Jordan chain

Aim lecture: We introduce the notions of Jordan chains & Jordan form tableaux which are the key notions to proving the Jordan canonical form theorem.

Throughout this lecture we fix the following notn. Let $T : V \longrightarrow V$ be linear & $\lambda \in \mathbb{F}$ be an e-value & $N = T - \lambda\,\mathrm{id}$.

Defn

Let $\mathbf{v} \in E_\lambda(n) - E_\lambda(n-1)$ so $N^n\mathbf{v} = \mathbf{0}$ but $N^{n-1}\mathbf{v} \neq \mathbf{0}$. A *Jordan chain* of *length* n (wrt T & λ) is a row matrix of the form $C = (N^{n-1}\mathbf{v} \quad N^{n-2}\mathbf{v} \quad \dots \quad \mathbf{v}) \in V^n$. We say $\mathbf{v}$ is a *seed* for the Jordan chain & $N^i\mathbf{v}$ are vectors of the chain. The *Jordan chain space* associated to C is $\mathrm{im}\,(C : \mathbb{F}^n \longrightarrow V) \leq V$.

E.g. $T = J_3(\lambda)$

T-invariance of Jordan chain spaces

Prop 1

Let $C = (\mathbf{v}_{n-1}\ \mathbf{v}_{n-2}\ \dots \mathbf{v}_0)$ be a Jordan chain. The Jordan chain space im C is T-invariant. In fact

1. $T\mathbf{v}_i = \mathbf{v}_{i+1} + \lambda \mathbf{v}_i$ for $i = 0, \dots, n-2$.
2. $T\mathbf{v}_{n-1} = \lambda \mathbf{v}_{n-1}$.

Proof. By our criterion for T-invariance lect. 20, suffice prove 1) & 2). But $\mathbf{v}_i = N^i \mathbf{v}_0$ so

$$T\mathbf{v}_i = (N + \lambda \,\mathrm{id}\,)\mathbf{v}_i = N\mathbf{v}_i + \lambda \mathbf{v}_i = NN^i \mathbf{v}_0 + \lambda \mathbf{v}_i = \mathbf{v}_{i+1} + \lambda \mathbf{v}_i$$

if we define $\mathbf{v}_n = N^n \mathbf{v}_0 = \mathbf{0}$. The propn follows.

Linear independence of Jordan chain

Prop 2

Let $C = (N^{n-1}\mathbf{v} \;\; N^{n-2}\mathbf{v} \;\; \dots \;\; \mathbf{v})$ be a Jordan chain of length n.

1. $N^i\mathbf{v} \in E_\lambda(n-i) - E_\lambda(n-i-1)$.
2. $C : \mathbb{F}^n \longrightarrow V$ is injective so the vectors of the Jordan chain form a basis for the Jordan chain space im C i.e. $C : \mathbb{F}^n \longrightarrow \text{im } C$ is a co-ord system.
3. im $C \cap E_\lambda(i) = \text{Span}(N^{n-1}\mathbf{v}, N^{n-2}\mathbf{v} \dots, N^{n-i}\mathbf{v})$

Proof. 1) Recall $\mathbf{v} \in E_\lambda(n) - E_\lambda(n-1)$ so in particular is lin indep modulo $E_\lambda(n-1)$. Key lemma lecture 28 shows that $N^i\mathbf{v} \notin E_\lambda(n-i-1)$. Also $N^{n-i}[N^i\mathbf{v}] = N^n\mathbf{v} = \mathbf{0}$ so $N^i\mathbf{v} \in E_\lambda(n-i)$ & 1) is proved.

2) We use downward induction to show that $C_i = \{N^{n-1}\mathbf{v}, N^{n-2}\mathbf{v}, \dots, N^i\mathbf{v}\}$ is lin indep. Suppose that C_i is lin indep. Note that 1) $\Longrightarrow \text{Span}(C_i) \subseteq E_\lambda(n-i)$. Hence 1) also shows $N^{i-1}\mathbf{v} \notin \text{Span}(C_i)$ so C_{i-1} is lin indep too.

For 3) just note $\mathbf{w} = \sum_j \beta_j N^j\mathbf{v} \in \text{im } C \cap E_\lambda(i)$ iff

$$\mathbf{0} = N^i\mathbf{w} = \sum_{j=0}^{n} \beta_j N^{i+j}\mathbf{v} = \sum_{j=0}^{n-i-1} \beta_j N^{i+j}\mathbf{v} \quad \text{iff} \quad 0 = \beta_0 = \dots = \beta_{n-i-1}.$$

Jordan chains & blocks

Prop 3

Let $C = (\mathbf{v}_{n-1} \ldots \mathbf{v}_0) : \mathbb{F}^n \longrightarrow V$ be a co-ord system. Then C is a Jordan chain iff the matrix representing T wrt C is $J_n(\lambda)$. In particular, $I = (\mathbf{e}_1 \ \ldots \ \mathbf{e}_n)$ is a Jordan chain for $J_n(\lambda)$.

Proof. This is an easy calculation. Recall from our propn on T-invariance of Jordan chain spaces that

$$T\mathbf{v}_i = \mathbf{v}_{i+1} + \lambda \mathbf{v}_i \text{ for } i = 0, \ldots, n-2 \quad \& \quad T\mathbf{v}_{n-1} = \lambda \mathbf{v}_{n-1}$$

Hence the first column of $C^{-1} \circ T \circ C$ is $C^{-1}(T(C\mathbf{e}_1)) = C^{-1}\lambda\mathbf{v}_{n-1} = (\lambda, 0, \ldots, 0)^T$ whilst the $(n-i)$-th column (for $i < n-1$) of $C^{-1} \circ T \circ C$ is

$$C^{-1}(\mathbf{v}_{i+1} + \lambda \mathbf{v}_i) = (0, \ldots, 0, 1, \lambda, 0, \ldots, 0)^T$$

where the λ is the $(n-i)$-th entry. Hence $C^{-1} \circ T \circ C = J_n(\lambda)$.

The computation reverses to give the converse.

Role in existence of Jordan canonical forms

The propn implies in particular, that $A \in M_{nn}(\mathbb{F})$ is similar to a Jordan block iff $\mathbb{F}^n$ has a Jordan chain of length n wrt A (and some λ).

The existence part of the Jordan canonical form thm states that if $\dim V < \infty$ & $\mathbb{F}$ is alg closed, then wrt some co-ord system, T is a direct sum of Jordan blocks. By the thm on decomposition into gen e-spaces, in proving this, we may as well assume that $V = E_\lambda(\infty)$ is a single gen e-space.

Upshot

The existence of Jordan canonical forms thus reduces to showing that each gen e-space $E_\lambda(\infty)$ is a direct sum of Jordan chain spaces.

Jordan form tableau

Suppose that $V = E_\lambda(\infty)$ & T has a Jordan canonical form so V is a direct sum of Jordan chain spaces. Let $C_1, \dots, C_r$ be the Jordan chains listed in weakly decreasing (= non-increasing) lengths $m_1 \geq m_2 \dots, \geq m_r$.

Defn

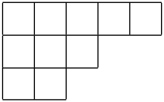
Assuming existence of the Jordan canonical form & $V = E_\lambda(\infty)$, the *Jordan form tableau* of T is the diagram consisting of a column of m_1 boxes on the left, another column of m_2 boxes to the right of that with the same top, followed by another column of m_3 boxes to the right of that, and so on.
We may fill the tableau by putting the vectors of the Jordan chains in the boxes of the corresponding columns, in descending order so the seeds are at the bottom.

E.g. $T = J_1(\lambda) \oplus J_3(\lambda) \oplus J_2(\lambda) \oplus J_3(\lambda) \oplus J_1(\lambda)$

A We first write down the Jordan chains.

Example cont'd

The Jordan form tableau here is



Note the upside-down staircase shape.

Prop 4

1. The vectors in the first i rows of the filled-in Jordan form tableau form a basis for $E_\lambda(i)$.
2. $\dim E_\lambda(i)$ is equal to the number of boxes in the first i rows of the Jordan form tableau.
3. The i-th row of the Jordan form tableau is a basis for a vector space complement to $E_\lambda(i-1)$ in $E_\lambda(i)$.

E.g. above

Proof. Note that 1) $\Longrightarrow$ 2) & 3) so we only need prove 1).

Proof of proposition

In Prop 2, we proved 1) in the case of a single Jordan chain. The case of a finite direct sum of Jordan chain spaces follows from the next lemma applied to $V_i = i$-th Jordan chain space.

Lemma (Kernel of direct sums)

Suppose $T = T_1 \oplus \ldots \oplus T_r : \oplus_i V_i \longrightarrow \oplus_i V_i$ *for some endomorphisms* $T_1, \ldots, T_r$ *of* $V_1, \ldots, V_r \leq V$. *Let* $f(x) \in \mathbb{F}[x]$. *Then*

1. $\ker f(T) = \ker f(T_1) \oplus \ldots \oplus \ker f(T_r)$.
2. *In particular,*

$$E_\lambda(i) = \ker(T_1 - \lambda\, id)^i \oplus \ldots \oplus \ker(T_r - \lambda\, id)^i.$$

Proof. We saw in lecture 26 that $f(T) = f(T_1) \oplus \ldots \oplus f(T_r)$ so $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_r)^T \in \ker f(T)$ iff

Uniqueness of Jordan canonical form

Theorem

Suppose T has two Jordan forms $J^{(1)}$ & $J^{(2)}$. The the Jordan blocks of $J^{(1)}$ & $J^{(2)}$ are the same up to permutation.

Proof. Note that $J^{(1)}$ & $J^{(2)}$ are similar so it suffices to show that similarity invariants determine the Jordan blocks uniquely up to permutation.

Let $J_{n_1}(\lambda), \ldots, J_{n_r}(\lambda)$ be the Jordan blocks of $J^{(1)}$ with e-value λ. It suffices to show these are determined by the similarity invariants $\dim E_\lambda(i)$ for $i \in \mathbb{N}$. But these invariants determine uniquely the Jordan form tableau by our previous propn. Furthermore, the columns of the tableau give the sizes of the Jordan blocks so we are done.

The tableau shape from $E_\lambda(n)$

Suppose now that $\dim V < \infty$. Our previous propn suggests that even if we don't know if T has a Jordan canonical form, we may form the Jordan form tableau of T wrt e-value λ by placing

- a row of $\dim E_\lambda(1)$ boxes at the top
- a row of $\dim E_\lambda(2) - \dim E_\lambda(1)$ beneath that (with same left end)
- another row of $\dim E_\lambda(3) - \dim E_\lambda(2)$ beneath that and so forth.

The upside down staircase shape is guaranteed by the following

Lemma

$\dim E_\lambda(i+1) - \dim E_\lambda(i) \leq \dim E_\lambda(i) - \dim E_\lambda(i-1)$

Proof. Note that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subset E_\lambda(i+1)$ is a basis for a vector space complement to $E_\lambda(i)$ in $E_\lambda(i+1)$ then $\dim E_\lambda(i+1) - \dim E_\lambda(i) = r$. Our key lemma lecture 28 shows that $\{N\mathbf{v}_1, \ldots, N\mathbf{v}_r\} \subset E_\lambda(i)$ is lin indep modulo $E_\lambda(i-1)$. Hence

$$\dim E_\lambda(i) \geq \dim(E_\lambda(i-1) \oplus \mathbb{F}\, N\mathbf{v}_1 \oplus \ldots \oplus \mathbb{F}\, N\mathbf{v}_r) = \dim E_\lambda(i-1) + r.$$

Re-arranging gives the inequality of the lemma.