Adapted co-ordinate systems

Aim lecture: We continue our study of invariant subspaces & their usefulness.

Prop-defn

Consider an internal direct sum $V = V_1 \oplus \ldots \oplus V_r$. A co-ordinate system is *adapted* to the direct sum if it has the form $C : \mathbb{F}^N \longrightarrow V$ where $C = C_1 \oplus \ldots C_r$ for some co-ordinate systems $C_i : \mathbb{F}^{n_i} \longrightarrow V_i$ for $i = 1, \ldots, r$ & $N = \sum_i n_i$. Note we have identified the internal and external direct sums here.

1. Let $T : V \longrightarrow V$ be a linear map given by the matrix $(T_{ij}) \in (L(V_j, V_i))$. Then the matrix representing T wrt C (for both domain & co-domain) is $(C_i^{-1} \circ T_{ij} \circ C_j)$.
2. In particular, if the direct sum is T-invariant, the representing matrix has block diagonal form

$$C^{-1} \circ T \circ C = (C_1^{-1} \circ T|_{V_1} \circ C_1) \oplus \ldots \oplus (C_r^{-1} \circ T|_{V_r} \circ C_r).$$

Proof. Just calculate e.g. for $r = 2$,

Testing invariance

The next lemma makes it easy to test if a subspace is invariant.

Lemma

Let $T : V \longrightarrow W$ be a linear map. Consider subspaces $V' = Span(\mathbf{v}_1, \ldots, \mathbf{v}_m) \leq V$, & $W' \leq W$.

1. *Then $T(V') \subseteq W'$ iff $T\mathbf{v}_1, \ldots, T\mathbf{v}_m \in W'$.*
2. *In particular, if $V = W, V' = W'$, we see that V' is T-invariant iff $T\mathbf{v}_1, \ldots, T\mathbf{v}_m \in V'$.*
3. *If $V = W$ & $V', V'' \leq V$ are invariant, so is $V' + V''$.*

Proof of 1). (3) is ex). ($\Longrightarrow$) is clear so we prove ($\Longleftarrow$). Any element of $T(V')$, can be written in the form $T\mathbf{v}'$ where $\mathbf{v}' = \sum_i \beta_i \mathbf{v}_i$ for some scalars $\beta_i \in \mathbb{F}$. Then

Examples

Space of sinusoidal functions of given frequency

Below we let $V_n = \mathbb{R}\sin(nx) + \mathbb{R}\cos(nx) = \{A\sin(nx+c) | A, c \in \mathbb{R}\}$.

E.g. Consider the linear map $T = \frac{d}{dx} : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$.

1. Show that V_n are T-invariant subspaces.
2. Show that the sum $V = V_1 + V_2$ is direct & so by restriction we get a linear map $T : V \longrightarrow V$ s.t. $V = V_1 \oplus V_2$ is a T-invariant direct sum.
3. By picking co-ordinates for V_1, V_2, write a 4×4-matrix representing T which has block diagonal form.

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Example continued

For part 3), pick co-ordinates $C_n = (\cos(nx)\ \sin(nx)) : \mathbb{R}^2 \longrightarrow V_n$ and let $\Phi = (\text{id}\,|_{V_1}\ \ \text{id}\,|_{V_1}) : V_1 \oplus V_2 \longrightarrow V$ be the natural isomorphism.
The adapted co-ordinates for the external direct sum is.

Hence the adapted co-ordinate system for the internal direct sum is

Formula for adapted co-ordinate systems

The above example illustrates the following general

Prop

Consider an internal direct sum $V = V_1 \oplus \ldots \oplus V_r$ and co-ord systems $C_i : \mathbb{F}^{n_i} \longrightarrow V_i$ for $i = 1, \ldots, r$ & $N = \sum_i n_i$. Suppose these co-ord systems are given by row matrices $C_i = (\mathbf{v}_{i1} \ldots \mathbf{v}_{in_i})$. Then the adapted co-ord system $C : \mathbb{F}^N \longrightarrow V$ is given by the row matrix

$$(\mathbf{v}_{11} \ldots \mathbf{v}_{1n_1} \mathbf{v}_{21} \ldots \mathbf{v}_{2n_2} \ldots \mathbf{v}_{r1} \ldots \mathbf{v}_{rn_r}).$$

Note that the adapted co-ord system for the external direct sum is a block diagonal matrix.

Invariant subspaces & block upper triangular forms

Prop

Let $T : V \longrightarrow V$ be linear & $V_1 \leq V$ be a T-invariant subspace. Pick any vector space complement V_2 to V_1 in V so $V = V_1 \oplus V_2$. Then (identifying internal and external direct sums using the natural isomorphism), T is given by a matrix in $(L(V_j, V_i))$ with *block upper triangular form*

$$\begin{pmatrix} T|_{V_1} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

for some linear maps T_{12}, T_{22}.

Proof. Same as for invariant direct sums.

Remark

As for invariant direct sums, if we pick a co-ordinate system adapted to the direct sum $V_1 \oplus V_2$, then the representing matrix has block upper triangular form.

Examples

E.g. Consider the linear map $T = \frac{d}{dx} : \mathbb{R}[x]_{\leq 1} \longrightarrow \mathbb{R}[x]_{\leq 1}$. Show that $\mathbb{R}[x]_{\leq 0}$ is a T-invariant subspace & picking a complement to it, write a representing matrix in (block) upper triangular form.