Isomorphisms of inner product spaces

Aim lecture: We examine the notion of isomorphisms for inner product spaces.

In this lecture, we let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let V, W be inner product spaces with inner products denoted $(\cdot|\cdot)$.

Prop-Defn

A linear map $T : V \longrightarrow W$ *preserves inner products* if for all $\mathbf{v}, \mathbf{v}' \in V$ we have $(T\mathbf{v}|T\mathbf{v}') = (\mathbf{v}|\mathbf{v}')$. In this case T is injective. If T is also bijective, then we say that T is an *isomorphism of inner product spaces*. In this case T^{-1} also preserves inner products. A co-ordinate system $C : \mathbb{F}^n \longrightarrow V$ is said to be *orthonormal* if C is an isomorphism of inner product spaces.

Proof. Suppose T preserves inner products & $\mathbf{v} \in \ker T$. Then

$$0 = (T\mathbf{v}|T\mathbf{v}) = (\mathbf{v}|\mathbf{v})$$

so $\mathbf{v} = \mathbf{0}$ which shows T to be injective.

It is an easy ex to see T^{-1} preserves inner products.

Orthonormal co-ordinate systems

Prop

Let $C = (\mathbf{v}_1 \dots \mathbf{v}_n) : \mathbb{F}^n \longrightarrow V$ be linear. Then C preserves inner products iff $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is orthonormal. In particular, C is an orthonormal co-ordinate system iff B is an orthonormal basis for V.

Proof. Suppose first that B is orthonormal & let $\mathbf{v} = (\beta_1, \dots, \beta_n)^T, \mathbf{v}' = (\beta'_1, \dots, \beta'_n)^T \in \mathbb{F}^n$. Then

$$(C\mathbf{v}|C\mathbf{v}') = (\sum_i \beta_i \mathbf{v}_i | \sum_j \beta'_j \mathbf{v}_j) = \sum_{i,j} \overline{\beta_i}\beta'_j(\mathbf{v}_i|\mathbf{v}_j) = \sum_i \overline{\beta_i}\beta'_i = (\mathbf{v}|\mathbf{v}')$$

Thus C preserves inner products. The converse is an easy ex reversing the above computation.

E.g. Consider the $\mathbb{R}$-space $V = \text{Span}(\sin x, \cos x)$ with inner product $(f|g) = \int_{-\pi}^{\pi} f(t)g(t)dt$. Then the co-ord system $C = \frac{1}{\sqrt{\pi}}(\sin x \quad \cos x) : \mathbb{R}^2 \longrightarrow V$ is orthonormal.

Rem Every fin dim inner product space has an orthonormal co-ord system by Gram-Schmidt.

Isometries

Defn

Let $T : V \longrightarrow W$ be a (not necessarily linear) fn. We say T is an *isometry* of V if it preserves "distances" in the sense that $\|T\mathbf{v} - T\mathbf{v}'\| = \|\mathbf{v} - \mathbf{v}'\|$ for all $\mathbf{v}, \mathbf{v}' \in V$.

E.g. 1 Pick $\mathbf{v}_0 \in \mathbb{R}^n$. The translation by $\mathbf{v}_0$ map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n : \mathbf{v} \mapsto \mathbf{v} + \mathbf{v}_0$ is an isometry which is not linear unless $\mathbf{v}_0 = \mathbf{0}$.

Why?

E.g. 2 Reflection about a plane in $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ which is linear if the plane is a subspace.

E.g. 3 Rotation about a line in $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ which is linear if the line is a subspace.

Linear isometries

Prop

Suppose that the inner product space V is over $\mathbb{R}$

1. $(\mathbf{v}|\mathbf{v}') = \frac{1}{2}\left(\|\mathbf{v}+\mathbf{v}'\|^2 - \|\mathbf{v}\|^2 - \|\mathbf{v}'\|^2\right)$ for all $\mathbf{v}, \mathbf{v}' \in V$.
2. A linear map $T : V \longrightarrow W$ preserves inner products iff T is a linear isometry.

Proof. 1) Just calculate.

2) If T preserves inner products then

$$\|T\mathbf{v} - T\mathbf{v}'\|^2 = (T(\mathbf{v}-\mathbf{v}')|T(\mathbf{v}-\mathbf{v}')) = (\mathbf{v}-\mathbf{v}'|\mathbf{v}-\mathbf{v}') = \|\mathbf{v}-\mathbf{v}'\|^2$$

so T is an isometry. Conversely, if T is a linear isometry, then $\|T\mathbf{v}\| = \|T\mathbf{v} - T\mathbf{0}\| = \|\mathbf{v}-\mathbf{0}\| = \|\mathbf{v}\|$ for all $\mathbf{v} \in V$ so T preserves inner products by 1).

Rem The result 2) above is true over $\mathbb{C}$ but you need a more complicated version of 1).

Unitary operators

Prop-Defn

Let $U : V \longrightarrow W$ be linear. The following condns on U are equiv.

1. U is an isomorphism of inner product spaces.
2. U^* exists & $U^*U = \text{id}_V, UU^* = \text{id}_W$ (i.e. $U^* = U^{-1}$).

We say $U : V \longrightarrow V$ is *unitary* or a *unitary operator* if $U^* = U^{-1}$. A complex matrix $A \in M_{nn}(\mathbb{C})$ is *unitary* if $A^* = A^{-1}$. A real matrix $A \in M_{nn}(\mathbb{C})$ is *orthogonal* if $A^T = A^{-1}$. In both the complex & real case, a matrix is unitary, resp orthogonal iff the columns are an orthonormal basis of $\mathbb{C}^n$, resp $\mathbb{R}^n$.

Proof. We need only show 1) $\iff$ 2). Suppose first that U is an isomorphism of inner product spaces. Then 2) will follow if we can show U^{-1} is the adjoint. But for all $\mathbf{v} \in V, \mathbf{w} \in W$ we have

$$(U^{-1}\mathbf{w}|\mathbf{v}) = (UU^{-1}\mathbf{w}|U\mathbf{v}) = (\mathbf{w}|U\mathbf{v})$$

as required.

Suppose now that U^{-1} is an adjoint of U. We need to show 1) holds. Indeed,

Example

E.g. Show that the real matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

Rem Geometrically, we knew this since R represents

The unitary group

Let U_n denote the set of all unitary matrices in $M_{nn}(\mathbb{C})$

Prop-Defn

1. If $U, U' \in U_n$ then $UU' \in U_n$.
2. If $U \in U_n$ then $U^{-1} \in U_n$.

In particular, U_n is a group when endowed with matrix multiplication. It is called the *unitary group* & has group identity I_n.

Proof. This follows from the following easy ex

Lemma

If $T : V \longrightarrow W, S : W \longrightarrow X$ are linear maps between inner product spaces preserving inner products, then $S \circ T$ also preserves inner products.

Similary, the set of all orthogonal matrices in $M_{nn}(\mathbb{R})$ forms a group under matrix multn called the *orthogonal group* O_n.

Unitarily similar matrices

Defn

Two matrices $A, B \in M_{nn}(\mathbb{C})$ are *unitarily similar* if there is a unitary matrix $U \in M_{nn}(\mathbb{C})$ such that $A = U^{-1}BU$. In this case $A = U^*BU$. Two matrices $A, B \in M_{nn}(\mathbb{R})$ are *orthogonally similar* if there is an orthogonal matrix $U \in M_{nn}(\mathbb{R})$ such that $A = U^{-1}BU$. In this case $A = U^TBU$.

If $B \in M_{nn}(\mathbb{C})$ & A is some matrix representing it wrt some orthonormal change of co-ord systems, then A & B are unitarily similar.

Prop

Let $U : W \longrightarrow W$ be linear & $T : V \longrightarrow W$ be an isomorphism of inner product spaces. Then $(T^*UT)^* = T^*U^*T$. In particular, if U is unitary, so is $T^*UT : V \longrightarrow V$.

Proof. Indeed we just compute

Basic properties of unitary matrices

Prop

Let $U \in U_n$.

1. $|\det U| = 1$.
2. If U is orthogonal then $\det(U) = \pm 1$
3. The e-values of U also have modulus 1.

Proof. 1) & 2) Just note that

$$1 = \det I = \det(UU^*) = \det(U)\det(\overline{U}^T)$$
$$= \det(U)\det(\overline{U}) = \det(U)\overline{\det(U)} = |\det(U)|^2.$$

3) Let $\mathbf{v}$ be an e-vector with e-value λ. Then

$$\lambda(\mathbf{v}|\mathbf{v}) = (\mathbf{v}|\lambda\mathbf{v}) = (\mathbf{v}|U\mathbf{v}) = (U^*\mathbf{v}|\mathbf{v}) = (U^{-1}\mathbf{v}|\mathbf{v}) = (\lambda^{-1}\mathbf{v}|\mathbf{v}) = \overline{\lambda^{-1}}(\mathbf{v}|\mathbf{v}).$$

Hence $\overline{\lambda} = \lambda^{-1}$ & $|\lambda|^2 = 1$ which gives the result.