Rank & nullity

Aim lecture: We further study vector space complements, which is a tool which allows us to decompose linear problems into smaller ones. We give an algorithm for finding complements & an application to the rank-nullity theorem.

Defn

Let $T : V \longrightarrow W$ be linear. We define the *rank* of T to be $\operatorname{rank} T = \dim \operatorname{im} T$ & the *nullity* of T to be $\operatorname{null} T = \dim \ker T$.

E.g. Recall from MATH1241/1251 how to compute the rank & nullity of the linear map associated to

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 2 & 4 & -2 \end{pmatrix}$$

N.B. If U is row echelon form of A, then rank is no. leading columns of U & nullity is no. non-leading columns.

Some technical lemmas

To prove the rank-nullity theorem, we need some technical lemmas.

Lemma

Let $T : V \longrightarrow W$ *be linear &* $V' \leq V$*. Then* $T(V' + \ker T) = T(V')$

Proof. We know $T(V' + \ker T) \supseteq T(V')$ so we prove $T(V' + \ker T) \subseteq T(V')$. Let $\mathbf{v}' \in V', \mathbf{w} \in \ker T$. Then

$$T(\mathbf{v}' + \mathbf{w}) = T\mathbf{v}' + T\mathbf{w} = T\mathbf{v}' + \mathbf{0} = T\mathbf{v}' \in T(V')$$

& we are done.

Lemma

If $T : V \longrightarrow W$ *is an isomorphism, then* $\dim V = \dim W$*.*

Proof. This follows since if $C : \mathbb{F}^n \longrightarrow V$ is a co-ordinate system then so is $T \circ C : \mathbb{F}^n \longrightarrow V \longrightarrow W$, for a composite of isomorphisms is an isomorphism.

An isomorphism theorem

Theorem (Isomorphism)

Let $T : V \longrightarrow W$ be a linear map. If V' is any vector space complement to $\ker T$ in V, then the restriction $T|_{V'} : V' \longrightarrow T(V)$ is an isomorphism.

Proof. We know that $T|_{V'}$ is linear so it suffices to check it is surjective & injective.

For surjectivity, note that by the lemma above $T|_{V'}(V') = T(V') = T(V' + \ker T) = T(V)$. Hence $T|_{V'}$ is surjective (onto $T(V)$).

For injectivity we compute the kernel of $T|_{V'}$ which is the set of all vectors $\mathbf{v}' \in V'$ such that $T\mathbf{v}' = \mathbf{0}$. Hence $\mathbf{v}' \in V' \cap \ker T = \mathbf{0}$. This shows that $\ker T|_{V'} = \mathbf{0}$ & the theorem is proved.

Geometric view of isomorphism theorem

Sometimes we can easily visualise the isomorphism theorem.
E.g. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be orthogonal projection onto a 1-dim subspace $L = \mathbb{R}\,\mathbf{v}$.
im $T =$
ker $T =$

Rank-nullity theorem

Theorem (Rank-nullity)

Let $T : V \longrightarrow W$ *be a linear map with* $\dim V < \infty$. *Then*

$$null\ T + rank\ T = \dim V.$$

Proof. We know there exists a vector space complement V' to $\ker T$ in V so $V = V' \oplus \ker T$. Also $T|_{V'} : V' \longrightarrow T(V)$ is an isomorphism so $\dim V' = \dim T(V) = \text{rank } T$. We then find

$$\dim V = \dim \ker T + \dim V' = \text{null } T + \text{rank } T.$$

This proves the theorem.

E.g. In the example of orthogonal projn on the previous page we find

An "abstract" example of rank-nullity theorem

It is often useful to use the rank-nullity thm to shorten computations, but usually we can get away without it since the 3 dimensions involved are usually computable. However, there are cases where it really is much nicer to use. First note

Prop

Let $S_r : V' \longrightarrow V, S_l : W \longrightarrow W'$; be linear maps. Then so are

$$L(V,W) \longrightarrow L(V',W) : T \mapsto T \circ S_r \ , \quad L(V,W) \longrightarrow L(V,W') : T \mapsto S_l \circ T$$

Proof. (ex) but it's just the same as for multn by fns using the distributive & associative laws.

E.g. Let $B \in M_{nn}(\mathbb{F})$ be a given square matrix & consider the map $T : M_{nn} \longrightarrow M_{nn}$ defined by $T(A) = BA - AB$. Note it is linear being the difference of two linear maps of the type given in the above propn. Show that T is not surjective.

A It suffices to show that rank $T = \dim \text{im } T < \dim M_{nn} = n^2$. By the rank-nullity theorem, this is equivalent to showing null $T = \dim M_{nn} - \text{rank } T >$

Algorithm for finding vector space complements

Algorithm

Let $C : \mathbb{F}^d \longrightarrow X$ be a co-ordinate system. For vectors $\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{w}_1, \dots, \mathbf{w}_n \in X$, let $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_m)$ & $W = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{w}_1, \dots, \mathbf{w}_n)$. We can find a basis for a vector space complement V' to V in W as follows.

1. Form the $d \times (m + n)$-matrix $A = (C^{-1}\mathbf{v}_1 \dots C^{-1}\mathbf{v}_m \ C^{-1}\mathbf{w}_1 \dots C^{-1}\mathbf{w}_n)$.
2. Let U be the row-echelon form for A.
3. A basis for V' is $\{\mathbf{w}_j | \text{the } (m + j) - \text{th column of } U \text{ is leading}\}$.

To see why the algorithm works, just note that by first year material (or appendix below), we know that the leading columns of U give a basis B_W for W whilst if we look at the first m columns of U, the leading columns there give a basis B_V for V. Hence we have just extended a basis for V to one for W. As in the proof of the existence of vector space complements, we see that $B_W - B_V$ is a basis for a vector space complement so we are done.

Example of finding vector space complement

E.g. Find a vector space complement W to $\text{Span}((1,0,2,1)^T, 0,1,1,1)^T)$ in $\mathbb{F}^4$. The answer should be given in terms of a basis or co-ord system for W.

A Since we are working in $\mathbb{F}^4$ we don't need to impose a new co-ord system (i.e. $C = \text{id}$ will do). Note $\mathbb{F}^4 = \text{Span}((1,0,2,1)^T, 0,1,1,1)^T, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ so we can apply the algorithm with this set up.

Another example of finding vector space complement

E.g. Find a vector space complement W to $\text{Span}(1 + 2x^2 + x^3, x + x^2 + x^3)$ in $\mathbb{F}[x]_{\leq 3}$. The answer should be given in terms of a basis or co-ord system for W.

A Use the co-ordinate system $C = (1 \ x \ x^2 \ x^3) : \mathbb{F}^4 \longrightarrow \mathbb{F}[x]_{\leq 3}$. Note $\mathbb{F}[x]_{\leq 3} = \text{Span}(1 + 2x^2 + x^3, x + x^2 + x^3, 1, x, x^2, x^3)$ so we may apply the algorithm.

Appendix: algorithm for reducing a spanning set to a basis

The algorithm above is based on the following fact you used in 1st year

Fact

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{F}^n$ and $A \in M_{nm}(\mathbb{F})$ be the matrix $A = (\mathbf{v}_1 \ldots \mathbf{v}_m)$. Let U be the row echelon form for A, $L \subseteq \{1, \ldots, m\}$ the set of indices i such that the i-th column of U is leading & $L' = \{1, \ldots, m\} - L$ i.e. L' gives non-leading columns. Then a basis for $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is $B = \{\mathbf{v}_i | i \in L\}$.

Why? Recall that the solns to $A\mathbf{x} = \mathbf{0} \iff U\mathbf{x} = \mathbf{0}$ are the vectors $\mathbf{x} = (x_1, \ldots, x_m)^T \in \mathbb{F}^m$ such that

$$x_1\mathbf{v}_1 + \ldots + x_m\mathbf{v}_m = \mathbf{0} \qquad (*).$$

Now B is linearly independent for omitting the i-th columns from A for $i \in L'$, we see the system of eqns has unique soln $\mathbf{0}$ so the only linear combn of elements in B which is zero is the trivial linear combn.

explanation continued

Also, B still has the same span as $\mathbf{v}_1, \ldots, \mathbf{v}_m$ for the following reason. The non-leading columns give parameters in the set of solns $\mathbf{x}$ so for any $j \in L'$ there is a soln $\mathbf{x}$ to (*) with $x_j = 1$ but $x_i = 0$ for all $i \neq j$ in L'. Then (*) allows us to write $\mathbf{v}_j$ as a lin combn of elements in B so we may omit it from the spanning set without reducing the span.