Matrix-valued functions

Aim lecture: We solve some first order linear homogeneous differential equations using exponentials of matrices.

Recall as in MATH2111, the any function $\mathbb{R} \longrightarrow M_{mn}(\mathbb{C}) : t \mapsto A(t)$ can be thought of as a matrix of functions $A(t) = (a_{ij}(t))_{ij}$ with each $a_{ij} : \mathbb{R} \longrightarrow \mathbb{C}$. We thus have

Defn

The *derivative* of the matrix-valued function $A(t)$ is

$$\frac{d}{dt}(A(t)) = \lim_{\Delta t \to 0} \frac{A(t+\Delta t) - A(t)}{\Delta t} = \left(\frac{da_{ij}}{dt}\right)_{ij}.$$

Similarly, we may *integrate*

$$\int_{t_0}^{t_1} A(t)dt = \left(\int_{t_0}^{t_1} a_{ij}(t)dt\right)_{ij}.$$

Rem In the special case that $A(t) = \mathbf{y}(t)$ is vector-valued, differentiation is also given by $\frac{d}{dt} \oplus \ldots \oplus \frac{d}{dt}$ so is linear.

First order matrix DE

We will be interested in solving DEs in the vector-valued function $\mathbf{y}(t)$ of the following form.

First-order DE with constant co-efficients

Let $A \in M_{nn}(\mathbb{C})$ & $\mathbf{v}(t)$ be a $\mathbb{C}^n$-valued fn.

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t) + \mathbf{v}(t)$$

is a *first order differential equation with constant co-efficients*. It is *homogeneous* if $\mathbf{v}(t) = 0$. We may also specify an initial condition such as $\mathbf{y}(0) = \mathbf{w}$ for some given $\mathbf{w} \in \mathbb{C}^n$.

We keep this as standard notation throughout.

Rem Let V be the space of infinitely differentiable functions from $\mathbb{R} \longrightarrow \mathbb{C}^n$ & suppose $\mathbf{v}(t) \in V$. Then the DE above is linear in the following sense. Note that $T = \frac{d}{dt} - A : V \longrightarrow V$ is linear & the eqn can be re-written as $T\mathbf{y} = \mathbf{v}$.

Hence homogenous solns correspond to ker T.

Product rule

Below we let $A(t)$ be a fn with values in $M_{lm}(\mathbb{C})$ & $B(t)$ be a fn with values in $M_{mn}(\mathbb{C})$.

Prop

$$\frac{d}{dt}(A(t)B(t)) = \frac{dA}{dt}B(t) + A(t)\frac{dB}{dt}.$$

In particular, for a constant matrix $B \in M_{mn}(\mathbb{C})$ we have $\frac{d}{dt}(A(t)B) = \frac{dA}{dt}B$.

Proof is just a computation e.g. if $l = m = 2, n = 1$ then given fns of t, $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ we have

$$\begin{aligned}
\frac{d}{dt}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right] &= \frac{d}{dt}\begin{pmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{pmatrix} \\
&= \begin{pmatrix} a'_{11}b_1 + a'_{12}b_2 + a_{11}b'_1 + a_{12}b'_2 \\ a'_{21}b_1 + a'_{22}b_2 + a_{21}b'_1 + a_{22}b'_2 \end{pmatrix} \\
&= \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix}
\end{aligned}$$

Properties of the exponential function

Prop

Let $A, B \in M_{nn}(\mathbb{C})$ be commuting matrices.

1. $\exp(0) = I$
2. $\exp(A+B) = \exp(A)\exp(B)$
3. $\exp(A)$ is invertible with $\exp(A)^{-1} = \exp(-A)$.

Proof. For 1) note $\exp(0) =$
Also 2) & 1) $\implies$ 3) so we prove 2) only. We assume the standard (Cauchy) product formula for power series.

$$\exp(A+B) = \sum_{k\geq 0} \frac{1}{k!}(A+B)^k = \sum_{k\geq 0} \frac{1}{k!} \sum_{\substack{i+j=k,\\ i,j\geq 0}} \binom{k}{i} A^i B^j$$

$$= \sum_{k\geq 0} \sum_{\substack{i+j=k,\\ i,j\geq 0}} \frac{1}{i!j!} A^i B^j = \left(\sum_{i\geq 0} \frac{1}{i!} A^i\right) \left(\sum_{j\geq 0} \frac{1}{j!} B^j\right) = \exp(A)\exp(B)$$

Derivative of the exponential function

Theorem

Let $A \in M_{nn}(\mathbb{C})$.

1. $\frac{d}{dt}(\exp(tA)) = A\exp(tA)$.
2. *For any $\mathbf{w} \in \mathbb{C}^n$, $\mathbf{y}(t) = \exp(tA)\mathbf{w}$ is a soln to $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t)$ with initial condition $\mathbf{y}(0) = \mathbf{w}$.*

Proof. For 1) all the entries $e_{ij}(t)$ of $\exp(tA)$ are analytic fns of t so we may differentiate term by term

$$\frac{d}{dt}(\exp(tA)) = \frac{d}{dt}\left(\sum_{k\geq 0}\frac{1}{k!}A^k t^k\right) = \sum_{k\geq 1}\frac{1}{(k-1)!}A^k t^{k-1}$$

$$= A\sum_{k\geq 1}\frac{1}{(k-1)!}A^{k-1}t^{k-1} = A\exp(tA)$$

For 2) note $\mathbf{y}(0) = \exp(0)\mathbf{w} = I_n\mathbf{w} = \mathbf{w}$ & the product rule gives

$$\frac{d\mathbf{y}}{dt} = \frac{d}{dt}(\exp(tA)\mathbf{w}) = \frac{d}{dt}(\exp(tA))\,\mathbf{w} = A\exp(tA)\mathbf{w} = A\mathbf{y}(t).$$

Example

E.g Solve the IVP $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t), \quad \mathbf{y}(0) = (2,0,1,1)^T$ if $A = C(J_1(5) \oplus J_3(2))C^{-1}$ &

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Example cont'd

Rem That this is the complete soln follows from the uniqueness result on the next slide.

Uniqueness of soln

Theorem

$\mathbf{y}(t) = \exp(tA)\mathbf{w}$ *gives the unique soln to the initial value problem,*

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t), \quad \mathbf{y}(0) = \mathbf{w}$$

Proof. Saw in thm, that $\mathbf{y}(t) = \exp(tA)\mathbf{w}$ solves the IVP. We show uniqueness by giving another method of solving the IVP.

We first triangularise A with some change of co-ords matrix $C \in GL_n(\mathbb{C})$ so that $C^{-1}AC = U$ for some upper triangular U. We change co-ords to $\mathbf{z}(t) = C^{-1}\mathbf{y}(t)$. The new initial condn is $\mathbf{z}(0) = C^{-1}\mathbf{w} = \mathbf{x}$ & the new DE is

$$\frac{d}{dt}(C\mathbf{z}(t)) = AC\mathbf{z}(t).$$

Left multiplying by C^{-1} & using the product rule this simplifies to

$$\frac{d\mathbf{z}}{dt} = C^{-1}\frac{d}{dt}(C\mathbf{z}(t)) = C^{-1}AC\mathbf{z}(t) = U\mathbf{z}(t).$$

Suffice show new IVP in $\mathbf{z}(t)$ has a unique soln.

Proof by cascaded system of DEs

We proceed by "back-subtitution". Write $U = (u_{ij})_{ij}$, $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t))^T$, $\mathbf{x} = (x_1, \ldots, x_n)^T$ & look first at the last row of $\frac{d\mathbf{z}}{dt} = U\mathbf{z}(t)$. We solve first uniquely for $z_n(t)$ which is possible as it is the soln to the first order linear ODE

$$\frac{dz_n}{dt} = u_{nn} z_n(t), \quad z_n(0) = x_n.$$

We now look at the second last row of $\frac{d\mathbf{z}}{dt} = U\mathbf{z}(t)$ to solve uniquely for $z_{n-1}(t)$. Indeed, this has a unique soln as it is the soln to

$$\frac{dz_{n-1}}{dt} = u_{n-1,n-1} z_{n-1}(t) + u_{n-1,n} z_n(t), \quad z_{n-1}(0) = x_{n-1}.$$

which can be re-written as the linear 1st order ODE

$$\frac{dz_{n-1}}{dt} - u_{n-1,n-1} z_{n-1}(t) = u_{n-1,n} z_n(t), \quad z_{n-1}(0) = x_{n-1}$$

& $z_n(t)$ has been determined uniquely.
Continuing in this fashion, we see that the IVP must have a unique soln. This completes the proof of the thm.

Space of homogeneous solns

Let V be the space of infinitely differentiable functions from $\mathbb{R} \longrightarrow \mathbb{C}^n$.

Prop

Let $A \in M_{nn}(\mathbb{C})$ and Y be the set of solns $\mathbf{y}(t)$ to the DE $\frac{d\mathbf{y}}{dt} = A\mathbf{y}(t)$. Then Y is n-dimensional with basis the columns of $\exp(tA)$, that is, if for $j = 1, \dots, n$ we write $\mathbf{y}^{(j)}(t) = (\exp(tA)_{1j}, \dots, \exp(tA)_{nj})^T$, then $\{\mathbf{y}^{(1)}(t), \dots, \mathbf{y}^{(n)}(t)\}$ is a basis for Y.

Proof. Any $\mathbf{y}(t) = (y_1(t), \dots, y_n(t))^T \in Y$ is a linear combn of columns of $\exp(tA)$ since

$$\mathbf{y}(t) = \exp(tA)\mathbf{y}(0) = y_1(0)\mathbf{y}^{(1)}(t) + \dots + y_n(0)\mathbf{y}^{(n)}(t).$$

Hence the columns span Y. Lin indep of the columns follows easily from the fact that $\exp(tA)$ is invertible so $\exp(tA) : \mathbb{C}^n \longrightarrow Y$ is an isomorphism.

Example of homogeneous DE, general soln

E.g. Find the general soln to the homogeneous ODE

$$\frac{dy_1}{dt} = 3y_1 + y_2$$
$$\frac{dy_2}{dt} = -y_1 + y_2$$

A Writing $\mathbf{y}(t) = \binom{y_1(t)}{y_2(t)}$ we see the above reduce to $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$ where
$A =$

& by lecture 24, A has Jordan canonical form

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$$

We compute

$\exp(tA) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} =$