Semisimple operators

Aim lecture: We extend the spectral thm to the normal operators. This gives a characterisation of when complex matrices can be unitarily diagonalised.

In this lecture, $\mathbb{F}$ will be an alg closed field, & later $\mathbb{F} = \mathbb{C}$.

Prop-Defn

Let $T : V \longrightarrow V$ be linear & $\dim V < \infty$. We say that T is *semisimple* or a *semisimple operator* if V is a direct sum of e-spaces of T, in other words, T can be diagonalised.

1. The direct sum of semisimple operators is semisimple.
2. If T is semisimple & $W \leq V$ is a T-invariant subspace then $T|_W : W \longrightarrow W$ is also semisimple.

Proof. 1) is clear from the theory of diagn.

2) We argue by contradn. Let T be semisimple so that all the gen e-spaces are equal to the corresponding e-spaces. Suppose that $T|_W$ is not semisimple & its λ-e-space $E_\lambda \leq W$ is not equal to the gen λ-e-space $E_\lambda(\infty) \leq W$. Then we can find $\mathbf{w} \in E_\lambda(\infty) - E_\lambda$. But then the gen e-space λ-e-space of T contains $\mathbf{w}$ so by semisimplicity of T, $\mathbf{w}$ is also a λ-e-vector. This contradn proves the propn.

E-spaces of commuting operators

For this result we do not need $\mathbb{F}$ to be alg closed.

Prop

Let $X, Y : V \longrightarrow V$ be linear maps. Let E_λ^X, E_μ^Y denote e-spaces of X, Y resp. If X, Y *commute*, that is $X \circ Y = Y \circ X$, then E_λ^X is Y-invariant & E_μ^Y is X-invariant.

Rem Hence the e-spaces are *both* X & Y-invariant.

Proof. Just check axioms.

Simultaneous diagonalisation

Theorem

Let $X, Y : V \longrightarrow V$ be commuting semisimple operators. Let $E^X_{\lambda_1}, \ldots, E^X_{\lambda_r}$ & $E^Y_{\mu_1}, \ldots, E^Y_{\mu_s}$ be the e-spaces of X & Y resp. Then

$$V = \oplus_{i=1}^r \oplus_{j=1}^s E^X_{\lambda_i} \cap E^Y_{\mu_j}$$

In particular, if $C : \mathbb{F}^n \longrightarrow V$ is some co-ord system adapted to this direct sum decomposition, we may simultaneously diagonalise X & Y in the sense that both the representing matrices $C^{-1} \circ X \circ C$ & $C^{-1} \circ Y \circ C$ are diagonal.

Proof. By the propn, we know that $E^X_{\lambda_i}$ is Y-invariant so by the propn-defn, Y restricted to $E^X_{\lambda_i}$ is semisimple & hence $E^X_{\lambda_i}$ is a direct sum of e-spaces wrt Y. But these e-spaces are just $E^X_{\lambda_i} \cap E^Y_{\mu_j}$. Hence

$$V = \oplus_{i=1}^r E^X_{\lambda_i} = \oplus_{i=1}^r \oplus_{j=1}^s E^X_{\lambda_i} \cap E^Y_{\mu_j}.$$

Simultaneous diagn follows from the fact that $E^X_{\lambda_i} \cap E^Y_{\mu_j}$ is both X-invariant & Y-invariant.

Normal operators

Assume from now on that $\mathbb{F} = \mathbb{C}$ & V is an inner product space.

Defn

A linear map $T : V \longrightarrow V$ is *normal* if T^* exists & $T \circ T^* = T^* \circ T$. A matrix $A \in M_{nn}(\mathbb{C})$ is *normal* if $A^*A = AA^*$.

E.g. 1 Any hermitian operator T is normal.
Why?

E.g. 2 Any unitary operator $U : V \longrightarrow V$ is also normal since

Spectral theorem for normal operators

Theorem

Let $T : V \longrightarrow V$ be a linear map on a fin dim inner product space V.

1. *If T is normal, then there is an orthonormal co-ord system $U : \mathbb{C}^n \longrightarrow V$ such that $U^{-1} \circ T \circ U$ is diagonal.*
2. *Conversely, if there is an orthonormal co-ord system $U : \mathbb{C}^n \longrightarrow V$ such that $U^{-1} \circ T \circ U$ is diagonal, then T is normal.*

In particular, a complex matrix $A \in M_{nn}(\mathbb{C})$ is unitarily diagonalisable iff it is normal.

Proof. We prove 2) first. Let D be the diagonal "matrix" $D = U^{-1} \circ T \circ U = U^* \circ T \circ U$. Note that D^* is also a diagonal matrix so $DD^* = D^*D$. Hence

$$\begin{aligned} T \circ T^* &= U \circ D \circ U^* \circ (U \circ D \circ U^*)^* = U \circ D \circ U^* \circ U \circ D^* \circ U^* \\ &= U \circ D \circ D^* \circ U^* = U \circ D^* \circ D \circ U^* = T^* \circ T \end{aligned}$$

We prove the converse after noting

Proof cont'd

Lemma

Let $T : V \longrightarrow V$ be a linear map where V is a fin dim inner product space $\mathbb{C}$. Let $X = \frac{1}{2}(T + T^*), Y = \frac{1}{2i}(T - T^*)$. Then

1. *both X, Y are hermitian operators such that $T = X + iY$.*
2. *If T is normal then X, Y commute.*

Proof. This is an easy calculation.

We return to the proof of thm 1). By the spectral thm for self-adjoint operators, X, Y are semisimple. Furthermore, if their respective e-spaces are $E^X_{\lambda_1}, \ldots, E^X_{\lambda_r}$ & $E^Y_{\mu_1}, \ldots, E^Y_{\mu_s}$, then the $E^X_{\lambda_i}$'s are mutually orthogonal as are the $E^Y_{\mu_j}$'s. The thm on simultaneous diagn shows that we can find an orthonormal co-ord system $U : \mathbb{C}^n \longrightarrow V$ adapted to the orthog direct sum decomp

$$V = \oplus_{i=1}^r \oplus_{j=1}^s E^X_{\lambda_i} \cap E^Y_{\mu_j}$$

such that both $U^{-1} \circ X \circ U, U^{-1} \circ Y \circ U$ are both diagonal. Then

$$U^{-1} \circ T \circ U = U^{-1} \circ (X + iY) \circ U = U^{-1} \circ X \circ U + i(U^{-1} \circ Y \circ U)$$

is also diagonal.

Example of unitary diagonalisation

E.g. Show that $A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ is normal and unitarily diagonalise it.