Jordan blocks

Aim lecture: Even over $\mathbb{F} = \mathbb{C}$, endomorphisms cannot always be represented by a diagonal matrix. We give Jordan's answer, to what is the best form of the representing matrix.

Defn

Let $\lambda \in \mathbb{F}, n \in \mathbb{Z}_+$. The *size n Jordan block with e-value* λ is the $n \times n$ upper triangular matrix

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix}$$

Example

E.g. Write down the block diagonal matrix $J_2(\sqrt{3}) \oplus J_3(-2) \oplus J_1(7)$.

Defn

A matrix is in *Jordan canonical form* if it is a direct sum of Jordan blocks.

E-values of Jordan blocks

Prop

The Jordan block $J_n(\lambda_0)$ has λ_0 as its only e-value. The geometric multiplicity is 1 but the algebraic multiplicity is n.

Rem When $\mathbb{F} = \mathbb{C}$, Jordan blocks are as far from diagonalisable as can be.

Proof. $J_n(\lambda_0) - \lambda I$ is upper triangular with $\lambda_0 - \lambda$ in all the diagonal entries so $\text{cp}_{J_n}(\lambda) = (\lambda_0 - \lambda)^n$ & the alg mult of λ_0 is n.

$$J_n(\lambda_0) - \lambda_0 I = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

which has kernel $\mathbb{F}\,\mathbf{e}_1$ so the geom mult is 1.

Commutativity of direct sums

Prop

Let $V, W = \mathbb{F}$-spaces. Then $S : V \oplus W \longrightarrow W \oplus V : (\mathbf{v}, \mathbf{w})^T \mapsto (\mathbf{w}, \mathbf{v})^T$ is an isomorphism.

Proof. Indeed, S is linear being given by the matrix

$$\begin{pmatrix} 0 & \text{id}_W \\ \text{id}_V & 0 \end{pmatrix}$$

and it is invertible with inverse

$$\begin{pmatrix} 0 & \text{id}_V \\ \text{id}_W & 0 \end{pmatrix}.$$

Permuting diagonal blocks

Prop

Let $A_1, \dots, A_r$ be square matrices over $\mathbb{F}$ & $\sigma \in S_r$ be a permutation. Then the block diagonal matrices

$$A_1 \oplus \dots \oplus A_r \quad \& \quad A_{\sigma(1)} \oplus \dots \oplus A_{\sigma(r)}$$

are similar. In other words, permuting the diagonal blocks of a matrix does not change its similarity class.

Proof. We prove the case $r = 2$, i.e. $A_1 \oplus A_2$ & $A_2 \oplus A_1$ are similar. The general case can be proved similarly by induction on the $r = 2$ case or using general permutation matrices. Suppose that $A_i \in M_{n_i n_i}$ for $i = 1, 2$. Then

$$\begin{pmatrix} 0 & I_{n_1} \\ I_{n_2} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} 0 & I_{n_1} \\ I_{n_2} & 0 \end{pmatrix}$$

Statement of Jordan's canonical form theorem

Theorem (Jordan's canonical form)

Let $\mathbb{F} = \mathbb{C}$ (or any alg closed field).

1. *Any square matrix A (over $\mathbb{F}$) is similar to a direct sum of Jordan blocks J. Such a J is called a* Jordan canonical form *for A.*
2. *If $J_{m_1}(\lambda_1) \oplus \ldots \oplus J_{m_r}(\lambda_r)$ & $J_{n_1}(\mu_1) \oplus \ldots \oplus J_{n_s}(\mu_s)$ are similar then $r = s$ & the Jordan blocks are a permutation of each other i.e. there's $\sigma \in S_r$ such that $J_{n_i}(\mu_i) = J_{m_{\sigma(i)}}(\lambda_{\sigma(i)})$.*
3. *Hence the set of Jordan blocks with multiplicity, can be considered a similarity invariant.*
4. *Two matrices are similar iff their Jordan blocks are the same up to permutation. We express this fact by saying the Jordan blocks give a* complete *similarity invariant.*

Proof in general will be delayed for several lectures when we have more tools.

E.g. The following matrices are not similar.

Proof of Jordan's theorem for $n = 2$

Proof $n = 2$ case. We prove part 1) (existence of Jordan canonical forms), the other parts follow easily from using the similarity invariants of the e-values & their geometric & algebraic multiplicity.

Note that the diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = J_1(\lambda_1) \oplus J_2(\lambda_2)$ so any diagonalisable matrix is similar to a direct sum of Jordan blocks. Hence the theorem is proved in the case of a diagonalisable matrix.

Suppose now that $A \in M_{22}(\mathbb{F})$ is not diagonalisable so by our criterion for diagonalisability, there is an e-value β whose alg mult a is strictly greater than its geom mult m. But $1 \leq m < a \leq 2$ so $m = 1, a = 2$. It suffices to prove in this case the following lemma.

Lemma

(Suppose A is not diagonalisable). Pick any $\mathbf{v}_2 \in \mathbb{F}^2 - \ker(A - \beta I)$ & let $\mathbf{v}_1 = (A - \beta I)\mathbf{v}_2$. Then the matrix representing A wrt the co-ord system $C = (\mathbf{v}_1 \ \mathbf{v}_2) : \mathbb{F}^2 \longrightarrow \mathbb{F}^2$ is $J_2(\beta)$.

Proof lemma

Proof. Note $\dim \ker(A - \beta I) = 1$ so we may find $\mathbf{v}_2 \in \mathbb{F}^2$ such that $\mathbf{v}_2 \notin \ker(A - \beta I)$. Also $a = 2$ so $cp_A(\lambda) = (\beta - \lambda)^2$. Note that $\mathbf{v}_1$ is an e-vector since the Cayley-Hamilton thm ensures

$$(A - \beta I)\mathbf{v}_1 = (A - \beta I)^2 \mathbf{v}_2 = (\beta I - A)^2 \mathbf{v}_2 = 0\mathbf{v}_2 = \mathbf{0}$$

i.e. $\mathbf{v}_1 \in \ker(A - \beta I)$. Now $\mathbf{v}_2 \notin \ker(A - \beta I)$ ensures $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent.

We show that the matrix B representing A wrt the co-ord system C is $J_2(\beta)$. The first column of B is

$$C^{-1}AC\mathbf{e}_1 = C^{-1}A\mathbf{v}_1 = C^{-1}\beta\mathbf{v}_1 = \beta\mathbf{e}_1.$$

The second column is

$$\begin{aligned} C^{-1}AC\mathbf{e}_2 &= C^{-1}A\mathbf{v}_2 = C^{-1}[(A - \beta I) + \beta I]\mathbf{v}_2 \\ &= C^{-1}(\mathbf{v}_1 + \beta\mathbf{v}_2) = \mathbf{e}_1 + \beta\mathbf{e}_2 \end{aligned}$$

Hence A is similar to

$$B = \begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix} = J_2(\beta)$$

which completes the proof of the thm for $n = 2$.

Example of computing Jordan canonical form

E.g. Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$. Write $A = CJC^{-1}$ for some $C \in GL_2(\mathbb{C})$ & J a direct sum of Jordan blocks.