Discrete time systems

Aim lecture: Show how Jordan canonical forms can be useful to study some discrete time systems.

Recall from first year the following important example of a discrete system.

Example of first order discrete time system

Let $\mathbf{v}(0), \mathbf{v}(1), \ldots \in \mathbb{C}^n$ be a sequence of vectors which evolve according to the equation $\mathbf{v}(k+1) = A\mathbf{v}(k)$ for some fixed $A \in M_{nn}(\mathbb{C})$ and all $k \geq 0$.

Question Given initial condition $\mathbf{v}(0)$ can we find a nice formula for $\mathbf{v}(k)$ as a fn of k.

First answer As in 1st year, $\mathbf{v}(k) = A^k\mathbf{v}(0)$.

Role of Jordan forms The question thus reduces to finding a nice formula for A^k as a fn of k. Now we know there is a Jordan canonical form $J = C^{-1}AC$ for some $C \in GL_n(\mathbb{C})$. Hence $A = CJC^{-1}$ and

$$A^k = CJ^kC^{-1}$$

so we are reduced to computing a nice formula for J^k.

Powers of direct sums

The following result reduces the computation of J^k, to the case of Jordan blocks.

Prop

For $i = 1, \ldots, r$, let $T_i, S_i : V_i \longrightarrow V_i$ be linear maps. Consider the direct sums $T = T_1 \oplus \ldots \oplus T_r, S = S_1 \oplus \ldots \oplus S_r : \oplus_i V_i \longrightarrow \oplus_i V_i$.

1. $S \circ T = (S_1 \circ T_1) \oplus \ldots \oplus (S_r \circ T_r)$
2. $T^k = T_1^k \oplus \ldots \oplus T_r^k$

Proof. 1) $\Longrightarrow$ 2) by induction. To see 1), let $(\mathbf{v}_1, \ldots, \mathbf{v}_r)^T \in \oplus_i V_i$ & just observe

$$\begin{aligned}(S \circ T)(\mathbf{v}_1, \ldots, \mathbf{v}_r)^T &= S(T_1\mathbf{v}_1, \ldots, T_r\mathbf{v}_r)^T \\ &= ((S_1 \circ T_1)\mathbf{v}_1, \ldots, (S_r \circ T_r)\mathbf{v}_r)^T \\ &= ((S_1 \circ T_1) \oplus \ldots \oplus (S_r \circ T_r))(\mathbf{v}_1, \ldots, \mathbf{v}_r)^T.\end{aligned}$$

Ex You can prove this by multiplying block diagonal matrices too.

E.g. $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)^k = ((2) \oplus (3))^k =$

Powers of the Jordan block $J_n(0)$

E.g. To get a feel of what's going on we compute
$J_3(0)^3 =$

Prop

Let $N = J_n(0)$ & $(b_{ij}) = N^k$ for some $k \in \mathbb{N}$. Then N^k has all entries zero except along the $j - i = k$ "diagonal" where we have $b_{ij} = 1$.

Proof. You can just compute this or note the following more enlightening argument.
$N = J_n(0) : \mathbb{F}^n \longrightarrow \mathbb{F}^n : (x_1, \dots, x_n)^T \mapsto (x_2, \dots, x_n, 0)^T$ is the "shift co-ords up by 1" linear map.
Iterating this k times gives
$N^k : \mathbb{F}^n \longrightarrow \mathbb{F}^n : (x_1, \dots, x_n)^T \mapsto (x_{k+1}, \dots, x_n, 0, \dots, 0)^T$ is the "shift co-ords up by k" linear map. One now matches up the matrix representing this lin map with the one in the propn.

Nilpotent matrices

We compute some powers of $J_4(0)$

Corollary-Defn

A square matrix $N \in M_{nn}(\mathbb{F})$ is *nilpotent* if for some k we have $N^k = 0$. The Jordan block $J_n(0)$ is nilpotent since $J_n(0)^n = 0$.

Ex Show that if N is nilpotent, then $I - N$ is invertible.

The binomial theorem for commuting matrices

Prop

The binomial formula

$$(A+B)^k = \sum_{l \geq 0} \binom{k}{l} A^l B^{k-l}$$

holds for *commuting matrices* $A, B \in M_{nn}(\mathbb{F})$ i.e. where $AB = BA$ as long as we interpret $\binom{k}{l} = 0$ for $l > k$.

Proof is easily seen from any example.
E.g. Note that $\lambda I_3, J_3(0)$ commute since $\lambda I_3 J_3(0) = \lambda J_3(0) = J_3(0)\lambda I_3$.
$(J_3(0) + \lambda I_3)^k =$

Powers of the Jordan block $J_n(\lambda)$

Applying the binomial formula to $J_n(\lambda) = N + \lambda I_n$ where $N = J_n(0)$ gives

Prop

Let $J_n(\lambda)^k = (b_{ij})$. Then $b_{ij} = 0$ for $i > j$ whilst on the $l = j - i \geq 0$ "diagonal", the entries are all $b_{ij} = \binom{k}{l}\lambda^{k-l}$.

Proof.

Example

Consider the DTS (discrete time system) $\mathbf{v}(k+1) = A\mathbf{v}(k)$ where $A \in M_{44}(\mathbb{R})$ has Jordan canonical form $C^{-1}AC = J = J_3(2) \oplus J_1(4)$. Solve for $\mathbf{v}(k)$ if

$$C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 2 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{v}(0) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Example continued

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