Data for a vector space

Aim lecture: We recall the notion of a vector space which provides the context for describing linear phenomena.

Throughout the rest of these lectures, $\mathbb{F}$ denotes a field.

Defn

A *vector space* over $\mathbb{F}$ (or $\mathbb{F}$-*space*) consists of a set V of elements called *vectors* and two maps

1. addition: $V \times V \longrightarrow V : (\mathbf{v}, \mathbf{v}') \mapsto \mathbf{v} + \mathbf{v}'$
2. scalar multiplication: $\mathbb{F} \times V \longrightarrow V : (\alpha, \mathbf{v}) \mapsto \alpha\mathbf{v}$

such that the following axioms hold

Axioms for a vector space

Axioms

For all $\mathbf{v}, \mathbf{v}' \in V, \beta, \beta' \in \mathbb{F}$ we have

1. $(V, +)$ is an abelian group.
2. $(\beta\beta')\mathbf{v} = (\beta(\beta'\mathbf{v}))$
3. $1\mathbf{v} = \mathbf{v}$
4. $(\beta + \beta')\mathbf{v} = \beta\mathbf{v} + \beta'\mathbf{v}$
5. $\beta(\mathbf{v} + \mathbf{v}') = \beta\mathbf{v} + \beta\mathbf{v}'$

Rem It is a good ex to show that (assuming 1) & 4)), axioms 2) & 3) above amount to saying that the group $\mathbb{F}^\times$ acts on V.

Simple examples of vector spaces

E.g. 1 $\mathbb{F}^n$ is a vector space over $\mathbb{F}$

E.g. 2 Geometric vectors in 3-dim space is a vector space over $\mathbb{F} = \mathbb{R}$.

E.g. 3 $M_{mn}(\mathbb{F})$ is an $\mathbb{F}$-space.

More examples

E.g. 4 $\mathbb{F}[x]$ = the set of polynomials in the indet x & co-eff from $\mathbb{F}$ is an $\mathbb{F}$-space.

E.g. 5 For a set X, the set of fns from $X \longrightarrow \mathbb{F}$, denoted $\text{Fun}(X, \mathbb{F})$ is a vector space with pointwise addn & scalar multn.
i.e. For $f, g \in \text{Fun}(X, \mathbb{F}), \beta \in \mathbb{F}, x \in X$ we have

$$(f + g)(x) = f(x) + g(x), \quad (\beta f)(x) = \beta(f(x)).$$

E.g. 6 **0** is a vector space over any $\mathbb{F}$.

Basic properties

You know the following (& should prove yourself if you've forgotten how)

Prop

For any $\mathbb{F}$-space V and $\mathbf{v} \in V, \beta \in \mathbb{F}$

1. The zero vector & negative of a vector are unique.
2. $\beta \mathbf{0} = \mathbf{0}$
3. $0\mathbf{v} = \mathbf{0}$
4. $(-1)\mathbf{v} = -\mathbf{v}$
5. $\beta \mathbf{v} = \mathbf{0} \implies \beta = 0$ or $\mathbf{v} = \mathbf{0}$.

Vector arithmetic

One of the key points of the axioms of a vector space V, is that for $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V, \beta_1, \ldots, \beta_n \in \mathbb{F}$ one can form the vector

$$\beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \ldots + \beta_n \mathbf{v}_n.$$

Such a vector or expression is called a *linear combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Furthermore, axioms show such linear combinations can be manipulated arithmetically, in the way one handles n-tuples of reals.

E.g.

Subspaces

To get more examples of vector spaces

Prop-Defn

Let $V = \mathbb{F}$-space & W be a non-empty subset. Then the following are equivalent.

1. W is closed under addn & scalar multn i.e. for $\mathbf{w}, \mathbf{w}' \in W, \beta \in \mathbb{F}$ we have $\mathbf{w} + \mathbf{w}' \in W$ and $\beta\mathbf{w} \in W$.
2. For any $\mathbf{w}, \mathbf{w}' \in W, \beta \in \mathbb{F}$ we have $\beta\mathbf{w} + \mathbf{w}' \in W$.
3. The linear combination of any set of elements of W lies in W.

If these conditions hold, then the addn & scalar multn laws on V restrict to addn & scalar multn laws on W making W an $\mathbb{F}$-space too. In this case we say W is a *subspace* of V & write $W \leq V$.

E.g. Any vector space V has two trivial subspaces

Examples of subspaces

E.g. 1 *If* one identifies $\mathbb{R}[x]$ with the set of real polynomial fns, then $\mathbb{R}[x]$ is a subspace of $\text{Fun}(\mathbb{R}, \mathbb{R})$.

E.g. 2 Let $I \subseteq \mathbb{R}$ be an interval & $C^0(I)$ denote the set of continuous functions on I. Then $C^0(I)$ is a subspace of $\text{Fun}(I, \mathbb{R})$ since

E.g.3 Let i be a positive integer or ∞ & $I \subseteq \mathbb{R}$ be an open interval. Let $C^i(I)$ denote the set of i-times continuously differentiable fns on I. Then $C^i(I)$ is a subspace of $C^j(I)$ for $j < i$.

E.g. 4 For each $d \in \mathbb{N}$, the subset $\mathbb{F}[x]_{\leq d} \subset \mathbb{F}[x]$ of polynomials of degree $\leq d$ is a subspace.

Some basic results about subspaces

Prop

1. If $U \leq V, V \leq W$ then $U \leq W$. Sim if $U \leq W, V \leq W, U \subseteq V$ then $U \leq V$.
2. If $\mathbf{v}$ is a vector of an $\mathbb{F}$-space then $\mathbb{F}\,\mathbf{v} = \{\beta\mathbf{v} | \beta \in \mathbb{F}\}$ is a subspace.
3. Let V be a vector space and $V_1, \ldots, V_r$ be subspaces. Then $\cap_{i=1}^r V_i$ is also a subspace of V.
4. With the above notn, $\sum_{i=1}^r V_i = V_1 + \ldots + V_r = \{\sum_i \mathbf{v}_i | \mathbf{v}_i \in V_i\}$ is a subspace of V called the *sum* of the V_i.

Proof. All follow from checking closure axioms. We prove 4) as an example. Consider $\mathbf{w}, \mathbf{w}' \in \sum V_i, \beta \in \mathbb{F}$. Then we may write $\mathbf{w} = \sum_i \mathbf{v}_i, \mathbf{w}' = \sum_i \mathbf{v}'_i$ for some $\mathbf{v}_i, \mathbf{v}'_i \in V_i$. Closure axioms ensure $\beta\mathbf{v}_i + \mathbf{v}'_i \in V_i$ so

$$\beta\mathbf{w} + \mathbf{w}' = \sum_i (\beta\mathbf{v}_i + \mathbf{v}'_i) \in \sum_i V_i.$$

Lines and Planes

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be non-parallel vectors. Recall from MATH1141/MATH1151

Fact

$\mathbb{R}\,\mathbf{v}$ is the line in $\mathbb{R}^3$ with parametric equation

Fact

$\mathbb{R}\,\mathbf{v} + \mathbb{R}\,\mathbf{w}$ is the plane in $\mathbb{R}^3$ with parametric equation

Example of sums

E.g. Let $W = \mathbb{R}(1,1,0)^T + \mathbb{R}(0,2,1)^T$, $W' = \mathbb{R}(1,1,1)^T$. Does $W + W' = \mathbb{R}^3$?

Example of intersections

E.g. Find the intersection of the planes
$W = \mathbb{R}(1,2,1)^T + \mathbb{R}(1,1,1)^T, W' = \mathbb{R}(0,2,1) + \mathbb{R}(1,1,0)^T$.