Lemma on generalised eigenspaces

Aim lecture: For non-diagonalisable matrices, we need decomposition into direct sums of *generalised* e-spaces as opposed to just e-spaces.

Lemma

Let $T : V \longrightarrow V$ *be a linear map &* $\lambda \in \mathbb{F}$ *an e-value of* T*. For* $n \in \mathbb{N}$*, we define* $E_\lambda(n) = \ker(T - \lambda I)^n \leq V$.

1. *Then* $E_\lambda(0) = \mathbf{0} \leq E_\lambda(1) = E_\lambda \leq E_\lambda(2) \leq E_\lambda(3) \leq \ldots$.
2. *If* $E_\lambda(n) = E_\lambda(n+1)$ *then* $E_\lambda(n) = E_\lambda(n')$ *for all* $n' \geq n$ *& we define this to be the* generalised λ-eigenspace *of* T *& denote it* $E_\lambda(\infty)$.
3. *If* $\dim V < \infty$ *then 2) must occur for some* $n \gg 0$ *so* $E_\lambda(\infty)$ *is well-defined.*
4. $E_\lambda(n)$ *is* T*-invariant.*

Proof. 3) holds as $\dim E_\lambda(n)$ cannot increase indefinitely as n increases. 4) holds for if $\mathbf{v} \in E_\lambda(n)$ then $(T - \lambda I)^n[T\mathbf{v}] = T(T - \lambda I)^n\mathbf{v} = \mathbf{0}$ so $T\mathbf{v} \in E_\lambda(n)$.
1)

Proof lemma cont'd

2) Suppose now that $E_\lambda(n) = E_\lambda(n+1)$. By induction, it suffices to show that $E_\lambda(n+2) = E_\lambda(n+1)$. Let $\mathbf{v} \in E_\lambda(n+2)$ so that

$$\mathbf{0} = (T - \lambda I)^{n+2}\mathbf{v} = (T - \lambda I)^{n+1}(T - \lambda I)\mathbf{v}$$

so $(T - \lambda I)\mathbf{v} \in E_\lambda(n+1) = E_\lambda(n)$. Hence

$$\mathbf{0} = (T - \lambda I)^{n}(T - \lambda I)\mathbf{v} = (T - \lambda I)^{n+1}\mathbf{v}$$

which shows $\mathbf{v} \in E_\lambda(n+1)$. Thus $E_\lambda(n+2) \subseteq E_\lambda(n+1)$. We get $E_\lambda(n+2) = E_\lambda(n+1)$ from 1) now. This completes the proof.

As for the geometric multiplicity we have

Prop

The dimensions $\dim E_\lambda(n)$ are similarity invariants. Moreover,

$$\dim E_\lambda(0) \leq \dim E_\lambda(1) \leq \dim E_\lambda(2) \ldots.$$

Example

E.g. Find $\dim E_3(n)$ for the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Bézout's theorem

Factorisation theory for polynomials in $\mathbb{F}[x]$ works just like factorisation in $\mathbb{R}[x]$ or $\mathbb{Q}[x]$. In particular, we have the following which can be proved using the "generalised Euclidean algorithm" of MATH1081.

Theorem

Let $f(x), g(x) \in \mathbb{F}[x]$ be polynomials whose only common factors are constants. Then there are $a(x), b(x) \in \mathbb{F}[x]$ such that $a(x)f(x) + b(x)g(x) = 1$.

Proof. (Not examinable) Consider the set of poly $J = \{a(x)f(x) + b(x)g(x) | a(x), b(x) \in \mathbb{F}[x]\}$. We pick $d(x) \in J$ which has minimal degree amongst all non-zero poly in J. It suffices to show that $d(x) = d$ is a constant so dividing by $d = a(x)f(x) + b(x)g(x)$ gives the thm.

By hypothesis, it suffices to show that $d(x) = a(x)f(x) + b(x)g(x)$ divides both $f(x)$ & $g(x)$. We use the long division to write $f(x) = q(x)d(x) + r(x)$ where $q(x), r(x) \in \mathbb{F}[x]$ & $\deg r(x) < \deg d(x)$. Now

$$r(x) = f(x) - q(x)d(x) = f(x) - q(x)(a(x)f(x) + b(x)g(x)) \in J$$

so minimality of $\deg d(x)$ shows that $r(x) = 0$ & $d(x) | q(x)d(x) = f(x)$. Sim $d(x) | g(x)$ so we are done.

Primary decomposition

Theorem (Primary decomposition)

Let $f(x), g(x) \in \mathbb{F}[x]$ be polynomials whose only common factors are constants. Let $T : V \longrightarrow V$ be linear & such that $f(T)g(T) = 0$. Then

$$V = \ker f(T) \oplus \ker g(T)$$

Proof. As in Bézout's thm we pick $a(x), b(x) \in \mathbb{F}[x]$ with $a(x)f(x) + b(x)g(x) = 1$. We first show $\ker f(T) \cap \ker g(T) = \mathbf{0}$. Suppose $\mathbf{v} \in \ker f(T) \cap \ker g(T)$. Then

$$\mathbf{v} = (a(T)f(T) + b(T)g(T))\mathbf{v} = a(T)f(T)\mathbf{v} + b(T)g(T)\mathbf{v} = \mathbf{0}$$

so $\ker f(T) \cap \ker g(T) = \mathbf{0}$. N.B. $f(T)g(T) = 0$ not needed here.

We now show $\ker f(T) + \ker g(T) = V$. Let $\mathbf{v} \in V$ so as above $\mathbf{v} = a(T)f(T)\mathbf{v} + b(T)g(T)\mathbf{v}$. It suffices now to show that $a(T)f(T)\mathbf{v} \in \ker g(T), b(T)g(T)\mathbf{v} \in \ker f(T)$. But

$$g(T)[a(T)f(T)\mathbf{v}] = a(T)f(T)g(T)\mathbf{v} = a(T)0\mathbf{v} = \mathbf{0}$$

so $a(T)f(T)\mathbf{v} \in \ker g(T)$. Sim, $b(T)g(T)\mathbf{v} \in \ker f(T)$. The thm is proved.

Minimal polynomials

Let $V =$ fin dim $\mathbb{F}$-space and $T : V \longrightarrow V$ be a linear. Let $\mathcal{P}$ be the set of all non-zero polynomials $p(x)$ such that $p(T) = 0$. By Cayley-Hamilton, $\mathrm{cp}_T(x) \in \mathcal{P}$. The division algorithm argument used in the proof of the previous thm can also be used to prove

Prop-Defn

There is a unique monic polynomial $\mathrm{mp}_T(x)$ of minimal degree in $\mathcal{P}$. It is called the *minimal polynomial of* T and divides every other polynomial in $\mathcal{P}$.

E.g. Let $A \in M_{55}(\mathbb{R})$ be a matrix with minimal polynomial $\mathrm{mp}_A(x) = x^2 - 1$. Show that A is diagonalisable over $\mathbb{R}$ and determine the e-values.

Decomposition into direct sum of generalised e-spaces

We assume that $\mathbb{F} = \mathbb{C}$ or any alg closed field.

Theorem (Decomposition into generalised e-spaces)

Let $T : V \longrightarrow V$ be linear & $\dim V < \infty$. Let $cp_T(\lambda) = \pm \prod_{i=1}^{r} (\lambda - \lambda_i)^{a_i}$ where λ_i are the distinct e-values of T. Then

1. *We have a T-invariant direct sum $V = E_{\lambda_1}(\infty) \oplus \ldots \oplus E_{\lambda_r}(\infty)$.*
2. $E_{\lambda_i}(\infty) = E_{\lambda_i}(a_i)$

Proof. It suffices to prove 2), since the Cayley-Hamilton thm, primary decomp thm & induction give the T-invariant direct sum decompn

$$V = \ker(T - \lambda_1 I)^{a_1} \oplus \ldots \oplus \ker(T - \lambda_r I)^{a_r}.$$

Let $\mathbf{v} \in E_{\lambda_i}(\infty)$ so say $(T - \lambda_i I)^a \mathbf{v} = \mathbf{0}$. It suffices to show that $\mathbf{v} \in E_{\lambda_i}(a_i)$. Using the direct sum decompn above we may write $\mathbf{v} = \mathbf{v}_1 + \ldots + \mathbf{v}_r$ with $\mathbf{v}_j \in \ker(T - \lambda_j I)^{a_j}$ for every j.

Proof continued

Then

$$\mathbf{0} = (T - \lambda_i I)^a \mathbf{v} = (T - \lambda_i I)^a \mathbf{v}_1 + \ldots + (T - \lambda_i I)^a \mathbf{v}_r.$$

T-inv of $E_{\lambda_j}(a_j) \Longrightarrow (T - \lambda_i I)^a \mathbf{v}_j \in E_{\lambda_j}(a_j)$. Since the sum of the $E_{\lambda_j}(a_j)$ is direct, we must have $(T - \lambda_i I)^a \mathbf{v}_j = \mathbf{0}$. Hence for $j \neq i$ we have

$$\mathbf{v}_j \in \ker(T - \lambda_i I)^a \cap \ker(T - \lambda_j I)^{a_j} = \mathbf{0}$$

by the proof of primary decomposition.

Thus $\mathbf{v} = \mathbf{v}_i \in E_{\lambda_i}(a_i)$ and we are done.

Example

E.g.

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Linear independence modulo a subspace

Prop-Defn

Let $W \leq V$, We say $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ are *linearly independent modulo* W if any of the following equivalent condns hold.

1. The sum $W + \mathbb{F}\,\mathbf{v}_1 + \ldots + \mathbb{F}\,\mathbf{v}_m$ is direct.
2. $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent & spans a vector space complement to W in $W + \mathbb{F}\,\mathbf{v}_1 + \ldots + \mathbb{F}\,\mathbf{v}_m$.

Proof. Easy ex e.g. (2) $\Longrightarrow$ 1)). Lin indep $\Longrightarrow$ the sum $\mathbb{F}\,\mathbf{v}_1 + \ldots + \mathbb{F}\,\mathbf{v}_m$ is direct & the fact that it is a vector space complement to W means the sum $W + (\mathbb{F}\,\mathbf{v}_1 + \ldots + \mathbb{F}\,\mathbf{v}_m)$ is direct too.

Geometric examples of lin indep modulo a subspace

E.g. Let $W \leq \mathbb{R}^3$ be a plane. Then $\mathbf{v} \in \mathbb{R}^3$ is lin indep modulo W iff

E.g. Let $W \leq \mathbb{R}^3$ be a line. Then $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ is lin indep modulo W iff

Key lemma

Lemma

Let $T : V \longrightarrow V$ be linear & $N = T - \lambda I$ for some e-value $\lambda \in \mathbb{F}$. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \in V$ is lin indep modulo $E_\lambda(n)$ for $n \geq 1$, then $\{N^i\mathbf{v}_1, \ldots, N^i\mathbf{v}_r\}$ is lin indep modulo $E_\lambda(n-i)$ for any $i \leq n$.

Proof. It suffices to prove this in the case $i = 1$ by induction. Suppose by way of contradiction that the sum $E_\lambda(n-1) + \mathbb{F}\,N\mathbf{v}_1 + \ldots + \mathbb{F}\,N\mathbf{v}_r$ is not direct so $K = \ker[E_\lambda(n-1) \oplus \mathbb{F}\,N\mathbf{v}_1 \oplus \ldots \oplus \mathbb{F}\,N\mathbf{v}_r \longrightarrow E_\lambda(n-1) + \mathbb{F}\,N\mathbf{v}_1 + \ldots + \mathbb{F}\,N\mathbf{v}_r] \neq \mathbf{0}$. Consider a non-zero vector $\mathbf{x} = (\mathbf{w}, \beta_1 N\mathbf{v}_1, \ldots, \beta_r N\mathbf{v}_r)^T \in K$ so

$$\mathbf{0} = \mathbf{w} + \beta_1 N\mathbf{v}_1 + \ldots + \beta_r N\mathbf{v}_r.$$

Let $\mathbf{v} = \beta_1\mathbf{v}_1 + \ldots + \beta_r\mathbf{v}_r$ & note

$$N^n\mathbf{v} = N^{n-1}(\beta_1 N\mathbf{v}_1 + \ldots + \beta_r N\mathbf{v}_r) = -N^{n-1}\mathbf{w} = \mathbf{0}$$

so $\mathbf{v} = \beta_1\mathbf{v}_1 + \ldots + \beta_r\mathbf{v}_r \in E_\lambda(n)$ & lin indep modulo $E_\lambda(n) \implies \mathbf{v} = \mathbf{0}$. Hence also $\mathbf{w} = \mathbf{0}$ so $\mathbf{x} = \mathbf{0}$ too & our contradiction is obtained.