Invariance of eigenspaces

Aim lecture: We see how the theory of eigenvectors gives good co-ordinate systems & in particular, when we can represent endomorphisms by diagonal or upper triangular matrices

Lemma

Let $T : V \longrightarrow V$ be linear & E_λ be the λ-e-space.

1. *E_λ is T-invariant so restricts to a map $T|_{E_\lambda} : E_\lambda \longrightarrow E_\lambda$.*
2. *$T|_{E_\lambda} = \lambda\, id$.*
3. *If $C : \mathbb{F}^n \longrightarrow E_\lambda$ is a co-ord system for E_λ, then the matrix representing $T|_{E_\lambda}$ is λI_n where I_n is the $n \times n$-identity matrix.*

Proof. 1) For any $\mathbf{v} \in E_\lambda$ we have $T\mathbf{v} = \lambda\mathbf{v} \in E_\lambda$ since E_λ is closed under scalar multn.

2) The above eqn also shows $T = \lambda\,\mathrm{id}$ as endomorphisms of E_λ.

3) Just note

$$C^{-1} \circ T|_{E_\lambda} \circ C =$$

Sum of e-spaces is direct

Prop

Let $T : V \longrightarrow V$ be linear & $\lambda_1, \dots, \lambda_r \in \mathbb{F}$ be distinct. Then the sum of e-spaces $E_{\lambda_1} + \dots + E_{\lambda_r}$ is direct.

Proof by induction on r. We may assume that the sum $E_{\lambda_1} + \dots + E_{\lambda_{r-1}}$ is direct so by the inductive defn of internal direct sum, it suffices to show $(E_{\lambda_1} + \dots + E_{\lambda_{r-1}}) \cap E_{\lambda_r} = \mathbf{0}$.

Consider an element in the intersection so must have form $\mathbf{v} = \mathbf{v}_1 + \dots + \mathbf{v}_{r-1}$ for some $\mathbf{v}_i \in E_{\lambda_i}$. We wish to show that $\mathbf{v} = \mathbf{0}$. Now $\mathbf{v} \in E_{\lambda_r}$ so

$$\begin{aligned}\mathbf{0} = (T - \lambda_r \,\mathrm{id}\,)\mathbf{v} &= (T - \lambda_r \,\mathrm{id}\,)\mathbf{v}_1 + \dots + (T - \lambda_r \,\mathrm{id}\,)\mathbf{v}_{r-1} \\ &= (\lambda_1 - \lambda_r)\mathbf{v}_1 + \dots + (\lambda_{r-1} - \lambda_r)\mathbf{v}_{r-1}.\end{aligned}$$

Now the sum $E_{\lambda_1} \oplus \dots \oplus E_{\lambda_{r-1}}$ is direct so each component $(\lambda_i - \lambda_r)\mathbf{v}_i = \mathbf{0}$. Also, we are assuming $\lambda_i \neq \lambda_r$ for $i = 1, \dots, r-1$ so each $\mathbf{v}_i = \mathbf{0}$. This proves $\mathbf{v} = \mathbf{0}$ and we are done.

Representing with diagonal matrices

Theorem

Let $V = \mathbb{F}$-space of dim $n < \infty$ & $T : V \longrightarrow V$ be a linear map. Suppose that $\lambda_1, \ldots, \lambda_r$ are the e-values of T & $n_1, \ldots, n_r$ are their geometric multiplicities.

1. *$n = n_1 + \ldots + n_r$ if and only if $V = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_r}$.*
2. *In this case, if $C : \mathbb{F}^n \longrightarrow V$ is any co-ord system adapted to this direct sum decomposition, then the matrix representing T wrt C has diagonal form*

$$C^{-1} \circ T \circ C = \lambda_1 I_{n_1} \oplus \ldots \oplus \lambda_r I_{n_r}.$$

If T comes from a square matrix, then writing out the factorisation in 2) above, is called diagonalising T.

Proof. 1) ($\Longleftarrow$) clear so we prove ($\Longrightarrow$). We know $E = E_{\lambda_1} + \ldots + E_{\lambda_r}$ is a direct sum so it suffices to show that $E = V$. But E is a subspace of V of dimension $n_1 + \ldots + n_r = n$ so we are done.

2) This follows from our lemma & propn on adapted co-ord systems in lecture 20.
Rem Finding an adapted co-ord system in 2), involves finding a basis for each E_{λ_i} & putting these together to get a basis of e-vectors of V.

Example

Recall from lecture 21 that $T : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 2}$ defined by

$$(Tp)(x) = xp'(x) - 2p'(x) - p(x)$$

has e-values $-1, 0, 1$ & corresponding e-spaces

$$E_{-1} = \mathbb{R}\,1,\ E_0 = \mathbb{R}(x-2),\ E_1 = \mathbb{R}(x-2)^2.$$

The sum of geometric multiplicities is $1 + 1 + 1 = \dim \mathbb{R}[x]_{\leq 2}$ so

$$\mathbb{R}[x]_{\leq 2} = \mathbb{R}\,1 \oplus \mathbb{R}(x-2) \oplus \mathbb{R}(x-2)^2.$$

The co-ordinate system $(1 \quad (x-2) \quad (x-2)^2) : \mathbb{R}^3 \longrightarrow \mathbb{R}[x]_{\leq 2}$ is adapted to this direct sum decomposition &

$C^{-1} \circ T \circ C =$

E-values of diagonal matrices

Prop

Let A be diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of e-vectors for A whose corresponding e-values are $\lambda_1, \ldots, \lambda_n$.

Proof. Easy computation. We'll look at a simple example instead.

Criterion for diagonalisability

Thm-Defn

We say $A \in M_{nn}(\mathbb{F})$ is *diagonalisable* (over $\mathbb{F}$) if there is some $C \in GL_n(\mathbb{F})$ with $D = C^{-1}AC$ a diagonal matrix.

- A is diagonalisable iff the sum of the geometric multiplicities is n (or equiv, $\mathbb{F}^n$ has a basis of e-vectors or equiv, $\mathbb{F}^n$ is a sum of e-spaces).

Proof. ($\Longleftarrow$) has been proved so we prove ($\Longrightarrow$) & assume there is a diagonal matrix $D = C^{-1}AC$ as above. Propn lecture 21 (on computing e-vectors of endomorphisms) shows that C induces an isomorphism between the e-spaces of A & those of D, so they have the same geometric multiplicities & we need only prove the prop in the case $A = D = (d_{ij})$.

Let $n_1, \dots, n_r$ be the geometric multiplicities of the e-values $\lambda_1, \dots, \lambda_r$. Since the sum of e-spaces is direct we have $\sum n_j \leq n$ & we need only show that $\sum_j n_j \geq n$. Now $\mathbf{e}_i$ is an e-vector with e-value d_{ii} so n_j is at least equal to the number of diagonal entries d_{ii} with $d_{ii} = \lambda_j$. Summing over j gives the result.

Non-diagonalisable example

Not all matrices (even over $\mathbb{C}$) can be diagonalised.

E.g. Show that the matrix $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ is not diagonalisable.

Rem We will introduce the theory of Jordan canonical forms to study these anomalous matrices.

Remark on algebraic closure

$\mathbb{C}$ has the nice property that any polynomial of degree ≥ 1 over $\mathbb{C}$ factorises into linears (over $\mathbb{C}$) so it "has all its roots". We say that a field is *algebraically closed* if any non-constant polynomial $p(x) \in \mathbb{F}[x]$ factorises into linear factors (over $\mathbb{F}$).

Theorem

Given any field $\mathbb{F}$, there is a field $\bar{\mathbb{F}}$ containing $\mathbb{F}$ which is algebraically closed and whose addn & multn extend that of $\mathbb{F}$.

Proof is best left to a course on Galois theory.

Triangularising matrices

Theorem

Let $\mathbb{F} = \mathbb{C}$ (or any algebraically closed field) & V = fin dim $\mathbb{F}$-space. Let $T : V \longrightarrow V$ be linear. Then there is a co-ordinate system $C : \mathbb{F}^n \longrightarrow V$ such that the matrix (u_{ij}) representing T is upper triangular i.e. $u_{ij} = 0$ for $i > j$.

Rem If T is given by a square matrix, then writing out the factorisation $U = C^{-1}TC$ with U upper triangular will be called *triangularising T*.

Proof by induction on n. Since $\mathbb{F}$ is alg closed, the char poly $\text{cp}_T(\lambda)$ has a root, say β & hence T has an e-vector, say $\mathbf{v}$ with e-value β. Let $V_1 \leq V$ be a complement to $\mathbb{F}\,\mathbf{v}$ in V. Prop lecture 20 shows that for any co-ordinate system $C_0 : \mathbb{F}^n \longrightarrow V$ adapted to the direct sum decomposition $V = \mathbb{F}\,\mathbf{v} \oplus V_1$, we have

$$C_0^{-1} \circ T \circ C_0 = \begin{pmatrix} \beta & T_{12} \\ 0 & T_{22} \end{pmatrix}.$$

Proof continued

By induction we can find a co-ord system $C_1 : \mathbb{F}^{n-1} \longrightarrow \mathbb{F}^{n-1}$ with $C_1^{-1} T_{22} C_1$ upper triangular. Then $C = C_0 \circ (\mathrm{id}_{\mathbb{F}} \oplus C_1) : \mathbb{F}^n \longrightarrow \mathbb{F}^n \longrightarrow V$ is the desired co-ord system for

$$\begin{aligned} C^{-1} \circ T \circ C &= (\mathrm{id} \oplus C_1)^{-1} \circ C_0^{-1} \circ T \circ C_0 \circ (\mathrm{id} \oplus C_1) \\ &= \begin{pmatrix} \mathrm{id} & 0 \\ 0 & C_1^{-1} \end{pmatrix} \begin{pmatrix} \beta & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ 0 & C_1 \end{pmatrix} \\ &= \begin{pmatrix} \beta & T_{12} C_1 \\ 0 & C_1^{-1} T_{22} C_1 \end{pmatrix} \end{aligned}$$