Dual vector space

Aim lecture: We generalise the notion of transposes of matrices to arbitrary linear maps by introducing dual vector spaces.

In most of this lecture, we allow $\mathbb{F}$ to be a general field.

Defn

Let $V = \mathbb{F}$-space. The *dual* of V is the $\mathbb{F}$-space $V^* = L(V, \mathbb{F})$. The elements of V^* are called *linear functionals*.

E.g.1 $(\mathbb{F}^n)^* = L(\mathbb{F}^n, \mathbb{F}) = M_{1n}(\mathbb{F})$, the set of length n row vectors in $\mathbb{F}$. Note that the transpose map $(\cdot)^T : \mathbb{F}^n \longrightarrow (\mathbb{F}^n)^* : \mathbf{v} \mapsto \mathbf{v}^T$ & $\mathbb{F}^n$ is an isomorphism!

E.g. 2 If X is a set & $V = \text{Fun}(X, \mathbb{F})$, then for every $x \in X$ we obtain an element $ev_x \in V^*$.

E.g. 3 Given an inner product $(\cdot|\cdot)$ on V & $\mathbf{v} \in V$ we have $(\mathbf{v}|\cdot) \in V^*$.

Visualising linear functionals on $\mathbb{R}^n$

Let $l \in (\mathbb{R}^n)^*$ is in particular, a fn $l : \mathbb{R}^n \longrightarrow \mathbb{R}$ so can be visualised by drawing level curves. Suppose $l \neq 0$ so im $l = \mathbb{R}$.

The level curve $l = 0$ is just ker l which is a subspace of dim

Transpose

We can now generalise the transpose map for matrices.

Prop-Defn

Let $S : V \longrightarrow W$ be linear. The *transpose* of S is the fn $S^T : W^* \longrightarrow V^*$ defined by $S^T f = f \circ S : V \xrightarrow{S} W \xrightarrow{f} \mathbb{F}$ for $f \in W^*$. S^T is linear.

Proof. Linearity follows from the distributive law: for $f, g \in W^*$ we have $(f+g) \circ S = f \circ S + g \circ S$ & the fact that $(\beta f) \circ S = \beta(f \circ S)$ for $\beta \in \mathbb{F}$.

E.g. Let $S = \frac{d}{dx} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ so $S^T : \mathbb{R}[x]^* \longrightarrow \mathbb{R}[x]^*$. What's $S^T \mathrm{ev}_0 \in \mathbb{R}[x]^*$?

A

Connection with matrix transpose

Q Let $A \in M_{mn}(\mathbb{F})$ & $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ be the assoc lin map. What is $T_A^T : (\mathbb{F}^m)^* = M_{1m}(\mathbb{F}) \longrightarrow (\mathbb{F}^n)^* = M_{1n}(\mathbb{F})$?

Simplest answer T_A^T is pre-composition (= right composition) with T_A. Since composition corresponds with matrix multn, this is *right* multn by A. Note, T_A is *left* multn by A.

More formally, let $\mathbf{v}^T \in M_{1m}(\mathbb{F}) = (\mathbb{F}^m)^*, \mathbf{w} \in \mathbb{F}^n$. Then

$$(T_A^T \mathbf{v}^T)\mathbf{w} = (\mathbf{v}^T \circ T_A)\mathbf{w} = (\mathbf{v}^T A)\mathbf{w}$$

so $T_A^T \mathbf{v}^T = \mathbf{v}^T A$.

Q What's the matrix representing T_A^T wrt natural co-ordinate systems $(\cdot)^T : \mathbb{F}^m \longrightarrow (\mathbb{F}^m)^*, (\cdot)^T : \mathbb{F}^n \longrightarrow (\mathbb{F}^n)^*$.

Answer

The matrix representing T_A^T wrt the natural co-ord systems is A^T.

Why?

Functoriality

Prop

1. $(\mathrm{id}_V)^T = \mathrm{id}_{V^*} : V^* \longrightarrow V^*$.
2. Consider a composite of linear maps $S \circ T : U \xrightarrow{T} V \xrightarrow{S} W$. Then $(S \circ T)^T = T^T \circ S^T : W^* \longrightarrow U^*$.
3. In particular, if $S : V \longrightarrow W$ is an isomorphism, so is S^T.

Proof. For 1) note $(\mathrm{id}_V)^T : f \mapsto f \circ \mathrm{id}_V = f$.

2) For $f \in W^*$, we have

3) Suffice show $S^{-T} = (S^{-1})^T$ is inverse to S^T. But by 1) & 2)

$$S^T \circ (S^{-1})^T = (S^{-1} \circ S)^T = \mathrm{id}^T = \mathrm{id}$$

& sim $(S^{-1})^T \circ S^T = \mathrm{id}$ so we are done.

Dimension of the dual vector space

Prop

Let $V =$ fin dim $\mathbb{F}$-space. Then $\dim V^* = \dim V$.

Proof. Let $C : \mathbb{F}^d \longrightarrow V$ be a co-ord system so dim $= d$. Then $C^T : V^* \longrightarrow (\mathbb{F}^d)^*$ is also an isomorphism so

$$\dim V^* = \dim(\mathbb{F}^d)^* = d.$$

E.g. $\{\text{ev}_0, \text{ev}_1\}$ is a basis for $\mathbb{C}[x]^*_{\leq 1}$ since it is lin indep (ex) & $\dim \mathbb{C}[x]^*_{\leq 1} = \dim \mathbb{C}[x]_{\leq 1} = 2$.

Linearity of the transpose operator

The matrix transpose operator $(\cdot)^T : M_{mn}(\mathbb{F}) \longrightarrow M_{nm}(\mathbb{F}) : A \mapsto A^T$ is linear. This suggests

Prop

Let $\beta \in \mathbb{F}$ & $R, S : V \longrightarrow W$ be linear. Then

1. $(R + S)^T = R^T + S^T$
2. $(\beta S)^T = \beta S^T$

In particular the map $(\cdot)^T : L(V, W) \longrightarrow L(W^*, V^*) : S \mapsto S^T$ is linear.

Proof. For $f \in W^*$, this follows from

Inner products & dual vector spaces

Let now $V = \mathbb{F}$-space equipped with an inner product $(\cdot|\cdot)$ so $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Prop

1. The canonical map $D : V \longrightarrow V^* : \mathbf{v} \mapsto (\mathbf{v}|\cdot)$ is an injective conjugate linear map.
2. If V is fin dim then D is a conjugate linear isomorphism.

Proof. 1) (Ex) Conjugate linearity of $(\cdot|\mathbf{w})$ shows D is conjugate linear. To check injectivity suppose $D\mathbf{v} = D\mathbf{v}'$ so that by conjugate linearity

$$(\mathbf{v} - \mathbf{v}'|\mathbf{v} - \mathbf{v}') = (\mathbf{v}|\mathbf{v} - \mathbf{v}') - (\mathbf{v}'|\mathbf{v} - \mathbf{v}') = (D\mathbf{v})(\mathbf{v} - \mathbf{v}') - (D\mathbf{v}')(\mathbf{v} - \mathbf{v}') = 0$$

so $\mathbf{v} - \mathbf{v}' = \mathbf{0}$ by inner product axioms.

2) We now check D is onto & show any $l \in V^*$ is in the image of D. Now $D\mathbf{0} = \mathbf{0}$ so we can assume $l \neq \mathbf{0}$ so has image $\mathbb{F}$. Our isomorphism thm applied to $l : V \longrightarrow \mathbb{F} \implies$ for any vector space complement W to $\ker l$ is isomorphic to $\mathbb{F}$ via the restricted map $l|_W : W \longrightarrow \mathbb{F}$. This holds in particular for $W = (\ker l)^\perp$ so we can find $\mathbf{w} \in W$ with $l(\mathbf{w}) = 1$.

Proof completed

It suffices now to show $D\frac{\mathbf{w}}{\|\mathbf{w}\|^2} = l$. Now any vector in $V = W \oplus (\ker l)$ can be written in the form $\beta\mathbf{w} + \mathbf{v}$ with $\beta \in \mathbb{F}, \mathbf{v} \in \ker l$. Then

$$\left(D\frac{\mathbf{w}}{\|\mathbf{w}\|^2}\right)(\beta\mathbf{w} + \mathbf{v}) = (\frac{\mathbf{w}}{\|\mathbf{w}\|^2}|\beta\mathbf{w} + \mathbf{v}) = \beta(\frac{\mathbf{w}}{\|\mathbf{w}\|^2}|\mathbf{w}) = \beta = l(\beta\mathbf{w} + \mathbf{v})$$

& we are done.

E.g. Let $V = \mathbb{R}[x]_{\leq 1}$ with inner product $(f|g) = \int_0^1 f(t)g(t)dt$. Find $f \in V$ such that $(f|\cdot) = \mathrm{ev}_0$.