Limits of matrices

Aim lecture: We extend polynomial functions of matrices $/\mathbb{C}$ to power series. Unfortunately, the study power series of more general endomorphisms requires subtle convergence issues which are left to courses on functional analysis.

We consider a sequence of $m \times n$-matrices $A^{(0)}, A^{(1)}, A^{(2)}, \ldots \in M_{mn}(\mathbb{C})$. Note $A^{(k)}_{ij}$ is the (i,j)-th entry of $A^{(k)}$. As in MATH2111 we have the following

Defn

We say the sequence $\{A^{(k)}\}_k$ *converges* to $\lim_{k\to\infty} A^{(k)} = A \in M_{mn}(\mathbb{C})$ iff $A_{ij} = \lim_{k\to\infty} A^{(k)}_{ij}$ for every i, j.
We say the infinite sum *converges* if the sequence of partial sums converges in which case we write

$$\sum_{k=0}^{\infty} A^{(k)} = \lim_{N\to\infty} \sum_{k=0}^{N} A^{(k)}.$$

Properties of limits

Consider sequences $\{A^{(k)}\}_k \subseteq M_{mn}(\mathbb{C}), \{B^{(k)}\}_k \subseteq M_{rs}(\mathbb{C})$

Lemma

Suppose $\lim_{k\to\infty} A^{(k)} = A, \lim_{k\to\infty} B^{(k)} = B$.

1. *Then* $\lim_{k\to\infty} A^{(k)} \oplus B^{(k)} = A \oplus B$.
2. *If* $n = r$ *then* $\lim_{k\to\infty} A^{(k)} B^{(k)} = AB$.

Rem By induction, both results extend to a direct sum or product of an arbitrary finite number of sequences. There are immediate applications to infinite sums.

Proof sketch 1) Clear from defns on writing direct sums as block diagonal matrices.

2) Follows since matrix multiplication is continuous, indeed

$$(i, k) - \text{th entry of } (a_{ij})_{ij}(b_{jk})_{jk} = \sum_j a_{ij} b_{jk}$$

which is a polynomial function of the a_{ij}s & b_{ij}s.

Power series in matrices

Let $f(x) = \sum_{k=0}^{\infty} f_k x^k$ be a power series over $\mathbb{C}$. We let $\mathbb{C}[[x]]$ denote the set of these.

Defn

For $A \in M_{nn}(\mathbb{C})$ we define

$$f(A) = \sum_{k=0}^{\infty} f_k A^k$$

when it converges.

E.g. Consider $\exp(x) = e^x = \sum_{k\geq 0} \frac{1}{k!} x^k$ & $f(x) = -\ln(1-x) = \sum_{k\geq 1} \frac{1}{k} x^k$ &

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exponentials of Jordan blocks

Let $t, \lambda \in \mathbb{C}, n \in \mathbb{Z}_+$.

Theorem

$E = \exp(tJ_n(\lambda))$ *is a well-defined matrix with entries*

- $E_{ij} = \frac{t^l}{l!} e^{\lambda t}$ *when* $l = j - i \geq 0$
- $E_{ij} = 0$ *if* $i > j$.

Before proving we give an

E.g.

Proof of exponential formula

Note

$$E = \sum_{k=0}^{\infty} \frac{t^k}{k!} J_n(\lambda)^k.$$

We know that $J_n(\lambda)^k$ is upper triangular so $E_{ij} = 0$ if $i > j$.

Suppose now that $l = j - i \geq 0$. From propn lecture 25 we know $J_n(\lambda)^k_{ij} = \binom{k}{l}\lambda^{k-l}$ so

$$\begin{aligned} E_{ij} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \binom{k}{l} \lambda^{k-l} \\ &= \sum_{k=l}^{\infty} \frac{t^k}{l!(k-l)!} \lambda^{k-l} \\ &= \frac{t^l}{l!} \sum_{k=l}^{\infty} \frac{t^{k-l}}{(k-l)!} \lambda^{k-l} \\ &= \frac{t^l}{l!} e^{\lambda t} \end{aligned}$$

Calculating exponentials in general

Prop

Let $A, B \in M_{nn}(\mathbb{C}), t \in \mathbb{C}$ & $f \in \mathbb{C}[[x]]$. Then

1. $f(A \oplus B) = f(A) \oplus f(B)$ (with convergence on either side implying convergence on the other).
2. If $C \in GL_n(\mathbb{C})$ then $f(CBC^{-1}) = Cf(B)C^{-1}$ (with convergence on either side implying convergence on the other).

In particular, if A has Jordan canonical form $J_{n_1}(\lambda_1) \oplus \ldots \oplus J_{n_r}(\lambda_r) = C^{-1}AC$ then $\exp(A)$ is well-defined and is given by

$$\exp(tA) = C\left[\exp(tJ_{n_1}(\lambda_1)) \oplus \ldots \oplus \exp(tJ_{n_r}(\lambda_r))\right] C^{-1}.$$

Proof. 2) has been proved in lecture 23 for $f \in \mathbb{C}[x]$ so holds on taking limits using the lemma.

1) has been proved for $f(x) = x^k$ in lecture 25. The same argument extends to $f \in \mathbb{C}[x]$ & the general result holds on taking limits using the lemma.

Example

E.g. We compute $\exp(tA)$ if $A = C(J_1(2) \oplus J_3(2))C^{-1}$ &

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Exponentials for 2×2-matrices

E.g. Let's compute $exp(tA)$ if

$$A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}.$$

A We first determine the Jordan canonical form.

Example cont'd