Permuting variables & negation

Aim lecture: We define the determinant via permutations.

Fact

Let $f(x_1, \ldots, x_n)$ be an $\mathbb{R}$-valued function & $\sigma \in S_n$. Then

$$\sigma.(-f) = -(\sigma.f).$$

Proof.

Even & odd permutations

We begin with the difference product

$$\Delta_n(x_1, \ldots, x_n) = \prod_{i<j}(x_i - x_j)$$

e.g. $\Delta_2(x_1, x_2) = x_1 - x_2, \Delta_3$

Thm-Defn

Let $\sigma \in S_n$ be a product of m 2-cycles. Then $\sigma\,\Delta_n = (-1)^m \Delta_n$.

- We define the *sign* of σ to be $\text{sgn}(\sigma) = (-1)^m$ which is well-defined.
- We say σ is *even* if $\text{sgn}(\sigma)$ is 1 (so σ is a product of an even number of 2-cycles) and *odd* otherwise.

E.g. $(1\ 2).\Delta_3$

Proof Thm-Defn

It suffices to show that for any 2-cycle $(i\ j)$ we have $(i\ j).\Delta_n = -\Delta_n$ for repeating this m times & using fact page 1, gives the desired result.
We can assume $i < j$. We observe the effect of $(i\ j)$ on each factor of Δ_n

- $(i\ j).(x_i - x_j) = -(x_i - x_j)$ factor negated
- For r, s, i, j disjoint, $(i\ j)(x_r - x_s) = x_r - x_s$ no change
- For $r < i < j$, $(i\ j).(x_r - x_i)(x_r - x_j) = (x_r - x_j)(x_r - x_i)$ no change
- Sim no change for $(x_i - x_r)(x_r - x_j)$ when $i < r < j$ and for $(x_i - x_r)(x_j - x_r)$ when $i < j < r$.

Multiplying the above gives the thm.

Corollary

$\text{sgn}(\sigma\tau) = \text{sgn}(\sigma)\text{sgn}(\tau)$

Proof If σ, τ are products of m, m' 2-cycles resp then, $\sigma\tau$ is a product of $m + m'$ 2-cycles. Cor boils down to $(-1)^{m+m'} = (-1)^m(-1)^{m'}$.

Permuting terms in sums and products

Recall $J_n = \{1, \dots, n\}$. For $\sigma \in S_n$, we may re-arrange terms to obtain formulas like

$$\sum_{i \in J_n} a_i = \sum_{i \in J_n} a_{\sigma(i)}$$

$$\prod_{i \in J_n} a_i = \prod_{i \in J_n} a_{\sigma(i)}$$

We'll also need following

Lemma

Let $\tau \in S_n$. Then as σ runs through the elts of S_n (once)

1. $\tau\sigma$ runs through all the elts of S_n exactly once.
2. σ^{-1} runs through all the elts of S_n exactly once.

Proof. (1) follows from prop lect 3, on the regular action of S_n. (2) is easy ex.

Determinant

Defn

For $A = (a_{ij}) \in M_{nn}(\mathbb{F})$ we define the *determinant* of A to be

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i \in J_n} a_{i\,\sigma(i)}$$

E.g.1 $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$

E.g. 2 Suppose $A = (a_{ij})$ is upper triangular i.e. $a_{ij} = 0$ if $j < i$.
$\det(A) =$

Position of entries in term

Consider a monomial term $\prod_{i \in J_n} a_{i\,\sigma(i)}$ of the determinant of (a_{ij}).

Note that since i ranges over J_n, there's exactly one entry $a_{i\,\sigma(i)}$ from each row.

Since $\sigma(i)$ ranges over elements of J_n exactly once, there's exactly one entry from each column.

E.g.

Alternating & multi-linear functions

The new defn of determinant agrees with the one from first year. Need some properties to see this.

Defn

Let $f : M_{nn}(\mathbb{F}) \longrightarrow \mathbb{F}$ be a function.

1. We say f is *multi-linear* in the columns if for any two matrices of form $A_1 = (A\ \mathbf{v}_1\ A'), A_2 = (A\ \mathbf{v}_2\ A') \in M_{nn}(\mathbb{F})$ with i-th columns $\mathbf{v}_1, \mathbf{v}_2$ resp (& other columns the same) we have a)
$f(A(\mathbf{v}_1 + \mathbf{v}_2)A') = f(A_1) + f(A_2) \quad \& \quad f(A\ (c\mathbf{v}_1)\ A') = cf(A_1)$ for $c \in \mathbb{F}$.
Here A represents first $i-1$ columns of A_1, A_2 whilst A' represents last $n-i$.
2. We say f is *alternating* in the columns if for any 2-cycle τ we have $f(a_{i\tau(j)})_{ij} = -f(a_{ij})$ i.e. swapping two columns of a matrix negates the value of f (& furthermore in char 2, $f(A) = 0$ if two columns are equal).
3. There are similar defns for rows.

e.g. $\det : M_{22} \longrightarrow \mathbb{F}$.

Key properties of the determinant function

Theorem

The determinant function is multi-linear and alternating in the rows and columns.

Proof is mainly exercise. We check alternating in columns. For 2-cycle τ note that $\tau^{-1} = \tau$ has sign -1.

$$\begin{aligned} \det(a_{i\tau(j)}) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i \in J_n} a_{i\tau\,\sigma(i)} && (1) \\ &= \sum_{\rho \in S_n} \operatorname{sgn}(\tau^{-1}\rho) \prod_{i \in J_n} a_{i\rho(i)} && (2) \\ &= \operatorname{sgn}(\tau^{-1}) \sum_{\rho \in S_n} \operatorname{sgn}(\rho) \prod_{i \in J_n} a_{i\rho(i)} && (3) \\ &= -\det(a_{ij}) && (4) \end{aligned}$$

Note the use of the lemma in line 2 (where we changed var to $\rho = \tau\,\sigma$) and corollary in line 4.

Minors & cofactors

Defn

Let $A = (a_{ij}) \in M_{nn}$ and $i, j \in J_n$.

1. The (i,j)-th *minor* of A is the matrix $A(i,j) \in M_{n-1,n-1}$ obtained by deleting the i-th row & j-th columns from A.
2. The (i,j)-th *cofactor* of A is $(-1)^{i+j} \det A(i,j)$.

e.g.

Connection with Laplace exansions

The equivalence with the defn via Laplace expansions is given in

Thm

Let $A = (a_{ij})$ and C_{ij} denote its (i, j)-th cofactor. Then for any fixed i

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}.$$

The proof (omitted) is easily obtained by multi-linearity and best seen in an actual example.