Self-adjoint operators

Aim lecture: We examine an important class of endomorphisms on inner product spaces which has good e-value/e-vector theory.

In this lecture, we let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let V be an inner product spaces with inner product denoted $(\cdot|\cdot)$.

Defn

We say a linear map $T : V \longrightarrow V$ is *self-adjoint* or *hermitian* if T^* exists & $T^* = T$. A complex matrix $A \in M_{nn}(\mathbb{C})$ is *hermitian* if $A^* = A$.

Rem 1) Note that a real matrix is hermitian iff it is symmetric.

2) (ex) Suppose V is fin dim & $C : \mathbb{F}^n \longrightarrow V$ is an orthonormal co-ord system. Then T is hermitian iff the representing matrix $C^{-1} \circ T \circ C = C^* \circ T \circ C$ is hermitian.

E-values & e-vectors of self-adjoint operators

Prop

Let $T : V \longrightarrow V$ be an hermitian linear map with $\dim V < \infty$. Then $\mathrm{cp}_T(\lambda)$ factorises linearly over $\mathbb{R}$. Moreover, the e-spaces of T are mutually othogonal.

Proof. We first prove the case $\mathbb{F} = \mathbb{C}$. For e-values λ, μ pick e-vectors $\mathbf{v}_\lambda, \mathbf{v}_\mu$ with e-values λ, μ respectively. Note that since $T^* = T$ we have

$$\lambda(\mathbf{v}_\mu|\mathbf{v}_\lambda) = (\mathbf{v}_\mu|\lambda\mathbf{v}_\lambda) = (\mathbf{v}_\mu|T\mathbf{v}_\lambda) = (T\mathbf{v}_\mu|\mathbf{v}_\lambda) = (\mu\mathbf{v}_\mu|\mathbf{v}_\lambda) = \overline{\mu}(\mathbf{v}_\mu|\mathbf{v}_\lambda).$$

When $\lambda = \mu, \mathbf{v}_\lambda = \mathbf{v}_\mu$ we see that $\lambda = \overline{\lambda}$ so $\lambda \in \mathbb{R}$. Hence all e-values are real & $\mathrm{cp}_T(\lambda)$ factorises lin over $\mathbb{R}$. Suppose now that $\lambda \neq \mu$. Given that $\mu \in \mathbb{R}$, we see the above eqn reduces to $\lambda(\mathbf{v}_\mu|\mathbf{v}_\lambda) = \mu(\mathbf{v}_\mu|\mathbf{v}_\lambda)$ so we must have $(\mathbf{v}_\mu|\mathbf{v}_\lambda) = 0$. Hence the e-spaces E_λ, E_μ are orthogonal & we are done in the case $\mathbb{F} = \mathbb{C}$.

Suppose now that $\mathbb{F} = \mathbb{R}$. If T is given by a real matrix $A \in M_{nn}(\mathbb{R})$, we may apply our complex result to A considered as a complex matrix to prove the result. The general real case follows on representing T with some real matrix A wrt some orthonormal co-ord system C. Indeed in this case, $\mathrm{cp}_T = \mathrm{cp}_A$ & C preserves orthogonality & maps e-spaces to e-spaces.

Example

E.g. 1 $(a) \in M_{11}(\mathbb{C}) = \mathbb{C}$ is hermitian iff $a \in \mathbb{R}$. Hence the only e-value a is real.

E.g. 2 $A \in M_{22}(\mathbb{C})$ is hermitian iff

T-invariance of orthogonal complements

The theory of Jordan canonical forms is complicated $\because$ T-invariant subspaces do not necessarily have T-invariant vector space complements. For self-adjoint operators we have the following.

Prop

Let $T : V \longrightarrow V$ be an hermitian linear map. Let $W \leq V$ be a T-invariant subspace. Then $W^{\perp}$ is also T-invariant. In particular, if W is fin dim, then $V = W \oplus W^{\perp}$ is a T-invariant direct sum.

Proof. Let $\mathbf{w} \in W, \mathbf{w}' \in W^{\perp}$. It suffices to show that $T\mathbf{w}' \in W^{\perp}$. Now $T\mathbf{w} \in W$ so

$$(\mathbf{w}|T\mathbf{w}') = (T\mathbf{w}|\mathbf{w}') = 0 \Longrightarrow T\mathbf{w}' \perp W$$

& the propn is proved.

Spectral theorem

Theorem (Spectral)

Let $T : V \longrightarrow V$ be an hermitian linear map with $\dim V < \infty$. Let $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ be the e-values of T. Then

1. $V = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_r}$, *an orthogonal direct sum.*
2. *given orthonormal co-ord systems $C_i : \mathbb{F}^{n_i} \longrightarrow E_{\lambda_i}$, we obtain an orthonormal co-ord system $C = \oplus_i C_i : \mathbb{F}^n \longrightarrow V$ where $n = \sum_i n_i$. The corresponding representing matrix is the diagonal matrix*

$$C^{-1} \circ T \circ C = \lambda_1 I_{n_1} \oplus \ldots \oplus \lambda_r I_{n_r}.$$

Proof. It suffices to prove 1) by the general theory of diagn. Let W be the T-invariant subspace $\oplus_i E_{\lambda_i}$. We will be done if we can show $V = W$ or equiv, $W^\perp = \mathbf{0}$ for then $W = (W^\perp)^\perp = \mathbf{0}^\perp = V$. From the previous propn, we know $W^\perp$ is T-invariant so if $p_1(\lambda), p_2(\lambda)$ are the char poly of $T|_W$ & $T|_{W^\perp}$ then $\mathrm{cp}_T(\lambda) = p_1(\lambda)p_2(\lambda)$. Hence if $W^\perp \neq \mathbf{0}$, $p_2(\lambda)$ is a non-constant poly with real linear factors, so we can find an e-vector $\mathbf{v} \in W^\perp$. But then $\mathbf{v} \in W$, a contradiction as $W \cap W^\perp = \mathbf{0}$. This proves the thm.

Unitary diagonalisation

Let $A \in M_{nn}(\mathbb{C})$ be hermitian. Then the spectral thm says we can diagonalise A by $D = U^{-1}AU = U^*AU$ where $U \in U_n$ & D is diagonal.

Defn

Finding such a factorisation $D = U^{-1}AU = U^*AU$ is called *unitarily diagonalising* A. If A is furthermore real, then we may assume U is orthogonal so we will refer to this as *orthogonally diagonalising* A & note $D = U^TAU$.

In particular, any hermitian matrix is unitary similar to a diagonal one, whilst a real symmetric matrix is orthogonally similar to a diagonal one.

Example

E.g. Orthogonally diagonalise the matrix

$$A = \begin{pmatrix} -2 & 2 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$$

A One calculates $\text{cp}_A(\lambda) = (\lambda+3)^2(\lambda-6)$ so the e-values are $-3\&6$. Note A is symmetric so the spectral thm ensures A is orthogonally similar to the diagonal matrix $(-3)\oplus(-3)\oplus(6)$ & $\dim E_{-3} = 2, \dim E_6 = 1$.

$\underline{\lambda = -3}$ $E_{-3} = \ker(A+3I)$

$$= \ker \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}^{\perp}$$

We apply Gram-Schmidt to get an orthogonal basis

Example cont'd

$\underline{\lambda = 6}$ We can compute the e-space as per normal or just note here that
$E_6 = E_{-3}^{\perp} =$

CHECK THIS!
We can now orthogonally diagonalise $D = U^{-1}AU = U^T AU$ where
$U =$