Inner product spaces

Aim lecture: Vector spaces have some geometry but their data encodes no info about angles & lengths. For this we need inner products which we define here.

Throughout this lecture (& the next few) $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Defn

Let $V = \mathbb{F}$-space. An inner product on V is a function $(\cdot|\cdot) : V \times V \longrightarrow \mathbb{F} : (\mathbf{v}, \mathbf{w}) \mapsto (\mathbf{v}|\mathbf{w})$ such that for all $\mathbf{v}, \mathbf{w}, \mathbf{w}' \in V, \beta \in \mathbb{F}$ we have

1. $(\mathbf{v}|\beta\mathbf{w} + \mathbf{w}') = \beta(\mathbf{v}|\mathbf{w}) + (\mathbf{v}|\mathbf{w}')$
2. $(\mathbf{w}|\mathbf{v}) = \overline{(\mathbf{v}|\mathbf{w})}$ where the bar denotes complex conjugation.
3. $(\mathbf{v}|\mathbf{v})$ is a positive real if $\mathbf{v} \neq \mathbf{0}$.

An *inner product space* is a vector space V equipped with an inner product as above.

Rem 1 Axiom 1) means that the function $l_{\mathbf{v}} = (\mathbf{v}|\cdot) : V \longrightarrow \mathbb{F} : \mathbf{w} \mapsto (\mathbf{v}|\mathbf{w})$ is linear so in particular $(\mathbf{v}|\mathbf{0}) = 0$.

Rem 2 If $\mathbb{F} = \mathbb{R}$ then axiom 2) simplifies to $(\mathbf{w}|\mathbf{v}) = (\mathbf{v}|\mathbf{w})$

E.g. For $V = \mathbb{R}^n$, the dot product $(\mathbf{v}|\mathbf{w}) = \mathbf{v}.\mathbf{w} = \mathbf{v}^T\mathbf{w}$ is an inner product on V.

Conjugates & adjoints of complex matrices

Let $A = (a_{ij})_{ij} \in M_{mn}(\mathbb{C})$. The *conjugate* of A is $\overline{A} = (\overline{a_{ij}})_{ij}$. The *adjoint* of A is $A^* = \overline{A^T} = \overline{A}^T$. (Do not confuse with the classical adjoint!).

Prop

Let $A, B \in M_{mn}(\mathbb{C}), X \in M_{nl}(\mathbb{C}), \beta \in \mathbb{C}$. Then

1. $\overline{(A+B)} = \overline{A} + \overline{B}$
2. $\overline{\beta A} = \overline{\beta}\ \overline{A}$
3. $\overline{AX} = \overline{A}\ \overline{X}$
4. $(A+B)^* = A^* + B^*$
5. $(\beta A)^* = \overline{\beta} A^*$
6. $(AX)^* = X^* A^*$
7. $A^{**} = A$

Proof. These are good easy ex. For example

Standard inner product on $\mathbb{C}^n$

Prop

The following is an inner product on the $\mathbb{C}$-space $\mathbb{C}^n$

$$(\mathbf{v}|\mathbf{w}) = \mathbf{v}^*\mathbf{w}.$$

It is called the *standard inner product* on $\mathbb{C}^n$.

Proof.

E.g.

Function spaces

Fix an interval $[a, b] \subseteq \mathbb{R}$. Let $V =$ the vector space of continuous $\mathbb{F}$-valued funtions on $[a, b]$.

Defn

The *standard* inner product on V is for $f, g \in V$

$$(f|g) = \int_a^b \overline{f(t)}g(t)dt$$

Proof. Easy ex in checking axioms. For example,

E.g.

Subspaces

Prop

Let V be a $\mathbb{F}$-space equipped with an inner product $(\cdot|\cdot)$ & $W \leq V$. Then $(\cdot|\cdot)$ restricts to an inner product on W too.

Proof. Inner product axioms are immediate.

E.g. By identifying real & complex polys with the functions they induce on $[a, b]$ we get an inner product on $\mathbb{F}[x]$ or $\mathbb{F}[x]_{\leq d}$.

Lengths & orthogonality

Defn

Let V be an inner product space with inner product $(\cdot|\cdot)$. The *norm* of $\mathbf{v} \in V$ is $\|\mathbf{v}\| = \sqrt{(\mathbf{v}|\mathbf{v})}$. We say that $S, S' \subseteq V$ are *orthogonal* if $(\mathbf{v}|\mathbf{v}') = 0$ (so also $(\mathbf{v}'|\mathbf{v}) = 0$) for all $\mathbf{v} \in S, \mathbf{v}' \in S'$. In this case we write $S \perp S'$.

E.g. Let $V =$ the vector space of continuous $\mathbb{R}$-valued funtions on $[0, 2\pi]$ equipped with the standard inner product.

Geometry in inner product spaces

Prop

Let V be an inner product space with inner product $(\cdot|\cdot)$. Let $\mathbf{v}, \mathbf{w} \in V$.

1. (Cauchy-Schwarz inequality) $|(\mathbf{v}|\mathbf{w})| \leq \|\mathbf{v}\| \, \|\mathbf{w}\|$
   with equality iff $\mathbf{v}, \mathbf{w}$ are lin dep.
2. (Triangle inequality) $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$
3. (Pythagoras theorem) If $\mathbf{v} \perp \mathbf{w}$ then $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

Proof. Same as for dot products in $\mathbb{R}^n$ so we only prove 3)

E.g. $\left(\int_0^1 \overline{f(t)} g(t) dt\right)^2 \leq \left(\int_0^1 |f(t)|^2 dt\right) \left(\int_0^1 |g(t)|^2 dt\right)$
by

Conjugate linearity

Prop-Defn

Let $T : V \longrightarrow W$ be a fn between $\mathbb{F}$-spaces. We say that T is *conjugate linear* if for $\mathbf{v}, \mathbf{w} \in V, \beta \in \mathbb{F}$ we have: i) $T(\mathbf{v} + \mathbf{w}) = T\mathbf{v} + T\mathbf{w}$ & ii) $T(\beta\mathbf{v}) = \overline{\beta}T\mathbf{v}$.

1. The composite of conjugate linear maps is linear.
2. The composite of a conjugate linear map with a linear one (in either order that makes sense) is conjugate linear.
3. If T is an invertible conjugate linear map, then T^{-1} is also conjugate linear & we say T is a *conjugate linear isomorphism*.

Proof. Easy ex.

E.g. 1 Given an inner product space V the map $(\cdot|\mathbf{w}) : V \longrightarrow \mathbb{F} : \mathbf{v} \mapsto (\mathbf{v}|\mathbf{w})$ is conjugate linear.

E.g. 2 $V = \mathbb{C}^n$