Orthogonal complement

Aim lecture: Inner products give a special way of constructing vector space complements.

As usual, in this lecture $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We also let V be an $\mathbb{F}$-space equipped with an inner product $(\cdot|\cdot)$.

Defn

Let $S \subseteq V$. We define the *orthogonal complement* to S to be

$$\begin{aligned} S^\perp &= \{\mathbf{v} \in V | \mathbf{v} \perp S\} \\ &= \cap_{\mathbf{w} \in S} \ker(\mathbf{w}|\cdot) \end{aligned}$$

Hence $S^\perp$ is a subspace orthogonal to S & in particular, is closed under addition.

Proof. Clear.

E.g. This concept is easily understood in $\mathbb{R}^3$

Orthogonal complements of spans

Lemma

Let $S \subseteq V$. Then $Span(S)^{\perp} = S^{\perp}$

Proof.

$$\begin{aligned} \mathbf{w} \in S^{\perp} &\iff \mathbf{v} \perp \mathbf{w} \text{ for all } \mathbf{v} \in S \\ &\iff \mathbf{w} \perp \mathbf{v} \text{ for all } \mathbf{v} \in S \\ &\iff S \subseteq \ker(\mathbf{w}|\cdot) \\ &\iff \mathrm{Span}(S) \subseteq \ker(\mathbf{w}|\cdot) \\ &\iff \mathbf{w} \in \mathrm{Span}(S)^{\perp} \end{aligned}$$

This completes the proof.

E.g. The orthogonal complement to $S = \mathrm{Span}((1,1,0)^T, (0,1,1)^T)$ is

Orthogonal (internal) direct sums

Prop-Defn

Let $W_1, \ldots, W_r \leq V$ be mutually orthogonal subspaces i.e. $W_i \perp W_j$ whenever $i \neq j$. Then the sum $\sum_{i=1}^r W_i$ is direct & we say the internal direct sum $\oplus_i W_i$ is *orthogonal*.

Proof. The lemma ensures that W_r is orthogonal to $W_{<r} = \sum_{i=1}^{r-1} W_i$ so by induction, it suffices to show that any $\mathbf{w} \in W_r \cap W_{<r} \subseteq W_r \cap W_r^{\perp}$ must be $\mathbf{0}$. But $\mathbf{w} \perp \mathbf{w}$ so $(\mathbf{w}|\mathbf{w}) = 0$ & $\mathbf{w} = \mathbf{0}$. This completes the proof.

E.g.

Vector space complement

Prop

Let $W \leq V$ & $\dim W < \infty$ then $W + W^{\perp} = V$ so $V = W \oplus W^{\perp}$.

Proof. We prove this here only in the case where $\dim V < \infty$. The propn on orthog direct sums ensures the sum $W + W^{\perp}$ is direct. Pick a basis $\mathbf{w}_1, \ldots, \mathbf{w}_r \in W$ for W. For $i = 1, \ldots, r$ we have $l_i = (\mathbf{w}_i | \cdot) \in L(V, \mathbb{F})$ so we may form the $r \times 1$-matrix T whose i-th entry is l_i. Note $T : V \longrightarrow \mathbb{F}^r : \mathbf{v} \mapsto (l_1(\mathbf{v}), \ldots, l_r(\mathbf{v}))^T$.

By the lemma

$$W^{\perp} = \cap_i \ker l_i = \ker T.$$

Since $\text{im } T \leq \mathbb{F}^r$, rank-nullity ensures that

$$\dim W^{\perp} = \dim V - \dim \text{im } T \geq \dim V - r = \dim V - \dim W.$$

However, the sum $W + W^{\perp}$ is direct, so we must have $\dim W + W^{\perp} = \dim W + \dim W^{\perp} = \dim V$ which ensures $V = W + W^{\perp}$ as desired.

Examples

E.g. Let $V = \mathbb{C}[x]_{\leq 1}$ with inner product $(f|g) = \int_0^1 \overline{f(t)} g(t) dt$. Find the orthogonal complement to $W = \mathbb{C}(1 + ix)$.

Orthogonal projections

Given an internal direct sum $V = W + W'$, let
$\Phi : W \oplus W' \longrightarrow V : \binom{\mathbf{w}}{\mathbf{w}'} \mapsto \mathbf{w} + \mathbf{w}'$ be the natural isomorphism. Note that we have the following linear projn map

$$\begin{pmatrix} \mathrm{id}\,|_W & 0 \\ 0 & 0 \end{pmatrix} : W \oplus W' \longrightarrow W \oplus W' : \binom{\mathbf{w}}{\mathbf{w}'} \mapsto \binom{\mathbf{w}}{\mathbf{0}}.$$

Prop-Defn

1. Suppose that $V = W \oplus W^\perp$ (e.g. when $\dim W < \infty$). The *orthogonal projection* onto W is the linear map defined by the composite

$$\mathrm{proj}_W = \Phi \circ \begin{pmatrix} \mathrm{id}\,|_W & 0 \\ 0 & 0 \end{pmatrix} \circ \Phi^{-1} : V \longrightarrow W \oplus W^\perp \longrightarrow W \oplus W^\perp \longrightarrow V.$$

2. For $\mathbf{v} \in V$ we can uniquely write $\mathbf{v} = \mathbf{w} + \mathbf{w}'$ with $\mathbf{w} \in W, \mathbf{w}' \in W^\perp$. Then $\mathrm{proj}_W : \mathbf{v} \mapsto \binom{\mathbf{w}}{\mathbf{w}'} \mapsto \binom{\mathbf{w}}{\mathbf{0}} \mapsto \mathbf{w}$. Equivalently, $\mathrm{proj}_W \mathbf{v} = \mathbf{w}$ is the unique vector in W such that $\mathbf{v} - \mathrm{proj}_W \mathbf{v} \in W^\perp$.

Proof is easy. More instructive to look at an

E.g. Consider the orthogonal direct sum $\mathbb{R}^2 = \mathbb{R}\binom{1}{1} \oplus \mathbb{R}\binom{1}{-1}$.

Orthogonal bases

Prop-Defn

1. If $V = W_1 \oplus \ldots \oplus W_r$ is an orthogonal direct sum then $W_i^\perp = \sum_{j \neq i} W_j$. In particular, if $V = W \oplus W^\perp$ then $(W^\perp)^\perp = W$.
2. We say a set $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_r\} \subseteq V - \mathbf{0}$ is *orthogonal* if $\mathbf{w}_i \perp \mathbf{w}_j$ for $i \neq j$. Equivalently, the sum $\sum_i \mathbb{F}\,\mathbf{w}_i$ is an orthogonal direct sum. In particular, S is lin indep in this case. Thus if S spans V we call it an *orthogonal basis*.
3. An orthogonal set $S \subseteq V$ is *orthonormal* if furthermore, $\|\mathbf{w}_i\| = 1$ for all i. If S spans V we call it an *orthonormal basis*.

Proof. We prove 1) first noting that $W_{\neq i} = \sum_{j \neq i} W_j$ is orthogonal to W_i. It thus suffices to show $W_i^\perp \subseteq W_{\neq i}$ so suppose $\mathbf{w} \perp W_i$. We may write $\mathbf{w} = \mathbf{w}_i' + \mathbf{w}_i$ with $\mathbf{w}_i \in W_i, \mathbf{w}_i' \in W_{\neq i}$. Then

$$0 = (\mathbf{w}_i|\mathbf{w}) = (\mathbf{w}_i|\mathbf{w}_i' + \mathbf{w}_i) = (\mathbf{w}_i|\mathbf{w}_i') + (\mathbf{w}_i|\mathbf{w}_i) = (\mathbf{w}_i|\mathbf{w}_i)$$

so $\mathbf{w}_i = \mathbf{0}$ & $\mathbf{w} = \mathbf{w}_i' \in W_{\neq i}$.

Existence of orthonormal bases

Theorem

Let V be fin dim. Then

1. *V is the orthogonal direct sum of 1-dimensional vector spaces.*
2. *V has an orthonormal basis.*

Proof. 1) We argue by induction on $d = \dim V$, the cases $d = 0, 1$ being clear so suppose that $d > 1$. We may thus pick a non-zero subspace $W \neq V$ e.g. $\mathbb{F}\,\mathbf{w}$ for any non-zero $\mathbf{w} \in W$. Now $V = W \oplus W^\perp$ & $\dim W, \dim W^\perp < d$. By induction, each of W & $W^\perp$ are orthogonal direct sums of 1-dimensional $\mathbb{F}$-spaces, say $W = \oplus_i W_i, W^\perp = \oplus_j V_j$. Clearly, the subspaces $\{W_i, V_j\}$ are still mutually orthog so V is the orthogonal direct sum of them.

2) By 1), it suffices to find an orthonormal basis for a 1-dim $\mathbb{F}$-space $\mathbb{F}\,\mathbf{v}$. Just pick $\frac{\mathbf{v}}{\|\mathbf{v}\|}$.

Orthogonal projection formula

Prop

1. Given a 1-dim $\mathbb{F}$-space $W = \mathbb{F}\mathbf{w}$ we have $V = W \oplus W^\perp$ & $\text{proj}_W \mathbf{v} = \frac{(\mathbf{w}|\mathbf{v})}{\|\mathbf{w}\|^2}\mathbf{w}$ for any $\mathbf{v} \in V$.
2. Suppose that $V = W_i \oplus W_i^\perp$ for $i = 1, \ldots, r$ so we have orthog projn maps proj_{W_i}. Suppose further that the W_i are mutually orthog so we may consider the orthogonal direct sum $W = \oplus W_i$. Then $V = W \oplus W^\perp$ & $\text{proj}_W = \sum_i \text{proj}_{W_i}$.
3. ("Fourier decomposition") In particular, if W is spanned by the orthog set $\{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$, then

$$\text{proj}_W \mathbf{v} = \sum_{i=1}^{r} \frac{(\mathbf{w}_i|\mathbf{v})}{\|\mathbf{w}_i\|^2}\mathbf{w}_i.$$

Rem This gives a proof of the propn on vector space complements in general.

Proof. Note 3) follows immediately from 1) & 2)

Proof propn

1) We need only show $\mathbf{v}' = \mathbf{v} - \frac{(\mathbf{w}|\mathbf{v})}{\|\mathbf{w}\|^2}\mathbf{w} \in W^\perp$ for then we see $\mathbf{v} = \frac{(\mathbf{w}|\mathbf{v})}{\|\mathbf{w}\|^2}\mathbf{w} + \mathbf{v}' \in W + W^\perp$. But this holds since

$$(\mathbf{w}|\mathbf{v}') = (\mathbf{w}|\mathbf{v} - \frac{(\mathbf{w}|\mathbf{v})}{\|\mathbf{w}\|^2}\mathbf{w}) = (\mathbf{w}|\mathbf{v}) - \frac{(\mathbf{w}|\mathbf{v})}{\|\mathbf{w}\|^2}(\mathbf{w}|\mathbf{w}) = 0$$

2) As above $V = W \oplus W^\perp$ follows from showing $\mathbf{v} - \sum_i \text{proj}_{W_i}\mathbf{v} \perp W$, which in turn follows on showing $\mathbf{v} - \sum_i \text{proj}_{W_i}\mathbf{v} \perp W_j$ for all j (ex). In this case we have an orthog direct sum $V = W_1 \oplus \ldots W_r \oplus W^\perp$. For $\mathbf{v} \in V$, write uniquely $\mathbf{v} = \mathbf{w}_1 + \ldots + \mathbf{w}_r + \mathbf{w}'$ with $\mathbf{w}_i \in W_i, \mathbf{w}' \in W^\perp$. Note $\text{proj}_{W_i}\mathbf{v} = \mathbf{w}_i$ so

$$\text{proj}_W\mathbf{v} = \mathbf{w}_1 + \ldots + \mathbf{w}_r = \sum_i \text{proj}_{W_i}\mathbf{v}.$$

Example

E.g. Consider the orthonormal basis $f_1(x) = 1, f_2(x) = 2\sqrt{3}x - \sqrt{3}$ for $W = \mathbb{R}[x]_{\leq 1}$ (wrt $(f|g) = \int_0^1 \overline{f(t)}g(t)dt$). Find $\text{proj}_W x^2$.