Proof outline

Aim lecture: We may now complete the proof of the Jordan canonical form thm by proving existence.

For this lecture, we shall assume that $T : V \longrightarrow V$ is a linear map & $V = E_\lambda(\infty)$ is fin dim. Let $\mathcal{T}$ be the Jordan form tableau of T (wrt e-value λ) so $\mathcal{T}$ has dim V boxes. From the material in the last few lectures, existence of Jordan canonical forms follows from the following more precise result.

Theorem

We may fill the columns of $\mathcal{T}$ with vectors of Jordan chains in such a way that the sum of the corresponding Jordan chain spaces is direct.

In fact, we will exhibit an algorithm for filling in the tableau with such Jordan chains, & so give an effective method of finding Jordan canonical forms. We prove the theorem by induction with the inductive step on the next slide.

Algorithm for finding Jordan canonical forms

Suppose that the tableau has been partially filled in. We let $\mathbf{v}_{ij}$ denote the vector in the (i, j)-th box (i.e. box in the i-th row & j-th column). The thm follows by induction from

Lemma

Suppose that the first $j-1$ columns of the tableau have been filled with vectors of Jordan chains so that the sum of the corresponding Jordan chain spaces is direct. Suppose the bottom of the j-th column is the i-th row. Then

1. *We can find $\mathbf{v}_{ij} \in E_\lambda(i)$ such that the new i-th row $\{\mathbf{v}_{i1}, \ldots, \mathbf{v}_{ij}\}$ is lin indep modulo $E_\lambda(i-1)$.*
2. *Given any $\mathbf{v}_{ij}$ as in 1), we may use it to seed a length i Jordan chain. Using this to fill in the j-th column of $\mathcal{T}$, the sum of these j Jordan chain spaces is direct.*

Before we prove this, we give an example of how to use this lemma to find a Jordan canonical form.

Example

E.g. Find the Jordan canonical form J for

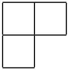
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

& find the change of co-ords matrix $C \in GL_3(\mathbb{C})$ such that $J = C^{-1}AC$.

A We first compute e-values

Compute $E_\lambda(n)$

Fill the tableau



End of example

A result on partially filled tableaux

Suppose as in the inductive step that the first $j-1$ columns of the tableau $\mathcal{T}$ have been filled with Jordan chains whose sum of Jordan chain spaces is direct.

Prop

The l-th row of the partially filled tableau is lin indep modulo $E_\lambda(l-1)$ for any l.

Proof. Let W be the sum of the $j-1$ Jordan chain spaces & $W^{<l}$ be the span of all vectors in the first $l-1$ rows of the tableau. By our lemma on kernels of direct sums (lecture 29) we know $W \cap E_\lambda(l-1) = W^{<l}$.

We know that the vectors in each column are lin indep & we are assuming that the sum of Jordan chain spaces is direct so the set of all vectors in the (partially filled) tableau form a basis for W. In particular, the vectors in the l-th row are lin indep.

Let $W^{(l)}$ be the span of the vectors in the l-th row. It remains only to show $W^{(l)} \cap E_\lambda(l-1) = \mathbf{0}$. Let

$$\mathbf{w} \in W^{(l)} \cap E_\lambda(l-1) \subseteq W^{<l}$$

so is a lin combn of vectors in the first $l-1$ rows. Lin indep of the vectors in the tableau ensures that $\mathbf{w} = \mathbf{0}$ so we are done.

Proof of lemma

Let i be the row at the bottom of the j-th column.

1) There exists $\mathbf{v}_{ij} \in E_\lambda(i)$ such that the new i-th row lin indep modulo $E_\lambda(i-1)$ since the i-th row has $\dim E_\lambda(i) - \dim E_\lambda(i-1)$ boxes only (& the previous propn guarantees the first $j-1$ vectors in the row are lin indep).

We now prove 2), that is, the sum of the first j Jordan chain spaces is direct. By assumption & our key lemma lecture 28, the l-th row of the new tableau (with j-columns filled) is still lin indep modulo $E_\lambda(l-1)$ for all l. Let $V^{<j}$ be the span of the first $j-1$ columns of vectors & $V^{(j)}$ be the span of the vectors in the j-th column. We suppose by way of contradiction that the sum is not direct & assume there is a non-zero vector $\mathbf{v} \in V^{<j} \cap V^{(j)}$. We may write

$$\mathbf{v} = \beta_1 \mathbf{v}_{1j} + \ldots + \beta_l \mathbf{v}_{lj}, \text{ with } \beta_l \neq 0.$$

Now we also have $\mathbf{v} \in V^{<j} \cap E_\lambda(l)$ so is a lin combn of vectors in the first l rows & $j-1$ columns of the tableau. We may thus re-write $\mathbf{v}_{lj}$ as lin combn of other vectors in the l-th row & a vector in $E_\lambda(l-1)$. This contradicts the lin indep of the l-th row modulo $E_\lambda(l-1)$. The lemma & Jordan canonical form thm are now proved.