $W \leq V$ we get linearity of the *inclusion map* $\text{id}|_W : W \longrightarrow V : \mathbf{w} \mapsto \mathbf{w}$.

Sums & scalar multiples

Prop-Defn

Let $S, T : V \longrightarrow W$ be $\mathbb{F}$-lin maps & $\beta \in \mathbb{F}$.

1. The *sum* of S & T is the lin map $S + T : V \longrightarrow W : \mathbf{v} \mapsto S\mathbf{v} + T\mathbf{v}$.
2. We also have a lin map $\beta T : V \longrightarrow W : \mathbf{v} \mapsto \beta T\mathbf{v}$ which is called a *scalar multiple* of T.

Proof. We check linearity

E.g. Let $A, B \in M_{mn}$ and consider assoc lin maps $T_A, T_B : \mathbb{F}^n \longrightarrow \mathbb{F}^m$. Then for $\mathbf{v} \in \mathbb{F}^n$
$(T_A + T_B)\mathbf{v} = T_A\mathbf{v} + T_B\mathbf{v} = A\mathbf{v} + B\mathbf{v} = (A + B)\mathbf{v} = T_{A+B}\mathbf{v}$
i.e. sum of lin maps corresponds to the matrix sum $A + B$.
Sim **ex** for $\beta \in \mathbb{F}$, $\beta T_A = T_{\beta A}$.

Vector Space structure

Theorem

Let $V, W = \mathbb{F}$-spaces. Let $L(V, W)$ denote the set of all $\mathbb{F}$-lin maps from V to W. Then with the sum & scalar multn defined above, $L(V, W)$ is an $\mathbb{F}$-space.

Proof. Omitted, but an easy long check of axioms. E.g. given $\beta \in \mathbb{F}, T, S \in L(V, W)$ we check the vector distributive law $\beta(S + T) = \beta S + \beta T$ by checking the two fns have the same output for every possible input $\mathbf{v} \in V$,

$$(\beta(S + T))\mathbf{v} = \beta((S + T)\mathbf{v}) = \beta(S\mathbf{v} + T\mathbf{v}) = \beta S\mathbf{v} + \beta T\mathbf{v} = (\beta S)\mathbf{v} + (\beta T)\mathbf{v} = (\beta S + \beta T)\mathbf{v}.$$

E.g. Consider the linear maps $\frac{d}{dx}, \mathrm{id} \in L(\mathbb{R}[x]_{\leq 1}, \mathbb{R}[x]_{\leq 1})$. Show $T : \mathbb{R}[x]_{\leq 1} \longrightarrow \mathbb{R}[x]_{\leq 1} : a + bx \mapsto (2b - a) - bx$ is linear by showing it is a lin combn of $\frac{d}{dx}, \mathrm{id}$.

Projection onto vectors

Let $\mathbf{u}$ be a geometric vector in 3-dim space V. We recall the (orthogonal) *projection* of a vector $\mathbf{v}$ onto $\mathbf{u}$ can be obtained by dropping a perpendicular from $\mathbf{v}$ to the line $\mathbb{R}\,\mathbf{u}$ as in the picture below. The resulting projection map $P_{\mathbf{u}} : V \longrightarrow V$ can be seen to be linear geometrically.

Projection via co-ordinates

If we identify the space of geometric vectors with $\mathbb{R}^3$ we can obtain a formula for the projection map. Recall the dot product which is defined for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ by $\mathbf{u}.\mathbf{v} = \mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$.

Formula

Let $\mathbf{u} \in \mathbb{R}^3$ be a unit vector & $P_{\mathbf{u}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the projection onto $\mathbf{u}$ lin map. Then $P_{\mathbf{u}}$ is the lin map assoc to the matrix $\mathbf{u}\mathbf{u}^T$.

Proof. $P_{\mathbf{u}}\mathbf{v} = \mathbf{u}(\mathbf{v}.\mathbf{u}) = \mathbf{u}\mathbf{u}^T\mathbf{v}$.
This gives another proof of linearity of the projection map.

E.g. Find the matrix representing projection onto $(1, 1, 0)^T$.

Reflections

Let $\mathbf{u} \in \mathbb{R}^3$ be a unit vector & $\mathbf{u}^\perp \subseteq \mathbb{R}^3$ be the plane through $(0,0,0)^T$ normal to $\mathbf{u}$. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the map sending $\mathbf{v}$ to its reflection about $\mathbf{u}^\perp$. From the picture we see

Formula: T is the lin combn $\text{id} - 2P_{\mathbf{u}}$ of the linear maps $\text{id}, P_{\mathbf{u}}$ so is linear.

Example of reflection

E.g. Find the matrix representing reflection about the plane $x + y = 0$ in x, y, z-space.