Basic idea of variation of parameters

Aim lecture: We intro the method of variation of parameters to extend our DE techniques to some inhomogeneous eqns.

Let $A \in M_{nn}(\mathbb{C})$. Recall

Fact

For any $\mathbf{w} \in \mathbb{C}^n$, $\mathbf{y}(t) = \exp(tA)\mathbf{w}$ is a soln to $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$.

We think of $\mathbf{w}$ as a parameter & now vary it with time, i.e. replace it with the vector-valued fn $\mathbf{w}(t)$. The product rule gives

Variation of parameters formula

$$\frac{d}{dt}(\exp(tA)\mathbf{w}(t)) = \frac{d}{dt}[\exp(tA)]\mathbf{w}(t)+\exp(tA)\frac{d\mathbf{w}}{dt} = A\exp(tA)\mathbf{w}(t)+\exp(tA)\frac{d\mathbf{w}}{dt}$$

Hence $\mathbf{y}(t) = \exp(tA)\mathbf{w}(t)$ is a soln to the DE

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t) + \mathbf{v}(t) \qquad (*)$$

where

$$\mathbf{v}(t) = \exp(tA)\frac{d\mathbf{w}}{dt} \qquad (**).$$

To find a particular soln to (*), we just need to solve (**) for $\mathbf{w}(t)$ in terms of $\mathbf{v}(t)$ by integrating $\exp(-tA)\mathbf{v}(t)$. This gives

Formula

The soln to $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t) + \mathbf{v}(t), \mathbf{y}(0) = \mathbf{y}_0$ is given by

$$\mathbf{y}(t) = \exp(tA)\mathbf{y}_0 + \exp(tA)\int_0^t \exp(-sA)\mathbf{v}(s)ds.$$

Example

E.g. Find the Jordan canonical form $J = C^{-1}AC$ of

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

& hence solve

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t) + e^{4t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

given the initial condn $\mathbf{y}(0) = \mathbf{0}$.
A

Hence the decomposition into gen e-spaces has form $\mathbb{R}^3 = E_{-1}(1) \oplus E_2(2)$.

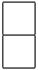
Compute gen e-spaces

$\underline{\lambda = -1}$

$$\ker(A + I) = \ker \begin{pmatrix} 0 & 0 & 0 \\ 1 & 4 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$\underline{\lambda = 2}$ $\ker(A - 2I) = \ker \begin{pmatrix} -3 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} =$

$$\ker(A - 2I)^2 = \ker \begin{pmatrix} 9 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$

Jordan form tableaux wrt $\lambda = -1$ & 2 are  and 

The Jordan canonical form & exponential matrix

For the length 2 Jordan chain use the seed $(0, 1, 0)^T$ & note

$$(A - 2I)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$$

We find $A = CJC^{-1}$ where $J = J_1(-1) \oplus J_2(2)$ & C is the change of co-ords matrix

$$\begin{pmatrix} 9 & 0 & 0 \\ -2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Hence

$$\exp(tA) = C\exp(tJ)C^{-1} = C\begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} C^{-1}$$

Soln to the DE

We compute $\mathbf{y}(t) = \exp(tA)\mathbf{y}(0) + \exp(tA)\int_0^t \exp(-sA)e^{4s}(0,1,1)^T ds = \exp(tA)\int_0^t \exp(-sA)e^{4s}(0,1,1)^T ds$ as follows.

$$\exp(-sA)e^{4s}\begin{pmatrix}0\\1\\1\end{pmatrix} = C\exp(-sJ)e^{4s}C^{-1}\begin{pmatrix}0\\1\\1\end{pmatrix} = e^{4s}C\exp(-sJ)\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= e^{4s}C\begin{pmatrix}0\\e^{-2s}\\0\end{pmatrix} = e^{2s}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\int_0^t \exp(-sA)e^{4s}\begin{pmatrix}0\\1\\1\end{pmatrix} ds = \frac{1}{2}(e^{2t}-1)\begin{pmatrix}0\\1\\1\end{pmatrix}$$

Hence

$$\mathbf{y}(t) = \exp(tA)\frac{1}{2}(e^{2t}-1)\begin{pmatrix}0\\1\\1\end{pmatrix} = \frac{1}{2}(e^{2t}-1)C\exp(tJ)C^{-1}\begin{pmatrix}0\\1\\1\end{pmatrix} =$$

E-vectors & solns to DEs

In the previous example, we did not need to compute the Jordan canonical form for A to solve $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t) + \mathbf{v}(t)$. This is because $\mathbf{v}(t)$ turned out to be a scalar function times an e-vector of A. We could then have used

Prop

Let $A \in M_{nn}(\mathbb{C})$ and $\mathbf{v} \in \mathbb{C}^n$ be an e-vector with e-value λ. Then

$$\exp(tA)\mathbf{v} = e^{\lambda t}\mathbf{v}.$$

Proof.

$$\begin{aligned}\exp(tA)\mathbf{v} &= \sum_{i \geq 0} \frac{1}{i!} t^i A^i \mathbf{v} \\ &= \sum_{i \geq 0} \frac{1}{i!} t^i \lambda^i \mathbf{v} = e^{\lambda t}\mathbf{v}.\end{aligned}$$

Example repeated

We re-compute $\mathbf{y}(t) = \exp(tA)\int_0^t \exp(-sA)e^{4s}\mathbf{v}ds$ for the e-vector $\mathbf{v} = (0,1,1)^T$ of A with e-value 2 as follows.

$$\exp(-sA)e^{4s}\mathbf{v} =$$

$$\int_0^t \exp(-sA)e^{4s}\mathbf{v}ds =$$

Hence

$$\mathbf{y}(t) =$$