Powers of elements

Aim lecture: We introduce the notion of group actions to generalise and formalise the notion of permuting objects such as the rows (or columns of a matrix).

Let $(G, *)$ be a group with identity e & $g \in G$. For $n \in \mathbb{Z}_+$ we define $g^n = g * g * \ldots * g$ (n copies of g), $g^0 = e$ and $g^{-n} = (g^n)^{-1}$. The following is easily proved by taking cases

Prop

For all $m, n \in \mathbb{Z}$ we have

1. $g^n * g^m = g^{n+m}$.
2. $(g^m)^n = g^{mn}$ so in particular $g^{-n} = (g^{-1})^n$ and $(g^{-1})^{-1} = g$.
3. For $h \in G$, $(g * h)^{-1} = h^{-1} * g^{-1}$.

These should be familiar formulae in the case that $g \in GL_n(\mathbb{C})$ is an invertible matrix.

Additive groups

Suppose given an abelian group $(A, +)$. Since the binary operation is both commutative & denoted $+$, we sometimes call such a group an *additive* group. This is then usually accompanied by the following change in terminology & notation.

Additive terminology/notn

Let $(A, +)$ be an additive group.

1. The group identity is called the *zero* & denoted 0_A or 0.
2. The group inverse of a is called the *negative* of a & is denoted $-a$.
3. For $n \in \mathbb{Z}$, the n-th power of $a \in A$ is called the *n-th multiple* & is denoted na.

Definition of group action

We start with a group G with group multn $*$. Let X be a set.

Defn

A *group action* of G on X is a function $\alpha : G \times X \longrightarrow X$, usually written $\alpha(g, x) = g.x$ such that

1. $e.x = x$ for all $x \in X$.
2. For $g, g' \in G, x \in X$ we have the associative law
$$(g * g').x = (g.(g'.x)).$$

In this case we also say G *acts on* X by α or α is a G-action.

Rem One can think of the group action as a type of "scalar" multn by "scalars" in a group. Also, associativity means we can "remove" brackets.

E.g. G acts on G by $g.h = g * h$. This is called the *regular action*. Indeed,

Permuting rows & columns of matrices

We can use S_n-actions to permute things like rows or columns of matrices.

Prop

The permutation group S_n acts on $X = M_{nn}(\mathbb{C})$ permuting columns in the following way. For $\sigma \in S_n, A = (a_{ij})_{ij} \in X$ we can define the group action $\sigma\,.(a_{ij})_{ij} = (a_{i\,\sigma^{-1}(j)})_{ij}$. There is a similar group action obtained by permuting rows.

Proof. We just check axioms as follows.

Basic properties

Prop

Let the group $(G, *)$ act on the set X & $g \in G$. As x runs through all the elements of X exactly once, $g.x$ runs through all the elements of X exactly once.

Proof. Any elt $x' = e.x' = (g * g^{-1}).x' = g.(g^{-1}.x')$ so has form $g.x$ where $x = g^{-1}.x'$.

No elts are repeated $\because$ $g.x_1 = g.x_2$ ensures $x_1 = g^{-1}.(g.x_1) = g^{-1}.(g.x_2) = x_2$.

Rem In fancy language, the propn says that the function $X \longrightarrow X : x \mapsto g.x$ is a bijection so just permutes elements of X.

Permuting variables

Defn

Let $f(x_1, \ldots, x_n)$ be a fn and $\sigma \in S_n$. We define

$$(\sigma . f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

i.e. the value of f after permuting variables.

e.g.

Permuting variables is a group action

Let X, Y be sets and F denote the set of all functions from $X^n \longrightarrow Y$.

Prop

There is an action of S_n on F defined by $S_n \times F \longrightarrow F : (\sigma, f) \mapsto \sigma.f$.

Proof. First note that $\text{id}.f = f$. Now let $\sigma, \tau \in S_n$. Note that

$$(\tau.f)(x'_1, \ldots, x'_n) = f(x'_{\tau(1)}, \ldots, x'_{\tau(n)})$$

even for $x'_i = x_{\sigma(i)}$ for $i = 1, \ldots, n$.

We now compute

$$\begin{aligned}
[\sigma.(\tau.f)](x_1, \ldots, x_n) &= (\tau.f)(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \\
&= (\tau.f)(x'_1, \ldots, x'_n) \\
&= f(x'_{\tau(1)}, \ldots, x'_{\tau(n)}) \\
&= f(x_{\sigma\tau(1)}, \ldots, x_{\sigma\tau(n)}) = [(\sigma\tau).f](x_1, \ldots, x_n)
\end{aligned}$$

so $\sigma.(\tau.f) = (\sigma\tau).f$ as desired.

Symmetric functions

Defn

A function of the form $f : X^n \longrightarrow Y$ is said to be *symmetric* if $\sigma .f = f$ for all

E.g.

Roots of polynomials

Prop

The co-efficients of monic complex polynomials of degree n are symmetric functions in the roots.

Proof. Suppose the roots of a polynomial $p(x)$ are $\beta_1, \dots, \beta_n$ (counted with multiplicity). Then writing $J_n = \{1, \dots, n\}$ we have

$$p(x) = \prod_{j \in J_n} (x - \beta_j) = \sum_{i=0}^{n} p_i(\beta_1, \dots, \beta_n) x^i$$

for some polynomial fns p_i in the roots β_i. We are required to show all the functions p_i are symmetric. Let $\sigma \in S_n$. Then $(\sigma . p_i)(\beta_1, \dots, \beta_n) = p_i(\beta_{\sigma(1)}, \dots, \beta_{\sigma(n)})$ is just the co-efficient of x^i in

$$\prod_{j \in J_n} (x - \beta_{\sigma(j)}) = \prod_{j \in J_n} (x - \beta_j).$$

Thus $\sigma . p_i = p_i$ & p_i is symmetric.