Some matrix lemmas

Aim lecture: We introduce the dimension which is an important geometric invariant of a vector space that helps us understand the theory of linear algebra. The basic idea is simple enough, an $\mathbb{F}$-space is n dimensional if it is isomorphic to $\mathbb{F}^n$.

Lemma

Let $T : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ be a linear map.

1. *If T is injective then $n \leq m$.*
2. *If T is an isomorphism then $n = m$.*

Proof. For 1), note that T is given by some matrix in $A = M_{mn}(\mathbb{F})$. We prove by contradiction & assume that $n > m$. Applying elementary row operations to A we obtain a row echelon form U which must have some non-leading columns $\because$ there are more columns than rows. Hence $A\mathbf{v} = \mathbf{0}$ has non-zero solns & $\ker T \neq \mathbf{0}$. This shows T is not injective.

To prove 2),

Prop-Defn

Let $C : \mathbb{F}^n \longrightarrow V$ be a co-ordinate system on an $\mathbb{F}$ −space V. Then we say that V is *finite dimensional* of *dimension* $\dim_{\mathbb{F}} V = n$. This is well-defined in the sense that given another co-ordinate system $C_0 : \mathbb{F}^m \longrightarrow V$, we must have $n = m$. The dimension is also the number of vectors in a basis.

Proof. We check the dimension is well-defined by showing $n = m$. Now the composite of isomorphisms $C_0^{-1} \circ C : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ is an isomorphism being linear & invertible (with inverse $C^{-1} \circ C_0$). Hence the lemma shows $n = m$.

E.g. 1 Since $\text{id} : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ is an isomorphism, $\dim \mathbb{F}^n = n$.
E.g. 2 The co-ordinate system $C = (1\ x\ x^2 \ldots x^n) : \mathbb{F}^{n+1} \longrightarrow \mathbb{F}[x]_{\leq n}$ shows that $\dim \mathbb{F}[x]_{\leq n} = n + 1$.

E.g. 3 $\dim M_{mn}(\mathbb{F}) = mn$ since we have the following basis

Size of linearly independent & spanning sets

Lemma

Let $T : V \longrightarrow V', T' : V' \longrightarrow V''$ be linear maps. Then $T' \circ T : V \longrightarrow V''$ is injective (resp surjective) if T, T' are.

Proof. Suppose T, T' are injective & $\mathbf{v} \in \ker T' \circ T$ so $T'(T\mathbf{v}) = \mathbf{0}$. Then T' injective ensures that $T\mathbf{v} = \mathbf{0}$ whilst T injective shows $\mathbf{v} = \mathbf{0}$ so $T' \circ T$ is injective too. The surjectivity result is proved similarly.

Corollary

Let V be an $\mathbb{F}-$space, $I \subset V$ a lin indep set & $S \subset V$ a spanning set. Then the no. of elements $|I|$ in I is less than or equal to the no. of elements $|S|$ in S.

Proof. It suffices to prove the case where S is finite (why?). Then S can be reduced to a basis so $|S| \geq \dim V$. It suffices now to show $m \stackrel{\text{def}}{=} |I| \leq \dim V$. Let $C : \mathbb{F}^n \longrightarrow V$ be a co-ordinate system where $n = \dim V$. If $I = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ then the row matrix $D = (\mathbf{v}_1 \ldots \mathbf{v}_m) : \mathbb{F}^m \longrightarrow V$ defines a linear map which is injective since I is linearly independent. Now C is bijective so the composite $C^{-1} \circ D : \mathbb{F}^m \longrightarrow \mathbb{F}^n$ is injective. The lemma on page 1 ensures $m \leq n$ so the corollary follows.

Alternate characterisation of bases

Since bases can be alternately described as minimal spanning sets or maximal linearly independent sets the corollary immediately gives

Prop

Let $V = \mathbb{F}$-space & $B \subset V$ be a finite set. Then B is a basis iff one of the following equivalent condns hold.

1. B spans V & $|B| = \dim V$.
2. B is linearly independent & $|B| = \dim V$.

Ex Consider a subspace $W \leq V$ with $\dim W = \dim V < \infty$. Show that $W = V$.

A Let B be a basis for W. Then

Extending linearly independent sets to bases

Rem Note that an $\mathbb{F}$-space is finitely spanned iff it is finite dimensional since any finite spanning set can be reduced to a basis. We look at the dual procedure of extending a linearly independent set to a basis.

Prop

Consider a subspace $W \leq V$ where V is finite dimensional. Any linearly independent set $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\} \subset W$ can be extended to a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ of W.

Proof. Note that $\dim W \leq \dim V < \infty$ so we may prove the propn by induction on $\delta = \dim W - m \geq 0$ (why?). The case $\delta = 0$ is given by our alternate charn of bases using dimension.

If $\delta > 0$, then $W \neq \text{Span}(\mathbf{w}_1, \ldots, \mathbf{w}_m)$ so we may pick $\mathbf{w}_{m+1} \in W - \text{Span}(\mathbf{w}_1, \ldots, \mathbf{w}_m)$. The lemma lecture 14 ensures that $\{\mathbf{w}_1, \ldots, \mathbf{w}_{m+1}\}$ is still linearly indep so, by induction we may extend this to a basis of W.

Example of extending a linearly independent set

Later, we will look at an algorithm for showing how to extend a linearly independent set to a basis via co-ordinates. However simple examples can be done by hand as we see below.

E.g. In the vector space $V = M_{22}(\mathbb{F})$, extend the linearly independent set below to a basis

$$I = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \right\}$$

A $\dim V = 4$ so we need only add one extra vector.

Existence of vector space complements

Theorem

Let V be a finite dimensional $\mathbb{F}$-space & W be a subspace. Then we can find a vector space complement W' to W in V (i.e. so $V = W \oplus W'$).

Proof. Pick a basis $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ for W and extend it to a basis of V say by adding $\mathbf{w}'_1, \dots, \mathbf{w}'_n$. We claim that $W' = \text{Span}(\mathbf{w}'_1, \dots, \mathbf{w}'_n)$ is a vector space complement to W.

Now $W + W' = V$ since any vector $\mathbf{v} \in V$ is a linear combn

$$\mathbf{v} = \beta_1 \mathbf{w}_1 + \dots + \beta_m \mathbf{w}_m + \beta'_1 \mathbf{w}'_1 + \dots + \beta'_n \mathbf{w}'_n$$

so in particular, is the sum of a vector in W with one in W'.

To see $W \cap W' = \mathbf{0}$ note that any vector in the intersection can be written in the forms

$$\beta_1 \mathbf{w}_1 + \dots + \beta_m \mathbf{w}_m = \beta'_1 \mathbf{w}'_1 + \dots + \beta'_n \mathbf{w}'_n$$

for some choice of scalars β_i, β'_i. Linear independence of $\{\mathbf{w}_1, \dots, \mathbf{w}_m, \mathbf{w}'_1, \dots, \mathbf{w}'_m\}$ ensures that this vector must be $\mathbf{0}$.

Dimension of direct sums

Ex Let V, W be $\mathbb{F}$-spaces. Show that $\dim V \oplus W = \dim V + \dim W$.

A Consider co-ordinate systems $C_V : \mathbb{F}^m \longrightarrow V, C_W : \mathbb{F}^n \longrightarrow W$. Propn lecture 14 shows that $C_V \oplus C_W : \mathbb{F}^{m+n} \longrightarrow V \oplus W$ is a co-ordinate system on $V \oplus W$. Hence

Dimension of vector space complements

Cor

Let $W \leq V$ be fin dim. Then any vector space complement W' to W in V has dimension $\dim V - \dim W$.

Why?

E.g. Let W be a 2-dim subspace of $\mathbb{R}^3 \oplus M_{23}(\mathbb{R})$. What is the dimension of any vector space complement to W?