Eigenvectors

Aim lecture: The simplest T-invariant subspaces are 1-dim & these give rise to the theory of eigenvectors. To compute these we introduce the similarity invariant, the characteristic polynomial.

Prop-Defn

Let $T : V \longrightarrow V$ be linear. The following are equivalent condns on $\mathbf{v} \in V$.

1. $\mathbb{F}\mathbf{v}$ is a T-invariant (automatically 1-dimensional) subspace.
2. $T\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{F}$.
3. $\mathbf{v} \in \ker(T - \lambda \,\mathrm{id}\,)$ for some $\lambda \in \mathbb{F}$.

If these hold & furthermore $\mathbf{v} \neq \mathbf{0}$ we say $\mathbf{v}$ is an *eigenvector* for T with *eigenvalue* λ.

Proof. Clearly 2) $\iff$ 3) since

$$T\mathbf{v} = \lambda\mathbf{v} \iff T\mathbf{v} - \lambda \,\mathrm{id}\, \mathbf{v} = \mathbf{0} \iff \mathbf{v} \in \ker(T - \lambda I).$$

2) $\Longrightarrow$ 1) by lemma on testing invariance, lecture 20.

For 1) $\Longrightarrow$ 2) note

Eigenvalues & eigenspaces of an endomorphism

Defn

Let $T : V \longrightarrow V$ be linear. An *eigenvalue* of T is a scalar $\lambda \in \mathbb{F}$ such that there is an e-vector $\mathbf{v}$ for T with eigenvalue λ. Given such an e-value, the λ-*eigenspace* of T is the subspace $E_\lambda = \ker(T - \lambda\,\mathrm{id}) \leq V$. The *geometric multiplicity* of λ is $\dim E_\lambda$.

E.g. Any real number λ is an e-value for $T = \frac{d}{dx} : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ since

In fact $E_\lambda =$

Characteristic polynomial of square matrices

To find e-values of square matrices we need

Prop-Defn

Let $A = (a_{ij}) \in M_{nn}(\mathbb{F})$. The *characteristic polynomial of* A is the function $\text{cp}_A(\lambda) = \det(A - \lambda I_n)$ where I_n is the $n \times n$-identity matrix. This function is a polynomial function of degree n with co-efficients in $\mathbb{F}$.

Proof. Note that $A - \lambda I_n = (a'_{ij})$ where $a'_{ij} = a_{ij}$ if $i \neq j$ whilst $a'_{ii} = a_{ii} - \lambda$. Now

$$\det(A - \lambda I_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a'_{i\,\sigma(i)}$$

which is clearly a polynomial function of λ.

The summand corresponding to $\sigma = \text{id}$ is $(a_{11} - \lambda)(a_{22} - \lambda)\dots(a_{nn} - \lambda)$ which has degree n.

Any other summand contains at least two non-diagonal entries so has degree $\leq n - 2$. Hence, $\deg \text{cp}_A(\lambda) = n$.
Scholium The co-efficient of λ^{n-1} in $\text{cp}_A(\lambda)$ is $(-1)^{n-1} \sum_i a_{ii}$.

Relation with e-values. Algebraic multiplicity

Prop-defn

Let $A \in M_{nn}(\mathbb{F})$. Then λ is an e-value for A iff it is a root of the characteristic polynomial of A. In this case, the multiplicity of the root is called the *algebraic multiplicity* of the e-value.

Proof. λ is an e-value iff A has an e-vector with e-value λ iff $\ker(A - \lambda I_n) \neq \mathbf{0}$ iff $A - \lambda I_n$ is not invertible iff $\det(A - \lambda I_n) = 0$.

E.g. Find the e-values of $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ & their algebraic & geometric multiplicities.

Similar matrices

Prop-Defn

Two matrices $A, B \in M_{nn}(\mathbb{F})$ are *similar* if there exists $C \in GL_n(\mathbb{F})$ such that $A = C^{-1}BC$ i.e. A is a matrix representing B wrt some co-ordinate system. Being similar is an equivalence relation. In particular,

1. if A is similar to B then B is similar to A.
2. if A is similar to B and B is similar to D, then A is similar to D.

The set of all matrices similar to A is called the *similarity class* of A.

Proof. Easy ex.

E.g. $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ are not similar for if $A = C^{-1}BC$ then

Defn

A function of the form $f : M_{nn}(\mathbb{F}) \longrightarrow X$ is a *similarity invariant* if $f(A) = f(B)$ whenever A, B are similar.

E.g. 1 The characteristic polynomial $\mathrm{cp} : M_{nn}(\mathbb{F}) \longrightarrow \mathbb{F}[\lambda] : A \mapsto \mathrm{cp}_A(\lambda)$ is a similarity invariant since if $B = C^{-1}AC$ for some $C \in GL_n(\mathbb{F})$ then

$$\det(B - \lambda I_n) = \det(C^{-1}AC - \lambda I_n) = \det(C^{-1}[A - \lambda I_n]C) =$$

E.g. 2 In particular, any of the co-efficients of the characteristic polynomial are similarity invariants. The important ones (up to sign) are the determinant $\det(A) = \mathrm{cp}_A(0)$ & the *trace* which is defined to be $\mathrm{tr}\,(A) = \sum_{i=1}^{n} a_{ii}$ where $A = (a_{ij})$.

E.g. 3 Similarly, the set of eigenvalues is a similarity invariant.

Sums & products of e-values

Let $A \in M_{nn}(\mathbb{C})$. Since $\mathbb{C}$ is alg closed, it has n e-values $\lambda_1, \ldots, \lambda_n$ when counted with (algebraic) multiplicity.

Formula

$$\operatorname{tr}(A) = \sum_i \lambda_i, \quad \det(A) = \prod_i \lambda_i$$

Proof. Note the following equality of polynomials in λ

$$\det(A - \lambda I_n) = \prod_i (\lambda_i - \lambda).$$

Equating constant terms gives $\det(A) = \prod_i \lambda_i$ while comparing co-effs of λ^{n-1} gives the trace formula.

E.g. Suppose you know two of the e-values of $A \in M_{33}(\mathbb{C})$ are $2, 3$ and A has diagonal entries $1, 1, 4$. Find the third e-value λ_3.

Characteristic polynomials of endomorphisms

Similarity invariants can be extended to endomorphisms of finite dimensional vector spaces. For example

Prop-Defn

Let $V =$ fin dim $\mathbb{F}$-space & $T : V \longrightarrow V$ be linear. For any co-ordinate system $C : \mathbb{F}^n \longrightarrow V$, we may define the *characteristic polynomial* of T, denoted $\text{cp}_T(\lambda)$, to be the characteristic polynomial of the representing matrix $C^{-1} \circ T \circ C$. This is well-defined since given any other co-ordinate system $C_1 : \mathbb{F}^n \longrightarrow V$, the characteristic polynomials of $C^{-1} \circ T \circ C$ & $C_1^{-1} \circ T \circ C_1$ are the same, so the definition is independent of the choice of co-ord system.

Proof. We need only check equality of characteristic polynomials by showing $C^{-1} \circ T \circ C$ & $C_1^{-1} \circ T \circ C_1$ are similar. Indeed

$$C_1^{-1} \circ T \circ C_1$$

Rem We similarly can define $\det(T), \text{tr}(T)$ etc.
E.g. We have seen that $T = \frac{d}{dx} : \mathbb{R}\cos x \oplus \mathbb{R}\sin x \longrightarrow \mathbb{R}\cos x \oplus \mathbb{R}\sin x$ is represented by the matrix

Computing e-values & e-vectors of endomorphisms

Using co-ordinates, we can calculate e-vectors, e-values & e-spaces using

Prop

Let $V =$ fin dim $\mathbb{F}$-space & $T : V \longrightarrow V$ be linear. Let $C : \mathbb{F}^n \longrightarrow V$ be a co-ord system & $A = C^{-1} \circ T \circ C$ be the representing matrix. Then

1. $\mathbf{x} \in \mathbb{F}^n$ is an e-vector of A with e-value λ iff $C\mathbf{x}$ is an e-vector of T with e-value λ.
2. The e-values of T & A are the same. They are the roots of $\text{cp}_A(\lambda) = \text{cp}_T(\lambda)$.
3. If E_λ is the λ-e-space of A then $C(E_\lambda)$ is the λ-e-space of T.

Proof. We just prove 1), as 2) & 3) readily then follow.

$$A\mathbf{x} = \lambda\mathbf{x} \iff C^{-1} \circ T \circ C\mathbf{x} = \lambda\mathbf{x} \iff$$

Example

E.g. We compute the e-vectors & e-values of $T : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 2}$ defined by

$$(Tp)(x) = xp'(x) - 2p'(x) - p(x).$$

Example continued