Definition of adjoint

Aim lecture: We generalise the adjoint of complex matrices to linear maps between fin dim inner product spaces.

In this lecture, we let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let V, W be inner product spaces with inner products denoted $(\cdot|\cdot)_V, (\cdot|\cdot)_W$. Let $D_V : V \longrightarrow V^*, D_W : W \longrightarrow W^*$ be the canonical maps.

Prop-Defn

Let $T : V \longrightarrow W, T' : W \longrightarrow V$ be linear.

1. $T^T \circ D_W = D_V \circ T' : W \longrightarrow V^*$ iff for all $\mathbf{v}, \in V, \mathbf{w} \in W$ we have $(T'\mathbf{w}|\mathbf{v})_V = (\mathbf{w}|T\mathbf{v})_W$.
2. Given T, there is at most one linear map $T' : W \longrightarrow V$ satisfying the condn in 1) above. In this case we write $T' = T^*$ & call it the *adjoint* of T.
3. If V is fin dim then T^* exists & $T^* = D_V^{-1} \circ T^T \circ D_W$.

Rem The defn of the adjoint depends critically on the inner products involved though the notation does not record this!

Proof. For 3), just note that D_V is invertible & $D_V^{-1} \circ T^T \circ D_W$ is linear being the composite of two conjugate linear maps with a linear one.

Proof cont'd

For 1) just note

$$\begin{aligned} T^T \circ D_W = D_V \circ T' &\iff \text{for all } \mathbf{w} \in W, T^T(D_W \mathbf{w}) = D_V(T'\mathbf{w}) \\ &\iff \text{for all } \mathbf{w} \in W, (D_W \mathbf{w}) \circ T = (T'\mathbf{w}|\cdot) \\ &\iff \text{for all } \mathbf{w} \in W, (\mathbf{w}|T(\cdot)) = (T'\mathbf{w}|\cdot) \end{aligned}$$

For 2), suppose T', T'' both satisfy the condn in 1). Then $D_V \circ T' = D_V \circ T''$. Suffice show for any $\mathbf{w} \in W$ that $T'\mathbf{w} = T''\mathbf{w}$. But $D_V(T'\mathbf{w}) = D_V(T''\mathbf{w})$ so the result follows from injectivity of D_V.

Fancy example (not examinable)

E.g. Let $V = C_c^\infty$ be the $\mathbb{R}$-space of compactly supported ($=$ zero outside of some compact set) infinitely differentiable functions on $\mathbb{R}$. Note $T = \frac{d}{dt}$ is a lin map from $V \longrightarrow V$. Note that we have an inner product $(f|g) = \int_{-\infty}^{\infty} f(t)g(t)dt$. What's T^*?

A Given $f, g \in V$ note that integration by parts gives

$$\begin{aligned}(f|\frac{dg}{dt}) &= \int_{-\infty}^{\infty} f(t)\frac{dg(t)}{dt}dt \\ &= [f(t)g(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df(t)}{dt}g(t)dt \\ &= -\int_{-\infty}^{\infty} \frac{df(t)}{dt}g(t)dt \\ &= (-\frac{df}{dt}|g)\end{aligned}$$

Hence $T^* = -\frac{d}{dt}$.

Connection with matrix adjoint

Q Let $A \in M_{mn}(\mathbb{F})$ & $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ be the assoc lin map. What is $T_A^* : \mathbb{F}^m \longrightarrow \mathbb{F}^n$?

Answer

$$T_A^* = T_{A^*}$$

Proof. It suffices to show that for any $\mathbf{v} \in \mathbb{F}^n, \mathbf{w} \in \mathbb{F}^m$ we have $(A^*\mathbf{w}|\mathbf{v}) = (\mathbf{w}|A\mathbf{v})$. Indeed

Conjugate linearity & functoriality

Prop

The following formulae hold whenever they make sense. S, T are appropriate linear maps.

1. $(S+T)^* = S^* + T^*$.
2. $(\beta S)^* = \overline{\beta} S^*$ for $\beta \in \mathbb{F}$.
3. $(S \circ T)^* = T^* \circ S^*$.
4. $(T^*)^* = T$.
5. $\mathrm{id}^* = \mathrm{id}$.

In particular, if V, W are fin dim, then the adjoint operator $(\cdot)^* : L(V,W) \longrightarrow L(W,V) : S \mapsto S^*$ is conjugate linear.

Proof. These all follow from defns & corresponding results for the transpose. For example,

Orthogonal complements to kernels

Prop

Let $T : V \longrightarrow W$ be a linear map between fin dim inner product spaces. Then

1. $(\ker T)^\perp = \text{im } T^*$
2. If V, W are fin dim then, $\operatorname{rank} T = \operatorname{rank} T^*$.

Proof. 1) We show equiv that $(\text{im } T^*)^\perp = \ker T$. Let $\mathbf{v} \in V$

$$\begin{aligned} \mathbf{v} \in (\text{im } T^*)^\perp &\iff (T^*\mathbf{w}|\mathbf{v}) = 0 \text{ for all } \mathbf{w} \in W \\ &\iff (\mathbf{w}|T\mathbf{v}) = 0 \text{ for all } \mathbf{w} \in W \\ &\iff 0 = (T\mathbf{v}|\cdot) = D_W(T\mathbf{v}) \\ &\iff T\mathbf{v} = \mathbf{0}, \text{ (recall } D_W \text{ is injective)} \\ &\iff \mathbf{v} \in \ker T \end{aligned}$$

For 2) we use rank-nullity

$$\begin{aligned} \operatorname{rank} T^* &= \dim(\text{im } T^*) = \dim((\ker T)^\perp)) \\ &= \dim V - \dim(\ker T) = \dim(\text{im } T) = \operatorname{rank} T. \end{aligned}$$

Kernel of $T^* \circ T$

Prop

Let $T : V \longrightarrow W$ be linear.

1. If $S : W \longrightarrow X$ is linear then $\ker(S \circ T) \supseteq \ker T$.
2. $\ker T^* \circ T = \ker T$.
3. If V, W are fin dim then $\operatorname{rank} T^* \circ T = \operatorname{rank} T$.

Proof. 1) is easy ex.

2) By 1), it suffices to show $\ker T^* \circ T \subseteq \ker T$ so let $\mathbf{v} \in \ker T^* \circ T$. Then

$$0 = (T^*(T\mathbf{v})|\mathbf{v}) = (T\mathbf{v}|T\mathbf{v})$$

so $T\mathbf{v} = \mathbf{0}$ & $\mathbf{v} \in \ker T$. Part 2) follows.

3) This follows from the rank-nullity thm since part 2) ensures the nullities are the same (whilst the domains are also the same).