Rotations & reflections

Aim lecture: We use the spectral thm for normal operators to show how any orthogonal matrix can be built up from rotations & reflections.

In this lecture we work over the fields $\mathbb{F} = \mathbb{R}$ & $\mathbb{C}$. We let

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

the 2×2-matrix which rotates anti-clockwise about $\binom{0}{0}$ through angle θ. Note that $\det R_\theta = 1$ & that R_π is the diagonal matrix $-I_2 = (-1) \oplus (-1)$.

Defn

Let $A \in M_{nn}(\mathbb{R})$. We say that A is a *rotation* matrix if it is orthogonally similar to $I_{n-2} \oplus R_\theta$ for some θ. We say that A is a *reflection* matrix if it is orthogonally similar to $I_{n-1} \oplus (-1)$.

Rem Note that this generalises the usual notions in dim 2 & 3. Also, rotations are orthogonal with determinant 1 whilst reflections are orthogonal with det -1.

Orthogonal matrices in O_2, O_3

Recall that orthogonal matrices have det ± 1. We defer the proof of

Theorem

1. *Let $A \in O_2$. If $\det A = 1$ then A is orthog sim to R_θ for some θ (so tr $A = 2\cos\theta$) & if $\det A = -1$ then A is a reflection.*
2. *Let $A \in O_3$. If $\det A = 1$ then A is orthog sim to $(1) \oplus R_\theta$ (so tr $A = 1 + 2\cos\theta$) & if $\det A = -1$ then A is orthog sim to $(-1) \oplus R_\theta$ (so tr $A = -1 + 2\cos\theta$) for some θ.*

Corollary

Let $A, B \in M_{22}(\mathbb{R})$ or $A, B \in M_{33}(\mathbb{R})$. If A, B are rotation matrices, then so is AB. If A, B are reflection matrices, then AB is a rotn.

Proof. In both cases, AB is an orthog matrix with det $(\pm 1)^2 = 1$ so the thm gives the result.

Example

E.g. Verify that the following is a rotation matrix & determine the axis of rotn & angle of rotn.

$$A = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$$

A

E-values & e-vectors of real matrices

Recall the conjugation map $\overline{(\cdot)} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a conjugate linear isomorphism. Furthermore, for $\mathbf{v}, \mathbf{v}' \in \mathbb{C}^n$ we have $\|\overline{\mathbf{v}}\| = \|\mathbf{v}\|$ & $\mathbf{v} \perp \mathbf{v}' \iff \overline{\mathbf{v}} \perp \overline{\mathbf{v}'}$.

Prop 1

Let $A \in M_{nn}(\mathbb{R})$ & $\lambda \in \mathbb{C}$.

1. The conjugation map restricts to a conjugate linear isomorphism $\overline{(\cdot)} : E_\lambda \longrightarrow E_{\overline{\lambda}}$, that is, $\overline{E_\lambda} = E_{\overline{\lambda}}$. In particular, λ is an e-value of A iff $\overline{\lambda}$ is.
2. The conjugation map sends any basis for E_λ to a basis for $E_{\overline{\lambda}}$.

Proof. 1) Let $\mathbf{v} \in E_\lambda$. Then

$$A\overline{\mathbf{v}} = \overline{A}\,\overline{\mathbf{v}} = \overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}}.$$

Hence $\overline{E_\lambda} \subseteq E_{\overline{\lambda}}$ & the reverse inclusion follows from this result applied to $\overline{\lambda}$. Conjugation is injective so 1) follows.

2) follows from the fact that conjugate lin isom take bases to bases (just as is the case for lin isom). The proof is a mild modification of the lin case.

E-theory of R_θ

To study rotations, we need to know the e-theory for R_θ well.

Prop 2

We may unitarily diagonalise (over $\mathbb{C}$) $R_\theta = U((e^{i\theta}) \oplus (e^{-i\theta}))U^*$ where

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}.$$

In particular, $E_{e^{i\theta}} = \mathbb{C}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$, $\quad E_{e^{-i\theta}} = \mathbb{C}\overline{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}} = \mathbb{C}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$.

Proof. easy ex. Let's just check the $e^{i\theta}$-e-space and use Prop 1.

General classification of orthogonal matrices

Theorem

Let $A \in O_n$.

1. *If* $\det A = 1$ *then* A *is orthogonally similar to* $I_r \oplus R_{\theta_1} \oplus \ldots \oplus R_{\theta_s}$ *for some* $r \in \mathbb{N}, \theta_1, \ldots, \theta_s \in \mathbb{R}$.
2. *If* $\det A = -1$ *then* A *is orthogonally similar to* $(-1) \oplus I_r \oplus R_{\theta_1} \oplus \ldots \oplus R_{\theta_s}$ *for some* $r \in \mathbb{N}, \theta_1, \ldots, \theta_s \in \mathbb{R}$.

Proof. This proof is a classic example of the use of complex numbers to answer questions about reals!

Note that A viewed as complex matrix is unitary & hence normal. Hence the e-values have modulus 1 & we may apply the spectral thm for normal operators to conclude there is an A-invariant orthogonal direct sum decomposition into e-spaces of the form

$$\mathbb{C}^n = E_1 \oplus E_{-1} \oplus E_{e^{i\theta_1}} \oplus \overline{E_{e^{i\theta_1}}} \oplus \ldots \oplus E_{e^{i\theta_s}} \oplus \overline{E_{e^{i\theta_s}}}$$

where we used the propn to re-write some of the e-spaces.

Alternate version of theorem

The theorem will follow if we can find an orthonormal basis of *real* vectors (abbrev to real orthonormal basis) such that wrt the corresp co-ord system $P \in O_n$, the representing matrix $P^T AP$ is a direct sum of rotation matrices & matrices of the form $\pm I$. Indeed, the only thing left to do is replace all copies of $-I_2$ with R_π to get the desired final form. The thm thus follows from the following more precise result.

Theorem (Version 2 for purposes of proof)

1. *There are real orthonormal bases for* E_1, E_{-1}.
2. *There is a real orthonormal basis for* $V_j = E_{e^{i\theta_j}} \oplus \overline{E_{e^{i\theta_j}}}$ *such that the matrix representing* A *restricted to* V_j *wrt the corresp the co-ord system is* $R_{\theta_j} \oplus \ldots \oplus R_{\theta_j}$ *(there are* $\dim E_{e^{i\theta_j}}$ *copies of* R_{θ_j}*).*

Proof. 1) Note $E_1 = \ker(A - I)$, the kernel of a matrix with real entries, so we may find an orthonormal basis for it consisting of real vectors. Sim E_{-1} has an orthonormal basis of real vectors. Part 1) follows.

Proof cont'd

2) To simplify notn, we drop the subscript j so $\theta = \theta_j, V = V_j$ etc. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be an orthonormal basis for $E_{e^{i\theta}}$ so $\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \overline{\mathbf{v}_1}, \ldots, \overline{\mathbf{v}_m}\}$ is an orthonormal basis for V. Hence V is an orthog direct sum of the subspaces $W_l = \text{Span}(\mathbf{v}_l, \overline{\mathbf{v}_l})$ so it suffices to find a real orthonormal co-ord system for W_l. Using the o/n co-ord system $C = (\mathbf{v}_l \ \overline{\mathbf{v}_l}) : \mathbb{C}^2 \longrightarrow W_l$, the representing matrix is

$$C^{-1} \circ T_A \circ C = (e^{i\theta}) \oplus (e^{-i\theta})$$

where $T_A : W_l \longrightarrow W_l$ is the left multn by A on W_l. Prop 2 then shows that for

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \quad \& \quad C_{\mathbb{R}} = CU^*$$

we have

$$C_{\mathbb{R}}^{-1} \circ T_A \circ C_{\mathbb{R}} = UC^{-1} \circ T_A \circ CU^* = U((e^{i\theta}) \oplus (e^{-i\theta}))U^* = R_\theta$$

Hence the thm follows from

Final lemma to complete proof

Lemma

The co-ord system $C_{\mathbb{R}}$ is 1) real and, 2) orthonormal.

Note that $C_{\mathbb{R}} = \begin{pmatrix} \frac{\mathbf{v}_l + \overline{\mathbf{v}_l}}{\sqrt{2}} & i\frac{\mathbf{v}_l - \overline{\mathbf{v}_l}}{\sqrt{2}} \end{pmatrix} = (\mathbf{w}_+ \mathbf{w}_-)$ say.
For 1), we just check $\overline{\mathbf{w}_\pm} = \mathbf{w}_\pm$.
For 2), just note that $C_{\mathbb{R}}$ is a composite of isomorphisms of inner product spaces, so is an isomorphism of inner product spaces itself.