Philosophy of studying linear maps $T : V \longrightarrow V$

Aim lecture: One of the main objects of study in linear algebra are *endomorphisms*, linear maps of the form $T : V \longrightarrow V$ where the domain & co-domain are equal. The key is that there are some co-ordinate systems $C : \mathbb{F}^n \longrightarrow V$ which make the study of T easier & are somehow preferred by T. In this lecture we explain an approach for finding these preferred co-ordinate systems.

From a theoretical point of view, it is better to re-phrase our question of finding good co-ordinate systems as follows. Note that a co-ordinate system is essentially a way of writing V as a direct sum of copies of $\mathbb{F}$. So we may simplify our question & ask, can we decompose $V = V' \oplus V''$ for some subspaces V', V'' in such a way as to better understand T. If so, we can try to repeat & further decompose $V' \& V''$.

If we're lucky, we eventually arrive at a decomposition of V into a direct sum of 1-dimensional vector spaces & we have a co-ordinate system. If not we can hopefully still write (in a useful way) V as a direct sum of subspaces V_i of smaller dimension & we get a co-ordinate system by arbitrarily picking bases for each V_i.

Generalisation of internal direct sums

We generalise the notion of internal direct sums to ≥ 2 subspaces, first inductively & then by relating to the (external) direct sum.

Prop-Defn

Let $V_1, \ldots, V_r \leq V$. We say the sum $V_1 + \ldots + V_r$ is *direct* if the sum of $r-1$ subspaces $V_1 + \ldots + V_{r-1}$ is direct, and also the sum of 2 subspaces $(V_1 + \ldots + V_{r-1}) + V_r$ is direct. This is equivalent to saying the natural linear map

$$\Phi = (\mathrm{id}\,|_{V_1} \ldots \mathrm{id}\,|_{V_r}) : V_1 \oplus \ldots \oplus V_r \longrightarrow V_1 + \ldots + V_r : (\mathbf{v}_1, \ldots, \mathbf{v}_r)^T \mapsto \mathbf{v}_1 + \ldots + \mathbf{v}_r$$

is an isomorphism. In this case, we will often write (abusing notation) the internal direct sum $V_1 + \ldots + V_r$ as $V_1 \oplus \ldots \oplus V_r$. Furthermore, for $\mathbf{v} \in \sum_i V_i$ we have $\Phi^{-1}(\mathbf{v}) = (\mathbf{v}_1, \ldots, \mathbf{v}_r)^T$ where $\mathbf{v} = \mathbf{v}_1 + \ldots + \mathbf{v}_r$ is the unique way of expressing $\mathbf{v}$ as a sum with $\mathbf{v}_1 \in V_1, \ldots, \mathbf{v}_r \in V_r$.

Proof. This is by induction using the case of two subspaces.

E.g. $\mathbb{F}^3$ is the internal direct sum of the 3 subspaces $\mathbb{F}\,\mathbf{e}_1, \mathbb{F}\,\mathbf{e}_2, \mathbb{F}\,\mathbf{e}_3$.

Definition of invariance

Given $T : V \longrightarrow V$ linear, the good way of decomposing V into an internal direct sum of subspaces is given in

Defn

A subspace $V' \leq V$ is T-*invariant* or *invariant* wrt T if $T(V') \subseteq V'$. In this case, T restricts to a linear endomorphism $T|_{V'} : V' \longrightarrow V'$. Suppose that $V = V_1 \oplus \ldots \oplus V_r$ is the internal direct sum of subspaces. We say the internal direct sum is T-*invariant* or *invariant* wrt T if each subspace $V_1, \ldots, V_r$ is T-invariant.

E.g. Let $T : \mathbb{F}^3 \longrightarrow \mathbb{F}^3$ be the linear map given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 4 & 5 \end{pmatrix}.$$

The subspace $V' = \mathbb{F}(1,1,0)^T$ is not invariant since

Example continued

Note that $\mathbb{F}^3$ is the internal direct sum of the subspaces $V' = \mathbb{F}\,\mathbf{e}_1, V'' = \mathbb{F}\,\mathbf{e}_2 + \mathbb{F}\,\mathbf{e}_3$. Moreover V', V'' are T-invariant since

Hence $\mathbb{F}^3$ is a T-invariant direct sum of V' & V''.

Example of 3-dim rotation

Let $\mathbf{v} \in \mathbb{R}$ be a unit vector & $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be rotation about the line $L = \mathbb{R}\,\mathbf{v}$. T is a linear map (as can be checked geometrically). Also $\mathbb{R}^3$ is the T-invariant direct sum of L & $P =$

Rem Geometrically it is obvious that the best co-ordinates to use for T is to make L one of the co-ordinate axes & have the other two axes on P. The general principle why this is true is the T-invariance of the direct sum.

Example of (not necessarily orthogonal) projections

Prop-Defn

Suppose that V is the internal direct sum of V_1 & V_2. The *projection onto* V_1 (wrt the direct sum $V_1 \oplus V_2$) is the linear map $P : V \longrightarrow V$ corresponding the map on external direct sums $\tilde{P} : V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$ given by the 2×2-matrix in $(L(V_j, V_i))_{ij}$

$$\begin{pmatrix} \mathrm{id}\,|_{V_1} & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, $P = \Phi \circ \tilde{P} \circ \Phi^{-1} : V \longrightarrow V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2 \longrightarrow V$ where $\Phi : V_1 \oplus V_2 \longrightarrow V : \binom{\mathbf{v}_1}{\mathbf{v}_2} \longrightarrow \mathbf{v}_1 + \mathbf{v}_2$ is the natural isomorphism.

1. More explicitly $P\mathbf{v} = \mathbf{v}_1$ where we have written uniquely $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_1 \in V_1, \mathbf{v}_2 \in V_2$.
2. In this case, $V = V_1 \oplus V_2$ is a P-invariant direct sum.

Proof is an easy ex. Check 1) first.
2) is also easy, but it's more instructive to look at a geometric

E.g. $V = \mathbb{R}^2 = \mathbb{R}\,\mathbf{v} \oplus \mathbb{R}\,\mathbf{w}$ for non-parallel vectors $\mathbf{v}, \mathbf{w}$.

Invariant sums & block diagonal form

Prop

Let $T : V \longrightarrow V$ be linear & $V = V_1 + \ldots + V_r$ be a T-invariant internal direct sum. Let $\Phi : V_1 \oplus \ldots \oplus V_r \longrightarrow V$ be the natural isomorphism with the external direct sum. Then $\Phi^{-1} \circ T \circ \Phi = T|_{V_1} \oplus \ldots \oplus T|_{V_r}$ as maps from $V_1 \oplus \ldots \oplus V_r \longrightarrow V_1 \oplus \ldots \oplus V_r$. Alternatively, this means that (on identifying internal & external direct sums) we may write T as a matrix in $(L(V_j, V_i))$ with *block diagonal form*

$$\begin{pmatrix} T|_{V_1} & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & T|_{V_r} \end{pmatrix}.$$

where we consider $T|_{V_i} : V_i \longrightarrow V_i$ as usual.

Proof. For ease of writing, we do the case $r = 2$. Recall the natural isomorphism $\Phi : V_1 \oplus V_2 \longrightarrow V_1 + V_2 : \binom{\mathbf{v}_1}{\mathbf{v}_2} \mapsto \mathbf{v}_1 + \mathbf{v}_2$.

Proof continued

We know that $\Phi^{-1} \circ T \circ \Phi = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$. For $\mathbf{v}_1 \in V_1$ we have

$$\begin{pmatrix} T\mathbf{v}_1 \\ \mathbf{0} \end{pmatrix} = \Phi^{-1}(T\mathbf{v}_1) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \Phi^{-1}\mathbf{v}_1 = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} T_{11}\mathbf{v}_1 \\ T_{21}\mathbf{v}_1 \end{pmatrix}.$$

Comparing LHS & RHS we see $T_{11} = T|_{V_1}$, $T_{21} = 0$. Similarly, $T_{22} = T|_{V_2}$, $T_{12} = 0$.