List of properties

Here are some additional properties of the determinant function.

Prop

Throughout let $A, B \in M_{nn}$.

1. If $A = (a_{ij})$ is upper triangular then $\det(A) = a_{11}a_{22}\dots a_{nn}$.
2. If a row or column of A is 0, so is $\det(A)$.
3. $\det(AB) = \det(A)\det(B)$.
4. $\det(A^T) = \det(A)$.
5. The determinant of A is unchanged if you add a scalar multiple of one row to a different row. Sim for columns.

Comments on proof. 1) has already been proved directly from defns. So can all the others, though sometimes it's easier to use multi-linearity and the alternating condition to help. We'll prove the hardest one, 3), the multiplicativity of det.
Comments on calculating det Always use the methods in first year which are legitimate by the above properties and the fact det is multi-linear and alternating in the rows and columns.

Proof multiplicativity of det

Recall $J_n = \{1, \ldots, n\}$ and F_n is the set of functions of form $f : J_n \longrightarrow J_n$. Let $A = (a_{ij}), B = (b_{ij}) \in M_{nn}$ so that $AB = (\sum_{k=1}^n a_{ik} b_{kj})_{ij}$.

$$\begin{aligned}
\det(AB) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i \in J_n} \sum_{k=1}^{n} a_{ik} b_{k\,\sigma(i)} \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(\sum_{k=1}^{n} a_{1k} b_{k\,\sigma(1)} \right) \ldots \left(\sum_{k=1}^{n} a_{nk} b_{k\,\sigma(n)} \right) \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{f \in F_n} \prod_{i \in J_n} a_{if(i)} b_{f(i)\,\sigma(i)} \qquad\qquad (*) \\
&= \sum_{\sigma, \tau \in S_n} \operatorname{sgn}(\sigma) \prod_{i \in J_n} a_{i\tau(i)} b_{\tau(i)\,\sigma(i)}
\end{aligned}$$

by the lemma below, we may assume $f = \tau \in S_n$ since other terms cancel.

Proof cont'd

$$
\begin{aligned}
\det(AB) &= \sum_{\sigma,\tau \in S_n} \operatorname{sgn}(\sigma) \prod_{i \in J_n} a_{\tau^{-1}(i)i} b_{i\,\sigma\,\tau^{-1}(i)} \\
&= \sum_{\tau,\rho \in S_n} \operatorname{sgn}(\rho\tau) \prod_{i \in J_n} a_{\tau^{-1}(i)i} b_{i\rho(i)} \\
&= \left(\sum_{\tau \in S_n} \operatorname{sgn}(\tau) \prod_{i \in J_n} a_{\tau^{-1}(i)i} \right) \left(\sum_{\rho \in S_n} \operatorname{sgn}(\rho) \prod_{i \in J_n} b_{i\rho(i)} \right) \\
&= \left(\sum_{\tau \in S_n} \operatorname{sgn}(\tau) \prod_{i \in J_n} a_{i\tau(i)} \right) \det(B) \\
&= \det(A)\det(B)
\end{aligned}
$$

which completes the proof modulo the lemma below.

Proof lemma

The following guarantees the cancellation of unwanted terms in (*) of the above proof.

Lemma

For $f \in F_n - S_n$ we have $\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i \in J_n} a_{if(i)} b_{f(i)\,\sigma(i)} = 0$.

Proof. If $f \notin S_n$ we can find distinct $r, s \in J_n$ with $f(r) = f(s)$. Consider the 2-cycle $(r\ s)$ and let $\bar{\sigma} = \sigma(r\ s)$. It suffices to show equality of the products

$$P = \prod_{i \in J_n} a_{if(i)} b_{f(i)\,\sigma(i)} \quad \text{and} \quad \bar{P} = \prod_{i \in J_n} a_{if(i)} b_{f(i)\bar{\sigma}(i)}$$

since the terms appear with opposite sign in the sum of the lemma. Now all factors of $P, \bar{P}$ are the same except possibly those b-terms involving $i = r, s$. We examine these factors. For P these terms give $b_{f(r)\,\sigma(r)} b_{f(s)\,\sigma(s)}$ whilst for $\bar{P}$ we get $b_{f(r)\bar{\sigma}(r)} b_{f(s)\bar{\sigma}(s)} = b_{f(r)\,\sigma(s)} b_{f(s)\,\sigma(r)}$. These are the same since $f(r) = f(s)$. The lemma is proved and multiplicativity of det is established.

Classical adjoint

Let $A = (a_{ij})$. Recall the cofactor $C_{ij} = (-1)^{i+j} \det A(i,j)$.

Defn

The *classical adjoint* of A is the matrix $\mathrm{adj}A$ with (i,j)-th entry C_{ji} (Note i,j swapped).

e.g.

Note that the (i,i)-th entry of $A(\mathrm{adj}A)$ is

$$\sum_{k=1}^{n} a_{ik} C_{ik} = \det(A)$$

by the Laplace expansion formula. Thus the diagonal entries of $A(\mathrm{adj}A)$ are all $\det(A)$.

Cramer's rule

On the other hand, the Laplace expansion formula also shows (i,j)-th entry of $A(\text{adj}A)$ is $\det(B)$ where B is the matrix obtained from A by replacing the j-th row of A with the i-th row. But this is zero since B has 2 rows the same. This proves

Theorem (Cramer's rule)

For $A \in M_{nn}$ we have $A(adjA) = \det(A)I$ where I is the identity matrix as usual.

Corollary

A is invertible iff $\det(A) \neq 0$ in which case $A^{-1} = \det(A)^{-1}(\text{adj } A)$.

Proof. $(\Longleftarrow)$ follows from Cramer's rule. $(\Longrightarrow)$ holds since $\det(A)\det(A^{-1}) = \det(I) = 1$.

Rem We almost never use Cramer's rule (or its corollary) to compute an actual inverse, it is very important for theory however.

An application

Prop

Let A be an $n \times n$-matrix with integer entries and $\det(A) = \pm 1$. Then A^{-1} is also a matrix with integer entries.

Why? We know the classical adjoint has integer entries since they are up to sign, determinants of minors which themselves have integer entries. Cramer's rule then shows A^{-1} also has integer entries.

Elementary matrices

The property that determinants are unchanged when you add a scalar multiple of one row to another also follows from multiplicativity of det and the determinant of elementary matrices.